**Open Journal of Discrete Mathematics, 2016, 6, 174-179**
Published Online July 2016 in SciRes. http://www.scirp.org/journal/ojdm
http://dx.doi.org/10.4236/ojdm.2016.63015

Scientific Research Publishing

# $\{C_k, P_k, S_k\}$-Decompositions of Balanced Complete Bipartite Multigraphs

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Received 16 June 2016; accepted 11 July 2016; published 14 July 2016



## Abstract

**Let $L=\{H_1,H_2,\cdots,H_r\}$ be a family of subgraphs of a graph *G*. An *L*-decomposition of *G* is an edge-disjoint decomposition of *G* into positive integer $\alpha_i$ copies of $H_i$, where $i\in\{1,2,\cdots,r\}$. Let $C_k$, $P_k$ and $S_k$ denote a cycle, a path and a star with *k* edges, respectively. For an integer $\lambda\ge 2$, we prove that a balanced complete bipartite multigraph $\lambda K_{n,n}$ has a $\{C_k,P_k,S_k\}$-decomposition if and only if *k* is even, $4\le k\le n$ and $\lambda n^2\equiv 0 \pmod{k}$.**

## Keywords

**Balanced Complete Bipartite Multigraph, Cycle, Path, Star, Decomposition**

## 1. Introduction

Let *F*, *G* and *H* be graphs. A *G*-decomposition of *F* is a partition of the edge set of *F* into copies of *G*. If *F* has a *G*-decomposition, we say that *F* is *G*-decomposable. Let $L=\{H_1,H_2,\cdots,H_r\}$ be a family of subgraphs of a graph *G*. An *L-decomposition* of *G* is an edge-disjoint decomposition of *G* into positive integer $\alpha_i$ copies of $H_i$, where $i\in\{1,2,\cdots,r\}$. If *G* has an *L*-decomposition, we say that *G* is *L*-decomposable.

For positive integers *m* and *n*, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes *m* and *n*. A complete bipartite graph is *balanced* if $m=n$. A *k-cycle*, denoted by $C_k$, is a cycle of length *k*. A *k-star*, denoted by $S_k$, is the complete bipartite graph $K_{1,k}$. A *k-path*, denoted by $P_k$, is a path with *k* edges. For a graph *G* and an integer $\lambda\ge 2$, we use $\lambda G$ to denote the multigraph obtained from *G* by replacing each edge *e* by $\lambda$ edges each of which has the same ends as *e*.

**How to cite this paper:** Lin, J.-J. and Jou, M.-J. (2016) $\{C_k,P_k,S_k\}$-Decompositions of Balanced Complete Bipartite Multigraphs. *Open Journal of Discrete Mathematics*, **6**, 174-179. http://dx.doi.org/10.4236/ojdm.2016.63015

Decompositions of graphs into *k*-stars have also attracted a fair share of interest (see [1]-[3]). Articles of $P_k$-decompositions of interest include [4] [5]. Decompositions of some families of graphs into *k*-cycles have been a popular topic of research in graph theory (see [6] [7] for surveys of this topic). The study of $\{G,H\}$-decomposition was introduced by Abueida and Daven in [8]. Abueida and Daven [9] investigated the problem of $\{K_k,S_k\}$-decomposition of the complete graph $K_n$. Abueida and O'Neil [10] settled the existence problem for $\{C_k,S_{k-1}\}$-decomposition of the complete multigraph $\lambda K_n$ for $k\in\{3,4,5\}$. In [11], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of a $\{G,H\}$-factorization of $\lambda K_n$ where $G,H\in\{C_n,P_{n-1},S_{n-1}\}$. Furthermore, Shyu [12] investigated the problem of decomposing $K_n$ into paths and stars with *k* edges, giving a necessary and sufficient condition for $k=3$. In [13], Shyu considered the existence of a decomposition of $K_n$ into paths and cycles with *k* edges, giving a necessary and sufficient condition for $k=4$. Shyu [14] investigated the problem of decomposing $K_n$ into cycles and stars with *k* edges, settling the case $k=4$. Recently, Lee [15] [16] established necessary and sufficient conditions for the existence of a $\{C_k,S_k\}$-decomposition of a complete bipartite graph and $\{P_k,S_k\}$-decomposition of a balanced complete bipartite graph. Lin and Jou [17] investigated the problems of the $\{C_k,P_k,S_k\}$-decomposition of the balanced complete bipartite graph $K_{n,n}$. It is natural to consider the problem of the $\{C_k,P_k,S_k\}$-decomposition of the balanced complete bipartite multigraph $\lambda K_{n,n}$ for $\lambda\ge 2$. In this paper, the necessary and sufficient conditions for the existence of such decomposition are given.

