

# On Polynomials $R_n(x)$ Related to the Stirling Numbers and the Bell Polynomials Associated with the $p$ -Adic Integral on $\mathbb{Z}_p$

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## Abstract

In this paper, one introduces the polynomials  $R_n(x)$  and numbers  $R_n$  and derives some interesting identities related to the numbers and polynomials:  $R_n$  and  $R_n(x)$ . We also give relation between the Stirling numbers, the Bell numbers, the  $R_n$  and  $R_n(x)$ .

## Keywords

The Euler Numbers and Polynomials, The Stirling Numbers, The Bell Polynomials and Numbers

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## 1. Introduction

Recently, many mathematicians have studied the area of the Stirling numbers, the Euler numbers and polynomials (see [1]-[11]). We studied some properties of the polynomials  $R_n(x)$  and numbers  $R_n$  in complex field (see [12]). In this paper, based on the Euler numbers and polynomials, we define the numbers  $R_n$  and polynomials  $R_n(x)$  by using the  $p$ -adic integrals on  $\mathbb{Z}_p$  in  $p$ -adic field. Then, we get some interesting properties and relations of the Stirling numbers, the  $R_n$ , and the Bell numbers. It is interesting that the Euler polynomials  $R_n(x)$  and  $R_n(x)$  to be define in this paper have a different structure (see [Figure 2]). Zeros of  $E_n(x)$  are a symmetric structure but zeros of  $R_n(x)$  are not.

Throughout this paper, we use the following notations. By  $\mathbb{Z}_p$ , we denote the ring of  $p$ -adic rational integers,  $\mathbb{Q}_p$  denotes the field of  $p$ -adic rational numbers,  $\mathbb{C}_p$  denotes the completion of algebraic closure of  $\mathbb{Q}_p$ ,  $\mathbb{N}$

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denotes the set of natural numbers,  $\mathbb{Z}$  denotes the ring of rational integers,  $\mathbb{Q}$  denotes the field of rational numbers,  $\mathbb{C}$  denotes the set of complex numbers, and  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$  and  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$  denote the binomial coefficient. Let  $v_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-v_p(p)} = p^{-1}$ .

For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  is defined by T. Kim as below:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-1}} \sum_{x=0}^{p^N-1} g(x) (-1)^x \quad (\text{cf. [5]}). \tag{1.1}$$

If we take  $g_1(x) = g(x+1)$  in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

From (1.2), we obtain

$$I_{-1}(g_n) + (-1)^{n-1} I_{-q}(g) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l), \tag{1.3}$$

where  $g_n(x) = g(x+n)$  (cf. [5]-[10]).

The classical Euler polynomials are defined by the following generating function

$$F(t, x) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \tag{1.4}$$

with the usual convention of replacing  $E^n(x)$  by  $E_n(x)$ . In generally, the original Euler numbers are when  $x = \frac{1}{2}$  and normalizing by  $2^n$  gives the Euler number as following:

$$E_n = 2^n E_n\left(\frac{1}{2}\right).$$

But in this paper, Euler numbers are when  $x=0$ . In other words,  $E_n(0) = E_n$  and in this paper, Euler numbers mean the Euler numbers having a generating function as below(cf. [5]-[10]):

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.5}$$

The Stirling number of the second kind  $S_2(n, r)$  is the number of partitions of  $n$  things into  $r$  non-empty sets; it is positive if  $1 \leq r \leq n$  and zero for other values of  $r$  (see [1]). It satisfies the recurrence relation

$$S_2(n+1, r) = S_2(n, r-1) + rS_2(n, r).$$

The generating function of the Stirling numbers is defined as below:

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S_2(n, k) \frac{x^n}{n!}. \tag{1.6}$$

As well known definition, the Bell polynomials are defined by Bell (1934) as below

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = e^{(e^t-1)x}. \tag{1.7}$$

Also, let  $S_2(n, k)$  be denote the Stirling numbers of the second kind. Then

$$B_n(x) = \sum_{k=0}^n S_2(n, k) x^k. \tag{1.8}$$

In the special case,  $B_n(1) = B_n$  are called the  $n$ -th Bell numbers.

