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On Mutually Orthogonal Graph-Path Squares

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Abstract

A *decomposition* $\mathcal{G}=\{G_0,G_1,\cdots,G_{s-1}\}$ of a graph *H* is a partition of the edge set of *H* into edge-disjoint subgraphs $G_0,G_1,\cdots,G_{s-1}$. If $G_i \cong G$ for all $i\in\{0,1,\cdots,s-1\}$, then $\mathcal{G}$ is a decomposition of *H* by *G*. Two decompositions $\mathcal{G}=\{G_0,G_1,\cdots,G_{n-1}\}$ and $\mathcal{F}=\{F_0,F_1,\cdots,F_{n-1}\}$ of the complete bipartite graph $K_{n,n}$ are orthogonal if, $\left|E(G_i)\cap(F_j)\right|=1$ for all $i,j\in\{0,1,\cdots,n-1\}$. A set of decompositions $\{\mathcal{G}_0,\mathcal{G}_1,\cdots,\mathcal{G}_{k-1}\}$ of $K_{n,n}$ is a set of *k* mutually orthogonal graph squares (MOGS) if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i,j\in\{0,1,\cdots,k-1\}$ and $i\neq j$. For any bipartite graph *G* with *n* edges, $N(n,G)$ denotes the maximum number *k* in a largest possible set $\{\mathcal{G}_0,\mathcal{G}_1,\cdots,\mathcal{G}_{k-1}\}$ of MOGS of $K_{n,n}$ by *G*. Our objective in this paper is to compute $N(n,G)$ where $G=\mathbb{P}_{d+1}(F)$ is a path of length *d* with *d* + 1 vertices (*i.e.* Every edge of this path is one-to-one corresponding to an isomorphic to a certain graph *F*).

Keywords

Orthogonal Graph Squares, Orthogonal Double Cover

1. Introduction

In this paper we make use of the usual notation: $K_{m,n}$ for the complete bipartite graph with partition sets of sizes *m* and *n*, P_{n+1} for the path on *n* + 1 vertices, $D\bigcup F$ for the disjoint union of *D* and *F*, $D\bigcup^{L_v} F$ for the union of *D* and *F* with L_v (set of vertices) that belong to each other (*i.e.* union of *D* and *F* with common vertices of the set L_v belong to *F* and *D*), K_n for the complete graph on *n* vertices, K_1 for an isolated vertex. The other terminologies not defined here can be found in [1].

A *decomposition* $\mathcal{G}=\{G_0,G_1,\cdots,G_{s-1}\}$ of a graph *H* is a partition of the edge set of *H* into edge-disjoint subgraphs $G_0, G_1, \cdots, G_{s-1}$. If $G_i \cong G$ for all $i \in \{0,1,\cdots,s-1\}$, then $\mathcal{G}$ is a decomposition of *H* by *G*. Two decompositions $\mathcal{G} = \{G_0, G_1, \cdots, G_{n-1}\}$ and $\mathcal{F} = \{F_0, F_1, \cdots, F_{n-1}\}$ of the complete bipartite graph $K_{n,n}$ are orthogonal if, $|E(G_i) \cap (F_j)| = 1$ for all $i, j \in \{0,1,\cdots,n-1\}$. Orthogonality requires that $|E(G_i)| = n = |E(F_i)|$ for all $i \in \{0,1,\cdots,n-1\}$. A set of decompositions $\{\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_{k-1}\}$ of $K_{n,n}$ is a set of *k* mutually orthogonal graph squares (MOGS) if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i, j \in \{0,1,\cdots,k-1\}$ and $i \neq j$. We use the notation $N(n,G)$ for the maximum number *k* in a largest possible set $\{\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_{k-1}\}$ of MOGS of $K_{n,n}$ by *G*, where *G* is a bipartite graph with *n* edges.

If two decompositions $\mathcal{G}$ and $\mathcal{F}$ of $K_{n,n}$ by *G* are orthogonal, then $\mathcal{G} \cup \mathcal{F}$ is an orthogonal double cover of $K_{n,n}$ by *G*. Orthogonal decompositions of graphs and orthogonal double covers (ODC) of graphs have been studied by several authors; see the survey articles [2] [3].