## 2. Preliminaries

Let *G* be a graph. The *degree* of a vertex *x* of *G*, denoted by $\deg_G x$, is the number of edges incident with *x*. The vertex of degree *k* in $S_k$ is the *center* of $S_k$. For $A\subseteq V(G)$ and $B\subseteq E(G)$, we use $G[A]$ and $G-B$ to denote the subgraph of *G* induced by *A* and the subgraph of *G* obtained by deleting *B*, respectively. When $G_1,G_2,\cdots,G_m$ are graphs, not necessarily disjoint, we write $G_1\cup G_2\cup\cdots\cup G_m$ or $\bigcup_{i=1}^m G_i$ for the graph with vertex set $\bigcup_{i=1}^m V(G_i)$ and edge set $\bigcup_{i=1}^m E(G_i)$. When the edge sets are disjoint, $G=\bigcup_{i=1}^m G_i$ expresses the decomposition of *G* into $G_1,G_2,\cdots,G_m$. $nG$ is the short notation for the union of *n* copies of disjoint graphs isomorphic to *G*. Let *H* be a subgraph of $K_{n,n}$ with vertex set $V(H)$ and edge set $E(H)$, and let *r* be a nonnegative integer. We use $H_{+r}$ to denote the graph with vertex set $\{a_i:a_i\in V(H)\}\cup\{b_{j+r}:b_j\in V(H)\}$ and edge set $\{a_ib_{j+r}:a_ib_j\in E(H)\}$ where the subscripts of *b* are taken modulo *n*. For any vertex *x* of a digraph *G*, the *outdegree* $\deg_G^+ x$ (respectively, *indegree* $\deg_G^- x$) of *x* is the number of arcs incident from (respectively, to) *x*. A *multistar* is a star with multiple edges allowed. We use $\overline{S}_k$ to denote a multistar with *k* edges. Let *G* be a multigraph. The *edge-multiplicity* of an edge in *G* is the number of edges joining the vertices of the edge. The *multiplicity* of *G*, denoted by $m(G)$, is the maximum edge-multiplicity of *G*.

**Lemma 1.** ([3]) *For integers m and n with* $m\ge n\ge 1$, *the graph* $K_{m,n}$ *has an* $S_k$*-decomposition if and only if* $m\ge k$ *and*

$$\begin{cases} m\equiv 0 \pmod k & \text{if } n<k,\\ mn\equiv 0 \pmod k & \text{if } n\ge k.\end{cases}$$

**Lemma 2.** ([18]) *Suppose that* $m(\overline{S}_{\lambda k})\le\lambda$. *Then* $\overline{S}_{\lambda k}$ *is* $S_k$*-decomposable.*

Let $a^{(s)}b^{(s)}$ denote the edge *ab* in the *s*-th copy $K_{n,n}$ of $\lambda K_{n,n}$ for $0\le s\le\lambda-1$.

**Lemma 3.** *If k is an even integer with* $k\ge 4$, *then there exist* $\lambda k/2$ *edge-disjoint* 2*k-cycles in* $\lambda K_{k,k}$.