The motivation of this paper is the Euler numbers and Bell numbers's generating function. From this idea, we induce some interesting properties related to the Stirling numbers, the Bell numbers, the Euler numbers and the  $R_n$ .

Our aim in this paper is to define analogue Euler numbers and polynomials. We investigate some properties which are related to  $R_n$ ,  $R_n(x)$ . Especially, we derive the relations of the Stirling numbers and the  $R_n$ , the  $R_n(x)$ .

## 2. An Introduction to Numbers $R_n$ and Polynomials $R_n(x)$

Our primary goal of this section is to define numbers  $R_n$  and polynomials  $R_n(x)$ . We also find the witt's formula for numbers  $R_n$  and polynomials  $R_n(x)$  by (1.2).

By (1.2) and using  $p$ -adic integral on  $\mathbb{Z}_p$ , we get as below:

$$\text{Let } g(x) = e^{(e^t-1)x}.$$

$$\begin{aligned} I_{-1}(g_1) + I_{-1}(g) &= \int_{\mathbb{Z}_p} e^{(e^t-1)(x+1)} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{(e^t-1)x} d\mu_{-1}(x) \\ &= \left( e^{(e^t-1)} + 1 \right) \int_{\mathbb{Z}_p} e^{(e^t-1)x} d\mu_{-1}(x) = 2. \end{aligned} \tag{2.1}$$

Hence, by (2.1) we get the following:

$$\frac{2}{e^{(e^t-1)} + 1} = \int_{\mathbb{Z}_p} e^{(e^t-1)x} d\mu_{-1}(x). \tag{2.2}$$

Also, Let  $g(y) = e^{(e^t-1)y+xt}$ . By the same method (2.1), we get the following:

$$\frac{2}{e^{(e^t-1)} + 1} e^{xt} = \int_{\mathbb{Z}_p} e^{(e^t-1)y+xt} d\mu_{-1}(y). \tag{2.3}$$

From (2.2) and (2.3), we define numbers and polynomials  $R_n$ ,  $R_n(x)$  as below:

$$\sum_{n=0}^{\infty} R_n \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(e^t-1)x} d\mu_{-1}(x), \tag{2.4}$$

$$\sum_{n=0}^{\infty} R_n(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(e^t-1)y+xt} d\mu_{-1}(y), \tag{2.5}$$

respectively.

From above definition, one easily has the Witt's formula as below:

$$R_n = \int_{\mathbb{Z}_p} \sum_{l=0}^n x^l S_2(n, l) d\mu_{-1}(x), \tag{2.6}$$

$$R_n(x) = \sum_{k=0}^n \binom{n}{k} x^k \int_{\mathbb{Z}_p} \sum_{l=0}^{n-k} y^l S_2(n-k, l) d\mu_{-1}(y), \tag{2.7}$$

with the usual convention of replacing  $R^n(x)$  by  $R_n(x)$  respectively. In the special case,  $x = 0$ ,  $R_n(0) = R_n$  are called the  $n$ -th  $R$ -numbers.

From (2.6) and  $\int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) = E_l$

$$R_n = \int_{\mathbb{Z}_p} \sum_{l=0}^n x^l S_2(n, l) d\mu_{-1}(x) = \sum_{l=0}^n S_2(n, l) \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) = \sum_{l=0}^n S_2(n, l) E_l.$$

Hence, we get the following;

$$R_n = \sum_{l=0}^n S_2(n, l) E_l \tag{2.8}$$

where  $E_i$  is the Euler numbers.

Also, from (2.5) and by simple calculus, one has

$$R_n(x) = \sum_{k=0}^n \binom{n}{k} R_{n-k} x^k. \tag{2.9}$$

From (2.8) and (2.9), we get some polynomials as below:

$$\begin{aligned} R_1(x) &= x - \frac{1}{2}, \\ R_2(x) &= x^2 - x - \frac{1}{2}, \\ R_3(x) &= x^3 - \frac{3x^2}{2} - \frac{3x}{2} - \frac{1}{4}, \\ R_4(x) &= x^4 - 2x^3 - 3x^2 - x + 1, \\ R_5(x) &= x^5 - \frac{5x^4}{2} - 5x^3 - \frac{5x^2}{2} + 5x + \frac{21}{4}, \\ R_6(x) &= x^6 - 3x^5 - \frac{15x^4}{2} - 5x^3 + 15x^2 + \frac{63x}{2} + \frac{29}{2}. \end{aligned}$$