It is well-known that orthogonal Latin squares exist for every $n \notin \{2,6\}$. A family of *k*-orthogonal Latin squares of order *n* is a set of *k* Latin squares any two of which are orthogonal. It is customary to denote $N(n) = \max\{k : \exists k \text{ } MOLS\}$ be the maximal number of squares in the largest possible set of mutually orthogonal Latin squares MOLS of side *n*. A decomposition of $K_{n,n}$ by nK_2 is equivalent to a Latin square of side *n*; two decompositions $\mathcal{G}$ and $\mathcal{F}$ of $K_{n,n}$ by nK_2 are orthogonal if and only if the corresponding Latin squares of side *n* are orthogonal; and thus $N(n, nK_2) = N(n)$. The computation of $N(n)$ is one of the most difficult problems in combinatorial designs; see the survey articles by Abel *et al.* [4] and Colbourn and Dinitz in [5]. Since $N(n,G)$ is a natural extension of $N(n)$, the study of $N(n,G)$ for general graphs is interesting. El-Shanawany [6] establishes the following: i) $N(n, K_{1,n}) = 2$; ii) $N(2, P^3) = 2, N(3, P^4) = 3, N(5, P_6) = 5$ and $N(7, P_8) = 7$; iii) let $p > 2$ be a prime number, then $N(p, K_2 + ((p-1)/2)P_3) = p$; iv) let *p* be a prime number, then $N(p, (p-2)K_2 + P_3) \geq p-1$. Based on ii), El-Shanawany [6] proposed:

Conjecturer 1. Let *p* be a prime number. Then $N(p, P_{p+1}) = p$.

Sampathkumar *et al.* [7] have proved El-Shanawany conjectured. In the following section, we present another technique to prove this conjecture as in Theorem 8.

The two sets $\{0_0, 1_0, \cdots, (n-1)_0\}$ and $\{0_1, 1_1, \cdots, (n-1)_1\}$ denote the vertices of the partite sets of $K_{n,n}$. The length of the edge $x_0 y_1$ of $K_{n,n}$ is defined to be the difference $y - x$, where $x, y \in \mathbb{Z}_n$. Note that sums and differences are carried over in $\mathbb{Z}_n$ (that is, sums and differences are carried modulo *n*). Let *G* be a subgraph of $K_{n,n}$ without isolated vertices and let $a \in \{0,1,\cdots,n-1\}$. The *a*-translate of *G*, denoted by $G + a$, is the edge-induced subgraph of $K_{n,n}$ induced by $\{(x+a)_0 (y+a)_1 : x_0 y_1 \in E(G)\}$. A subgraph *G* of $K_{n,n}$ is half-starter if $|E(G)| = n$ and the lengths of all edges in *G* are mutually different.

Lemma 2 (see [8]). If *G* is a half-starter, then the union of all translates of *G* forms an edge decomposition of $K_{n,n}$ (*i.e.* $E(K_{n,n}) = \bigcup_{a \in \mathbb{Z}_n} E(G+a)$).

In what follows, we denote a half-starter *G* by the vector $v(G) = (v_0, v_1, \cdots, v_{n-1}) \in \mathbb{Z}_n^n = \underbrace{\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{n \text{ times}}$, where $v_0, v_1, \cdots, v_{n-1} \in \mathbb{Z}_n$ and v_i can be obtained from the unique edge $(v_i)_0 (v_i + i)_1$ of length *i* in *G*.

Theorem 3 (see [8]). Two half-starters $v(G) = (v_0, v_1, \cdots, v_{n-1}) \in \mathbb{Z}_n^n$ and $v(F) = (u_0, u_1, \cdots, u_{n-1}) \in \mathbb{Z}_n^n$ are orthogonal if $\{v_i - u_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

If two half-starters $v(G)$ and $v(F)$ are orthogonal, then the set of translates of *G* and the set of translates of *F* are orthogonal.

A set of decompositions $\{\mathcal{G}_i = \bigcup_{a \in \mathbb{Z}_n} E(G_i + a) = E(K_{n,n}) : 0 \leq i \leq k-1\}$ of $K_{n,n}$ } is a set of *k* mutually orthogonal graph squares (MOGS) if $\mathcal{G}_i$ and $\mathcal{G}_j$ are orthogonal for all $i, j \in \{0,1,\cdots,k-1\}$ and $i \neq j$.