*Proof.* A decomposition of $\lambda K_{k,k}$ into 2*k*-cycles is given by the following $\lambda k/2$ cycles: $C_{+2r}^{(s)}$, where $0\le s\le\lambda-1$, $0\le r\le k/2-1$ and $C^{(s)}=\left(b_0^{(s)}a_0^{(s)}b_1^{(s)}a_1^{(s)}\cdots b_{k-2}^{(s)}a_{k-2}^{(s)}b_{k-1}^{(s)}a_{k-1}^{(s)}\right)$. □

Note that $C_{+2r}^{(s)}$ can be decomposed into two copies of *k*-paths: $P_{+2r}^{(s,0)}:b_{2r}^{(s)}a_0^{(s)}b_{1+2r}^{(s)}a_1^{(s)}\cdots b_{k/2-2+2r}^{(s)}a_{k/2-2}^{(s)}b_{k/2-1+2r}^{(s)}a_{k/2-1}^{(s)}$ and $P_{+2r}^{(s,1)}:b_{k/2+2r}^{(s)}a_{k/2}^{(s)}b_{k/2+1+2r}^{(s)}a_{k/2+1}^{(s)}\cdots b_{k-2+2r}^{(s)}a_{k-2}^{(s)}b_{k-1+2r}^{(s)}a_{k-1}^{(s)}$, that is, $\lambda K_{k,k}$ can be decomposed into $\lambda k$ copies of *k*-paths.

**Lemma 4.** ([4]) *There exists a* $P_k$*-decomposition of* $K_{m,n}$ *if and only if* $mn\equiv 0\pmod k$, *and one of the following* (*see* **Table 1**) *cases occurs.*

**Lemma 5.** ([19]) *For positive integers m, n and k, the graph* $K_{m,n}$ *has a* $C_k$*-decomposition if and only if* $m,n$ *and k are even*, $k\ge 4$, $\min\{m,n\}\ge k/2$, *and* $mn\equiv 0\pmod k$.

**Table 1.** The conditions of a $P_k$-decomposition of $K_{m,n}$.

| Case | k | m | n | Conditions |
|---|---|---|---|---|
| 1 | even | even | even | $k \le 2m, k \le 2n$, not both equalities |
| 2 | even | even | odd | $k \le 2m-2, k \le 2n$ |
| 3 | even | odd | even | $k \le 2m, k \le 2n-2$ |
| 4 | odd | even | even | $k \le 2m-1, k \le 2n-1$ |
| 5 | odd | even | odd | $k \le 2m-1, k \le n$ |
| 6 | odd | odd | even | $k \le m, k \le 2n-1$ |
| 7 | odd | odd | odd | $k \le m, k \le n$ |

## 3. Main Results

With the results ([17]) of the $\{C_k, P_k, S_k\}$-decomposition of the balanced complete bipartite graph $K_{n,n}$, it is assumed that $\lambda \ge 2$ in the sequel. In this section, we will prove the following result.

**Main Theorem.** *Let k and n be positive integers. The graph* $\lambda K_{n,n}$ *has a* $\{C_k, P_k, S_k\}$*-decomposition if and only if k is even,* $4 \le k \le n$ *and* $\lambda n^2 \equiv 0 \pmod{k}$.

We first give necessary conditions for a $\{C_k, P_k, S_k\}$-decomposition of $\lambda K_{n,n}$.

**Lemma 6.** *If* $\lambda K_{n,n}$ *has a* $\{C_k, P_k, S_k\}$*-decomposition, then k is even,* $4 \le k \le n$ *and* $\lambda n^2 \equiv 0 \pmod{k}$.

*Proof.* Since bipartite graphs contain no odd cycle, *k* is even. In addition, the minimum length of a cycle and the maximum size of a star in $\lambda K_{n,n}$ are 4 and *n*, respectively, we have $4 \le k \le n$. Finally, the size of each member in the decomposition is *k* and $\left|E(\lambda K_{n,n})\right| = \lambda n^2$; thus $\lambda n^2 \equiv 0 \pmod{k}$. □

Throughout this paper, let $(A, B)$ denote the bipartition of $\lambda K_{n,n}$, where $A = \{a_0, a_1, \cdots, a_{n-1}\}$ and $B = \{b_0, b_1, \cdots, b_{n-1}\}$. We now show that the necessary conditions are also sufficient. The proof is divided into cases $n = k$, $k < n < 2k$, and $n \ge 2k$, which are treated in Lemmas 7, 8, and 9, respectively.