### 3. Basic Properties for $R_n$ and $R_n(x)$ Related to the Stirling Numbers, the Bell Numbers and the Euler Numbers

From (2.5) and by the simple calculation

$$\begin{aligned} \sum_{n=0}^{\infty} R_n(x) \frac{t^n}{n!} &= \frac{2}{e^{e^t-1} + 1} e^{xt} = 2 \sum_{l=0}^{\infty} (-1)^l e^{(e^t-1)^l} e^{xt} \\ &= 2 \sum_{l=0}^{\infty} (-1)^l \sum_{m=0}^{\infty} \left( \sum_{k=0}^m \binom{m}{k} \sum_{n=0}^{m-k} l^n S_2(m-k, n) x^k \right) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left( 2 \sum_{l=0}^{\infty} (-1)^l (B(l) + x)^n \right) \frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

where  $B_n(x) = \sum_{l=0}^n x^l S_2(n, l)$  are the Bell polynomials.

By comparing the coefficients of  $\frac{t^n}{n!}$  on the both sides of the above equation, we get the following the theorem immediately.

**Theorem 1.** For  $n \in \mathbb{N}$  with  $n > 1$ , one has

$$R_n(x) = 2 \sum_{l=0}^{\infty} (-1)^l (B(l) + x)^n = 2 \sum_{l=0}^{\infty} (-1)^l \sum_{k=0}^n \binom{n}{k} \sum_{m=0}^{n-k} l^m S_2(n-k, m) x^k$$

where  $B_n(x)$  are the Bell polynomials.

From (2.5), one has

$$\frac{d}{dx} R_n(x) = n R_{n-1}(x).$$

Let  $f(x) = e^{(e^t-1)x}$ . Then from (1.2)

$$\begin{aligned} 2 &= I_{-1}(f_1) + I_{-1}(f) = \int_{Z_p} e^{(e^t-1)(x+1)} d\mu_{-1}(x) + \int_{Z_p} e^{(e^t-1)x} d\mu_{-1}(x) \\ &= \sum_{l=0}^{\infty} \left( \int_{Z_p} B_l(x+1) d\mu_{-1}(x) + \int_{Z_p} B_l(x) d\mu_{-1}(x) \right) \frac{t^l}{l!} = \sum_{l=0}^{\infty} \left( \sum_{n=0}^l S_2(l, n) (E_n(1) + E_n) \right) \frac{t^l}{l!}. \end{aligned} \tag{3.2}$$

By comparing the coefficients of  $\frac{t^l}{l!}$  on the both sides of the above equation, we get the following theorem immediately.

**Theorem 2.** For  $n \in \mathbb{N}$  with  $n > 1$ , let  $S_2(m, n)$  be the stirling numbers. Then, one has

$$\sum_{n=0}^l S_2(l, n)(E_n(1) + E_n) = \begin{cases} 2, & \text{if } l = 0, \\ 0, & \text{if } l \geq 1, \end{cases}$$

where  $E_n(x)$  and  $E_n$  are the Euler polynomials and the Euler numbers respectively. And

$$\int_{Z_p} B_l(x+1) + B_l(x) d\mu_{-1}(x) = \begin{cases} 2, & \text{if } l = 0, \\ 0, & \text{if } l \geq 1, \end{cases}$$

where  $B_n(x)$  is the  $n$ -th Bell polynomial.

Also, from (2.1) one has

$$\begin{aligned} 2 &= I_{-1}(f_1) + I_{-1}(f) = e^{(e^t-1)} \int_{Z_p} e^{(e^t-1)x} d\mu_{-1}(x) + \int_{Z_p} e^{(e^t-1)x} d\mu_{-1}(x) \\ &= e^{(e^t-1)} \sum_{n=0}^{\infty} R_n \frac{t^n}{n!} + \sum_{n=0}^{\infty} E_n R_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} S_2(n-k, l) R_k + R_n \right) \frac{t^n}{n!}. \end{aligned} \tag{3.3}$$

By comparing the coefficients of  $t^n$  on the both sides of the above equation, we get the following theorem immediately.

**Theorem 3.** For  $n \in \mathbb{N}$  with  $n > 1$ , one has

$$R_n = - \sum_{k=0}^n \binom{n}{k} S_2(n-k, l) R_k.$$

Let  $f(x) = e^{(e^t-1)x}$ . Then from (1.3), we derive the following:

Left side of (1.3) is as below:

$$\begin{aligned} I_{-1}(f_n) + (-1)^{n+1} I_{-1}(f) &= \int_{Z_p} e^{(e^t-1)(x+n)} d\mu_{-1}(x) + (-1)^{n+1} \int_{Z_p} e^{(e^t-1)x} d\mu_{-1}(x) \\ &= \sum_{k=0}^{\infty} (e^t - 1)^k \left( \int_{Z_p} (x+n)^k - (-1)^n \int_{Z_p} x^k \right) d\mu_{-1}(x) \frac{1}{k!} \\ &= \sum_{k=0}^{\infty} (e^t - 1)^k (E_k(n) - (-1)^n E_k) \frac{1}{k!} \\ &= \sum_{l=0}^{\infty} \left( \sum_{k=0}^l S_2(l, k) (E_k(n) - E_k) \right) \frac{t^l}{l!}, \end{aligned} \tag{3.4}$$

and right side of (1.3) is as below:

$$\begin{aligned} 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) &= 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} e^{(e^t-1)l} = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} \sum_{k=0}^{\infty} \left( \sum_{s=0}^k S_2(k, s) l^s \right) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left( 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} \sum_{s=0}^k S_2(k, s) l^s \right) \frac{t^k}{k!}. \end{aligned} \tag{3.5}$$

Hence, from (3.4) and (3.5), we get the following theorem.

**Theorem 4.** For  $n \in \mathbb{N}$  with  $n > 1$ , one has

$$\sum_{k=0}^m S_2(m, k) (E_k(n) - E_k) = \sum_{l=0}^{n-1} (-1)^{n-1-l} \sum_{s=0}^m S_2(m, s) l^s.$$

where  $E_k(x)$  and  $E_k$  are the Euler polynomials and numbers respectively.

By using the definition of  $R_n(x)$  and simple calculation, we get the following:

$$R_n(x+y) = \sum_{k=0}^n \binom{n}{k} R_k(x) y^{n-k} = \sum_{k=0}^n \binom{n}{k} R_{n-k}(x+y)^k.$$

and the equality above is expressed as follows:

$$R_n(x+y) = (R(x)+y)^n = (R+x+y)^n.$$

It is well known that  $B_n(x) = \sum_{l=0}^n x^l S_2(n,l)$ . By the definition  $R_n$  and some calculation, we get the following:

$$\sum_{n=0}^{\infty} R_n \frac{t^n}{n!} = \int_{Z_p} e^{(e^t-1)x} d\mu_{-1}(x) = \sum_{n=0}^{\infty} \int_{Z_p} \sum_{l=0}^n x^l S_2(n,l) d\mu_{-1}(x) \frac{t^n}{n!}. \tag{3.6}$$

Hence, one has the following theorem.

**Theorem 5.** For  $n \in \mathbb{N}$  with  $n > 1$ , one has

$$R_n = \int_{Z_p} \sum_{l=0}^n x^l S_2(n,l) d\mu_{-1}(x) = \int_{Z_p} B_n(x) d\mu_{-1}(x)$$

where  $B_n(x)$  are the Bell polynomials.

By the same method above Theorem 5, we get the corollary as follows:

**Corollary 6.** For  $n \in \mathbb{N}$  with  $n > 1$ , one has

$$R_n(x) = \sum_{k=0}^n \binom{n}{k} x^k \int_{Z_p} \sum_{l=0}^{n-k} y^l S_2(n-k,l) d\mu_l(y) = \int_{Z_p} (x+B(y))^n d\mu_l(y) \tag{3.7}$$

where  $B_n(y)$  are the Bell polynomials.