**Lemma 7.** *For an even integer* $k \ge 4$, *then* $\lambda K_{k,k}$ *has a* $\{C_k, P_k, S_k\}$*-decomposition.*

*Proof.* Note that $\lambda K_{k,k} = 2K_{k/2,k} \cup (\lambda - 1) K_{k,k}$. By Lemmas 1 and 4, $2K_{k/2,k}$ has a $S_k$-decomposition and a $P_k$-decomposition. In addition, by Lemma 5, $(\lambda - 1) K_{k,k}$ has a $C_k$-decomposition. Hence $\lambda K_{k,k}$ has a $\{C_k, P_k, S_k\}$-decomposition. □

**Lemma 8.** *Let k be a positive even integer and let n be a positive integer with* $4 \le k < n < 2k$. *If* $\lambda n^2$ *is divisible by k, then* $\lambda K_{n,n}$ *has a* $\{C_k, P_k, S_k\}$*-decomposition.*

*Proof.* Let $n = k + r$. From the assumption $k < n < 2k$, we have $0 < r < k$. Let $t = \lambda r^2 / k$. Since $k \mid \lambda n^2$, we have $k \mid \lambda r^2$, which implies that *t* is a positive integer. The proof is divided into two cases according to the values of *t*.

Case 1. $t \ge 2$.

Let $G = \lambda K_{n,n}\left[\{a_0, a_1, \cdots, a_{k-1}\} \cup \{b_0, b_1, \cdots, b_{k-1}\}\right]$, $H_1 = \lambda K_{n,n}\left[\{a_0, a_1, \cdots, a_{k-1}\} \cup \{b_k, b_{k+1}, \cdots, b_{k+r-1}\}\right]$, $H_2 = \lambda K_{n,n}\left[\{a_k, a_{k+1}, \cdots, a_{k+r-1}\} \cup \{b_k, b_{k+1}, \cdots, b_{k+r-1}\}\right]$ and $F = \lambda K_{n,n}\left[\{a_k, a_{k+1}, \cdots, a_{k+r-1}\} \cup \{b_0, b_1, \cdots, b_{k-1}\}\right]$.

Clearly $\lambda K_{n,n} = G \cup H_1 \cup H_2 \cup F$. Note that *G* is isomorphic to $\lambda K_{k,k}$, $H_1$ is isomorphic to $\lambda K_{k,r}$, $H_2$ is isomorphic to $\lambda K_{r,r}$ and *F* is isomorphic to $\lambda K_{r,k}$, which can be decomposed into $\lambda r$ copies of $S_k$ by Lemmas 1 and 2. In the following, we will show that $G \cup H_1 \cup H_2$ can be decomposed into $t-1$ copies of $P_k$, one copy of $C_k$ and $\lambda(k+r)$ copies of $S_k$.

Let $p = \lfloor t/2 \rfloor = c(k/2) + d$, where $0 \le c \le \lambda - 2$ and $0 \le d \le k/2 - 1$. Define a subgraph *W* of *G* as follows:

$$W = \begin{cases} \left(\bigcup_{s=0}^{c-1}\bigcup_{s=0}^{k/2-1} C_{+2r}^{(s)}\right) \cup \left(\bigcup_{r=0}^{d-1} C_{+2r}^{(c)}\right), & \text{if } t \text{ is even,} \\ \left(\bigcup_{s=0}^{c-1}\bigcup_{r=0}^{k/2-1} C_{+2r}^{(s)}\right) \cup \left(\bigcup_{r=0}^{d-1} C_{+2r}^{(c)}\right) \cup P_{+2d}^{(c,0)}, & \text{if } t \text{ is odd,} \end{cases}$$

and the subscripts of *b* are taken modulo *k*. Note that $\lambda k - 2p = \lambda k - t > 0$ for *t* is even, and $\lambda k - 2p - 2 = \lambda k - (t-1) - 2 = \lambda k - t - 1 > 0$ for *t* is odd, this assures us that there are enough edges for *W*.