It is well known that  $\frac{2}{e^t+1} e^{xt}$  is the generating function of the Euler polynomials. We substitute  $e^t - 1$  for  $t$  in the generating function of the Euler polynomials as below:

$$\sum_{n=0}^{\infty} E_n(x) \frac{(e^t-1)^n}{n!} = \frac{2}{e^{(e^t-1)}+1} e^{(e^t-1)x}. \tag{3.8}$$

The left-hand-side of (3.8) is

$$\sum_{n=0}^{\infty} E_n(x) \frac{(e^t-1)^n}{n!} = \sum_{n=0}^{\infty} E_n(x) \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!} \sum_{n=0}^{\infty} \left( \sum_{l=0}^n E_l(x) S_2(n,l) \right) \frac{t^n}{n!}. \tag{3.9}$$

The right-hand-side of (3.8) is

$$\begin{aligned} \frac{2}{e^{(e^t-1)}+1} e^{(e^t-1)x} &= \sum_{n=0}^{\infty} R_n \frac{t^n}{n!} \times \sum_{n=0}^{\infty} (e^t-1)^n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} R_n \frac{t^n}{n!} \times \sum_{n=0}^{\infty} \left( \sum_{l=0}^n S_2(n,l) x^l \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} R_{n-k} \sum_{l=0}^k S_2(k,l) x^l \right) \frac{t^n}{n!}. \end{aligned} \tag{3.10}$$

By (3.9),(3.10) and comparing the coefficient of both sides, we get the following theorem.

**Theorem 7.** For  $n \in \mathbb{N}$  with  $n > 1$ , one has

$$\sum_{l=0}^n E_l(x) S_2(n,l) = \sum_{k=0}^n \binom{n}{k} R_{n-k} \sum_{l=0}^k S_2(k,l) x^l = \sum_{k=0}^n \binom{n}{k} R_{n-k} B_k(x)$$

where  $E_l(x)$  and  $B_n(x)$  are the Euler polynomials and the Bell polynomials respectively.

It is not difficult to see that

$$e^{(e'-1)x} = \frac{1}{2} \left( \frac{2}{e^{(e'-1)} + 1} e^{(e'-1)(x+1)} + \frac{2}{e^{(e'-1)} + 1} e^{(e'-1)x} \right). \tag{3.11}$$

From the expression (3.11), one has

$$2 \sum_{l=0}^n S_2(n, l) x^l = \sum_{k=0}^n \binom{n}{k} R_{n-k} (B_k(x+1) + B_k(x)).$$

Specially, if  $x = 1$ ,

$$2 \sum_{l=0}^n S_2(n, l) = \sum_{k=0}^n \binom{n}{k} R_{n-k} (B_k(2) + B_k(1))$$

where  $B_n(x)$  are the  $n$ -th Bell polynomials.

### 4. Zeros of the Bell Polynomials $B_n(x)$ and the Polynomials $R_n(x)$

In this section, we investigate the zeros of the Bell, Euler, and  $R_n(x)$  polynomials by using a computer.

From (1.7), we get some polynomials as below:

$$\begin{aligned} B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x - \frac{1}{2}, \\ B_3(x) &= x^3 - \frac{3x^2}{2} - \frac{3x}{2} - \frac{1}{4}, \\ B_4(x) &= x^5 - \frac{5x^4}{2} - 5x^3 - \frac{5x^2}{2} + 5x + \frac{21}{4}, \\ B_5(x) &= x^6 - 3x^5 - \frac{15x^4}{2} - 5x^3 + 15x^2 + \frac{63x}{2} + \frac{29}{2}. \end{aligned}$$

We plot the zeros of  $B_n(x)$  for  $x \in \mathbb{C}$  (Figure 1). In Figure 1 (top-left), we choose  $n = 5$ . In Figure 1 (top-right), we choose  $n = 10$ . In Figure 1 (bottom-left), we choose  $n = 15$ . In Figure 1 (bottom-right), we choose  $n = 20$ .

Next, we plot the zeros of  $E_n(x), B_n(x), R_n(x)$  for  $x \in \mathbb{C}$  (Figure 2). In Figure 2 (left), we choose  $n = 20$  and plot of zeros of  $E_n(x)$ . In Figure 2 (middle), we choose  $n = 20$  and plot of zeros of  $B_n(x)$ . In Figure 2 (right), we choose  $n = 20$  and plot of zeros of  $R_n(x)$ .

Our numerical results for numbers of real and complex zeros of  $B_n(x)$  and  $R_n(x)$  are displayed in Table 1.

We observe a remarkably regular structure of the complex roots of the Bell polynomials  $B_n(x)$  and polynomials  $R_n(x)$ . We hope to verify a remarkably regular structure of the complex roots of the Bell polynomials  $B_n(x)$  and polynomials  $R_n(x)$  (Table 1). Prove that the numbers of complex zeros  $C_{B_n(x)}$  of  $B_n(x), Im(x) \neq 0$  is

$$C_{R_n(x)} = 0.$$

Next, we calculate an approximate solution satisfying  $B_n(x), x \in \mathbb{R}$ . The results are given in Table 2.

Stacks of zeros of  $B_n(x)$  for  $1 \leq n \leq 20$  from a 3-D structure are presented (Figure 3). Next, we present stacks of zeros of  $E_n(x), B_n(x), R_n(x)$  for  $1 \leq n \leq 20$  from a 3-D structure. In Figure 3 (left), stacks of zeros of  $E_n(x)$  for  $1 \leq n \leq 20$  from a 3D structure are presented. In Figure 3 (middle), stacks of zeros of  $B_n(x)$  for  $1 \leq n \leq 20$  from a 3D structure are presented. In Figure 3 (right), stacks of zeros of  $R_n(x)$  for  $1 \leq n \leq 20$  from a 3D structure are presented.

Since  $n$  is the degree of the polynomial  $R_n(x)$ , the number of real zeros  $R_{R_n(x)}$  lying on the real plane  $Im(x) = 0$  is then  $R_{R_n(x)} = n - C_{R_n(x)}$ , where  $C_{R_n(x)}$  denotes complex zeros. See Table 1 for tabulated values of  $R_{R_n(x)}$  and  $C_{R_n(x)}$ . Prove or disprove:  $R_n(x) = 0$  has  $n$  distinct solutions. Find the numbers of complex

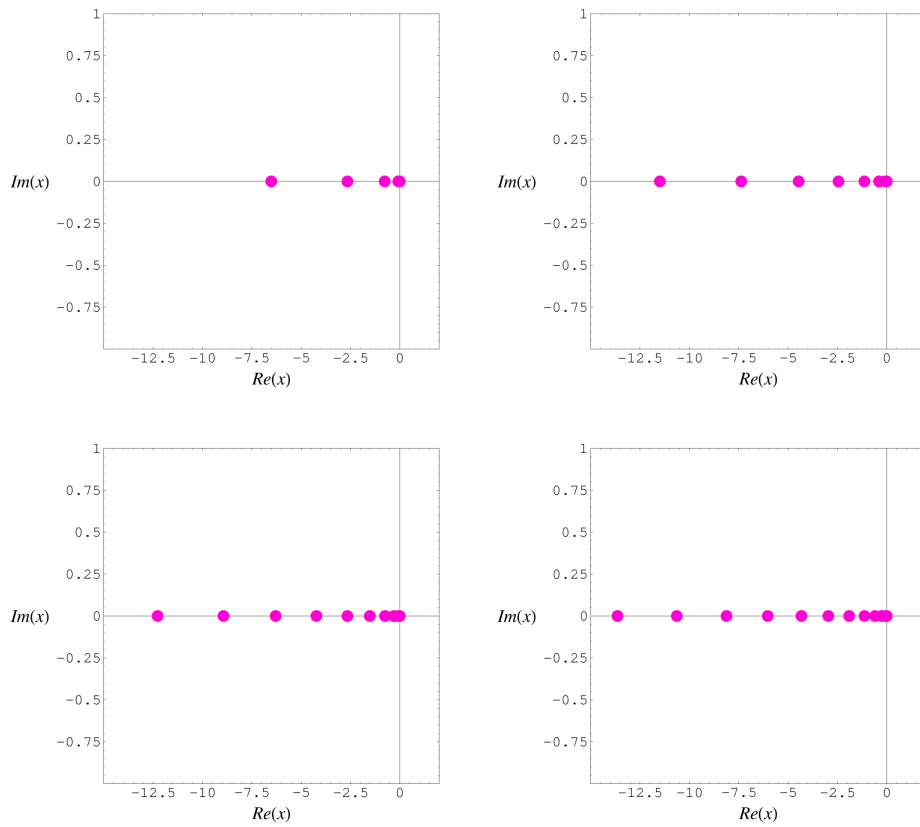


Figure 1. Zeros of  $B_n(x)$ .