Note that a $C_{2k}$ can be decomposed into 2 copies of $P_k$. In addition, $2p=t$ for $t$ is even as well as $2p+1=t$ for $t$ is odd, it follows that $W$ can be decomposed into $t$ copies of $P_k$. Since $t=\lambda r^2/k<\lambda k-1$, we interchange two edges $a_{k/2-1}^{(0)}b_{k/2}^{(0)}$ in $P^{(0,0)}$ and $a_{k/2-1}^{(\lambda-1)}b_0^{(\lambda-1)}$ in $P_{+2\lfloor(k+2)/4\rfloor}^{(\lambda-1,0)}$, then we obtain a new cycle $\left(b_0^{(0)}a_0^{(0)}b_1^{(s)}a_1^{(0)}\cdots b_{k/2-2}^{(0)}a_{k/2-2}^{(0)}b_{k/2-1}^{(0)}a_{k/2-1}^{(0)}\right)$. Hence $W\setminus\left\{a_{k/2-1}^{(0)}b_{k/2}^{(0)}\right\}\cup\left\{a_{k/2-1}^{(\lambda-1)}b_0^{(\lambda-1)}\right\}$ can be decomposed into $t-1$ copies of $P_k$ and one copy of $C_k$.

Let $G'$ be the graph obtained from $G$ by deleting the edges in $W$. For the case of $t$ is even, we have that

$$\deg_{G'}a_i=\lambda k-2p.$$

The other case of $t$ is odd, we have that

$$\deg_{G'}a_i=\begin{cases}\lambda k-2p-2, & \text{if } i=0,1,\cdots,k/2-1,\\ \lambda k-2p, & \text{if } i=k/2,k/2+1,\cdots,k-1,\end{cases}$$

Let $X_i=G'\left[\{a_i\}\cup\{b_0,b_1,\cdots,b_{k-1}\}\right]$ for $i=0,1,\cdots,k-1$. Then for $t$ is even $X_i=\bar{S}_{\lambda k-2p}$, and for $t$ is odd

$$X_i=\begin{cases}\bar{S}_{\lambda k-2p-2}, & \text{if } i=0,1,\cdots,k/2-1,\\ \bar{S}_{\lambda k-2p}, & \text{if } i=k/2,k/2+1,\cdots,k-1\end{cases}$$

with the center at $a_i$.

In the following, we will show that $H_1$ can be decomposed into $r$ copies of $\bar{S}_{\lambda(k-r)}$ with centers in $\{b_k,b_{k+1},\cdots,b_{k+r-1}\}$, and into $k$ copies of $\bar{S}_{2p}$ with centers in $\{a_0,a_1,\cdots,a_{k-1}\}$ for $t$ is even as well as $k/2$ copies of $\bar{S}_{2p+2}$ with centers in $\left\{a_0,a_1,\cdots,a_{\frac{k}{2}-1}\right\}$ and $k/2$ copies of $\bar{S}_{2p}$ with centers in $\left\{a_{\frac{k}{2}},a_{\frac{k}{2}+1},\cdots,a_{k-1}\right\}$ for $t$ is odd, that is, there exists an orientation of $H_1$ such that

$$\deg_{H_1}^{+}b_j=\lambda(k-r), \tag{1}$$

where $j=k,k+1,\cdots,k+r-1$, and for $t$ is even

$$\deg_{H_1}^{+}a_i=2p, \tag{2}$$

where $i=0,1,\cdots,k-1$, and for $t$ is odd

$$\deg_{H_1}^{+}a_{ii}=\begin{cases}2p+2, & \text{if } i=0,1,\cdots,k/2-1,\\ 2p, & \text{if } i=k/2,k/2+1,\cdots,k-1.\end{cases} \tag{3}$$

We first consider the edges oriented outward from $\{a_0,a_1,\cdots,a_{k-1}\}$. If $t$ is even, then the edges $a_ib_{(2p)i+k},a_ib_{(2p)i+k+1},\cdots,a_ib_{(2p)i+k+2p-1}$ are all oriented outward from $a_i$ where $i=0,1,\cdots,k-1$. If $t$ is odd, for $i=0,1,\cdots,k/2-1$, the edges $a_ib_{(2p+2)i+k},a_ib_{(2p+2)i+k+1},\cdots,a_ib_{(2p+2)i+k+2p+1}$ and $a_ib_{(p+2)k+(2p)i},a_ib_{(p+2)k+(2p)i+1},\cdots,a_ib_{(p+2)k+(2p)i+2p-1}$ are all oriented outward from $a_i$, where the subscripts of $b$ are taken modulo $r$ in the set $\{k,k+1,\cdots,k+r-1\}$. In both of the cases the subscripts of $b$ are taken modulo $r$ in the set of numbers $\{k,k+1,\cdots,k+r-1\}$. Since $2p=t<\lambda r$ for $t$ is even, and $2p+2=(t-1)+2=t+1<\lambda r$ for $t$ is odd, this assures us that there are enough edges for the above orientation. Finally, the edges which are not oriented yet are all oriented from $\{b_k,b_{k+1},\cdots,b_{k+r-1}\}$ to $\{a_0,a_1,\cdots,a_{k-1}\}$.