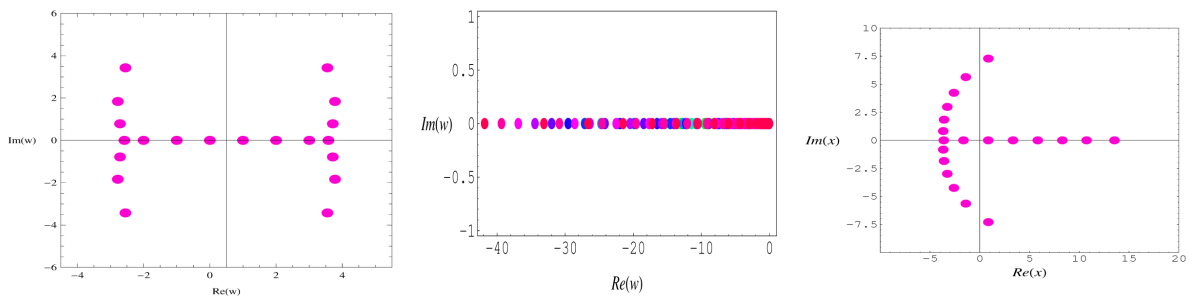


Figure 2. Zeros of  $E_n(x)$ ,  $B_n(x)$  and  $R_n(x)$ .

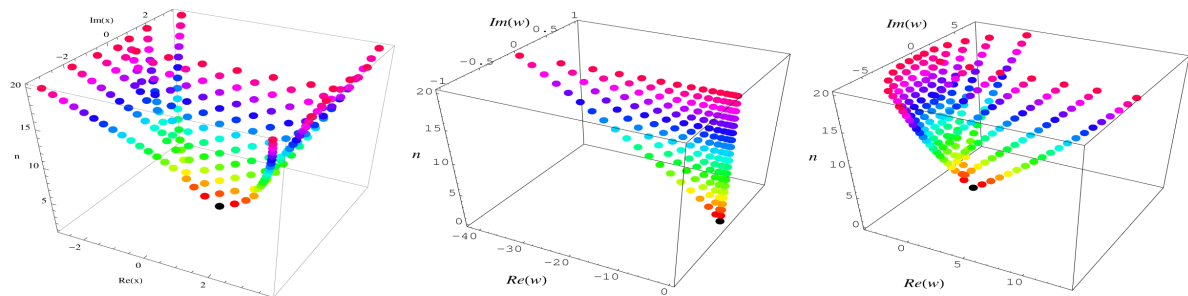


Figure 3. Zeros of  $E_n(x)$ ,  $B_n(x)$  and  $R_n(x)$ .



**Table 1.** Numbers of real and complex zeros of  $B_n(x)$  and  $R_n(x)$ .

Degree $n$	$B_n(x)$		$R_n(x)$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	2	2
5	5	0	3	2
6	6	0	4	2
7	7	0	3	4
8	8	0	4	4
9	9	0	5	4
10	10	0	4	6
11	11	0	5	6
12	12	0	6	6
13	13	0	5	8

**Table 2.** Approximate solutions of  $B_n(x) = 0$ .

Degree $n$	$x$
1	0
2	-1, 0
3	0, -2.6180, -0.3820
4	-4.491, -1.343, -0.1658
5	-6.51, -2.65, -0.762, -0.076
6	-8.63, -4.18, -1.70, -0.453, -0.04

zeros  $C_{R_n(x)}$  of  $R_n(x)$ ,  $Im(x) \neq 0$ . Using numerical investigation, we observed the behavior of complex roots of the Euler polynomials  $E_n(x)$ . By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the Euler polynomials  $E_n(x)$  (see [12]). The theoretical prediction on the zeros of  $R_n(x)$  is await for further study. These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the  $R_n(x)$ . For more studies and results in this subject, you may see [12]-[14].

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