From the construction of the orientation, it is easy to see that (2) and (3) are satisfied, and for all $b_j,b_{j'}\in\{b_k,b_{k+1},\cdots,b_{k+r-1}\}$, we have

$$\left|\deg_{H_1}^{-}b_j-\deg_{H_1}^{-}b_{j'}\right|\le 1. \tag{4}$$

So, we only need to check (1).

Since $\deg_{H_1}^{+}b_j+\deg_{H_1}^{-}b_j=\lambda k$ for $b_j\in\{b_k,b_{k+1},\cdots,b_{k+r-1}\}$, it follows from (4) that $\left|\deg_{H_1}^{+}b_j-\deg_{H_1}^{+}b_{j'}\right|\le 1$ for $b_j,b_{j'}\in\{b_k,b_{k+1},\cdots,b_{k+r-1}\}$. Note that $t$ is even, $\sum_{i=0}^{k-1}\deg_{H_1}^{+}a_i=(2p)k=tk$, and $t$ is odd,

$$\sum_{i=0}^{k-1} \deg^{+}_{H_1} a_i = k/2(2p+2)+k/2(2p) = (2p+1)k = tk.$$

Thus,

$$\sum_{j=k}^{k+r-1} \deg^{+}_{H_1} b_j = \left|E\left(\lambda K_{k,r}\right)\right| - \sum_{i=0}^{k-1} \deg^{+}_{H_1} a_i = \lambda kr - tk = \lambda kr - \lambda r^2 = \lambda r(k-r).$$

Therefore $\deg^{+}_{H_1} b_j = \lambda(k-r)$ for $b_j \in \{b_k, b_{k+1}, \cdots, b_{k+r-1}\}$. This proves (1). Hence, there exists the required decomposition $\mathscr{T}$ of $H_1$. Let $X'_i$ be the star with center at $a_i$ in $\mathscr{T}$ for $i = 0,1,\cdots,k-1$. Then $X_i + X'_i$ is an $\overline{S}_{\lambda k}$. Since $m(X_i + X'_i) \le \lambda$, by Lemma 2, we obtain that $X_i + X'_i$ can be decomposed into $\lambda$ copies of $S_k$ for $i = 0,1,\cdots,k-1$.

Let $U_j$ be the $\lambda(k-r)$-multistar with center at $b_j$ in $\mathscr{T}$ for $j = k, k+1, \cdots, k+r-1$. Let $U'_j = H_2\left[\{a_k, a_{k+1}, \cdots, a_{k+r-1}, b_j\}\right]$ for $k \le j \le k+r-1$. Then $H_2$ is decomposed into $U'_k$, $U'_{k+1}, \cdots, U'_{k+r-1}$, and each $U'_j = \overline{S}_{\lambda r}$. It follows that $U_j + U'_j = \overline{S}_{\lambda k}$. Since $m\left(U_j + U'_j\right) \le \lambda$, by Lemma 2, we obtain that $U_j + U'_j$ can be decomposed into $\lambda$ copies of $S_k$ for $j = k, k+1, \cdots, k+r-1$. Recall that $\lambda K_{n,n} = G + H_1 + H_2 + F$, we have that $\lambda K_{n,n}$ is $(C_k, P_k, S_k)$-decomposable.

Case 2. $t = 1$.

Let $G'_0 = K_{n,n}\left[\{a_0, a_1, \cdots, a_{k/2-1}\} \cup \{b_0, b_1, \cdots, b_{k-1}\}\right]$, $G'_1 = K_{n,n}\left[\{a_{k/2}, a_{k/2+1}, \cdots, a_{k-1}\} \cup \{b_0, b_1, \cdots, b_{k-1}\}\right]$, $H = \lambda K_{n,n}\left[\{a_0, a_1, \cdots, a_{k+r-1}\} \cup \{b_k, b_{k+1}, \cdots, b_{k+r-1}\}\right]$ and $F = \lambda K_{n,n}\left[\{a_k, a_{k+1}, \cdots, a_{k+r-1}\} \cup \{b_0, b_1, \cdots, b_{k-1}\}\right]$.

Then $\lambda K_{n,n} = (\lambda-1)K_{k,k} \cup G'_0 \cup G'_1 \cup F \cup H$. By similar arguments as in the proof of Case 1, we have that $G'_0 \cup G'_1 \cup F \cup H$ can be decomposed into one copy of $P_k$ and $k + 2\lambda r$ copies of $S_k$. On the other hand, by Lemma 5, $(\lambda-1)K_{k,k}$ has a $C_k$-decomposition. Hence $\lambda K_{n,n}$ has a $\{C_k, P_k, S_k\}$-decomposition. □

**Lemma 9.** *Let k be a positive even integer and let n be a positive integer with* $4 \le k \le n/2$. *If* $\lambda n^2$ *is divisible by k, then* $\lambda K_{n,n}$ *has a* $\{C_k, P_k, S_k\}$*-decomposition.*

*Proof.* Let $n = qk + r$ where *q* and *r* are integers with $0 \le r < k$. From the assumption of $k \le n/2$, we have $q \ge 2$. Note that

$$\lambda K_{n,n} = \lambda K_{qk+r,qk+r} = \lambda K_{(q-1)k,(q-1)k} \cup \lambda K_{k+r,(q-1)k} \cup \lambda K_{(q-1)k,k+r} \cup \lambda K_{k+r,k+r}.$$

Trivially, $\left|E\left(\lambda K_{(q-1)k,(q-1)k}\right)\right|$, $\left|E\left(\lambda K_{k+r,(q-1)k}\right)\right|$ and $\left|E\left(\lambda K_{(q-1)k,k+r}\right)\right|$ are multiples of *k*. Thus $\lambda(k+r)^2 \equiv 0 \pmod{k}$ from the assumption that $n^2$ is divisible by *k*. By Lemmas 1 and 2, $\lambda K_{(q-1)k,(q-1)k}$, $\lambda K_{k+r,(q-1)k}$ and $\lambda K_{(q-1)k,k+r}$ have $S_k$-decomposition.

In the case of $r = 0$, by Lemma 7, we obtain that $\lambda K_{k,k}$ has a $\{C_k, P_k, S_k\}$-decomposition. In addition, by Lemma 8, $\lambda K_{k+r,k+r}$ has a $\{C_k, P_k, S_k\}$-decomposition for $0 < r < k$. Hence there exists a $\{C_k, P_k, S_k\}$-decomposition of $\lambda K_{n,n}$. □

Now we are ready for the main result. It is obtained by combining Lemmas 6, 7, 8 and 9.

**Theorem 1.** *Let k and n be positive integers. The graph* $\lambda K_{n,n}$ *has a* $\{C_k, P_k, S_k\}$*-decomposition if and only if k is even,* $4 \le k \le n$ *and* $\lambda n^2 \equiv 0 \pmod{k}$.

**Remark.** Let *m* and *n* be positwe integers with $m \ge n$. Since bipartite graphs contain no odd cycle, *k* is even. In addition, the minimum length of a cycle and the maximum size of a star in $\lambda K_{m,n}$ are 4 and *m*, respectively, we have $4 \le k \le m$. Moreover, each *k*-cycle in $\lambda K_{m,n}$ uses $k/2$ vertices of each partite set, which implies that $k/2 \le n$. Finally, the size of each member in the decomposition is *k* and $\left|E\left(K_{m,n}\right)\right| = \lambda mn$, thus $\lambda mn \equiv 0 \pmod{k}$. Hence the obvious necessary conditions for the graph $\lambda K_{m,n}$ to have a $\{C_k, P_k, S_k\}$-decomposition are: 1) *k* is even, 2) $4 \le k \le \min\{m, n/2\}$, and 3) $\lambda mn \equiv 0 \pmod{k}$. It is natural to ask whether they are sufficient.

## Acknowledgements

The authors are grateful to the referees for the helpful comments.

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