Note that

$$\bigcup_{i=0}^{k-1} \mathcal{G}_i = \bigcup_{i=0}^{k-1} \left(\bigcup_{a \in \mathbb{Z}_n} E(G_i + a) \right) = kE(K_{n,n}).$$

In the following, we define a *G*-square over additive group $\mathbb{Z}_n$.

Definition 4 (see [6]). Let *G* be a subgraph of $K_{n,n}$ A square matrix $\mathcal{L}$ of order *n* is called an *G*-square if every element in $\mathbb{Z}_n$ occur exactly *n* times, and the graphs G_i, $i \in \mathbb{Z}_n$ with

$E(G_i)=\{(x,y):\mathcal{L}(x,y)=i, x,y\in\mathbb{Z}_n\}$ are isomorphic to graph G.

We have already from Lemma 2 and Definition 4 that every half starter vector $v(G)$ and its translates are equivalent to G-square. For more illustration, the first matrix $\mathcal{L}_0$ in equation (1) is equivalent to the first row in **Figure 3**, which represented by the half starter vector $v(G_{00})=(0,0,2)$ and its translates.

Definition 5. Two squares matrices $\mathcal{L}_0$ and $\mathcal{L}_1$ of order n are said to be orthogonal if for any ordered pair (a,b), there is exactly one position (x,y) for $\mathcal{L}_0(x,y)=a$ and $\mathcal{L}_1(x,y)=b$.

Now, we shall derive a class of mutually orthogonal subgraphs of $K_{n,n}$ by a given graph G as follow.

Definition 6. A set of matrices $\{\mathcal{L}_i: 0\le i\le k-1\}$ of $K_{n,n}$ is called a set of k mutually orthogonal graph squares (MOGS) if $\mathcal{L}_i$ and $\mathcal{L}_j$ are orthogonal for all $i,j\in\{0,1,\cdots,k-1\}$ and $i\ne j$.

Definition 7 (see [9]). Let F be a certain graph, the graph F-*path* denoted by $\mathbb{P}_{d+1}(F)$, is a path of a set of vertices $\mathbb{V}=\{\mathbb{V}_i: 0\le i\le d\}$ and a set of edges $\mathbb{E}=\{\mathbb{E}_i: 0\le i\le d-1\}$ if and only if there exists the following two bijective mappings:

1) $\Psi:\mathbb{E}\to\mathcal{F}$ defined by $\Psi(\mathbb{E}_i)=F_i$, where $\mathcal{F}=\{F_0,F_1,\cdots,F_{d-1}\}$ is a collection of d graphs, each one is isomorphic to the graph F.

2) $\phi:\mathbb{V}\to\mathcal{Y}$ defined by $\phi(\mathbb{V}_i)=X_i$, where $\mathcal{Y}=\{X_i: 0\le i\le d; \bigcap_i X_i=\varnothing\}$ is a class of disjoint sets of vertices (*i.e.*, $\mathcal{Y}$ decomposed into $d+1$ disjoint sets such that no two vertices within the same set are adjacent).

As a special case if the given graph F is isomorphic to $K_{1,1}$ then $\mathbb{P}_{d+1}(K_{1,1})$, is the natural path P_{d+1} that is, $\mathbb{P}_{d+1}(K_{1,1})=P_{d+1}$.

For more illustration, see **Figure 1**, **Figure 2**.

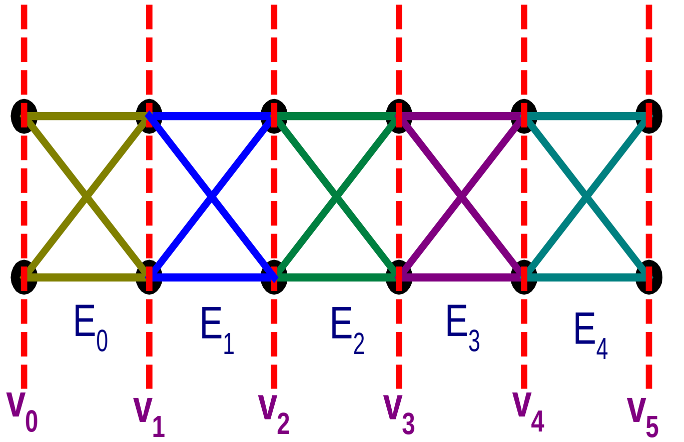

Figure 1. $\mathbb{P}_6(K_{2,2})$, the path of 6 sets of vertices (every sethas only 2 disjoint vertices) and 5 edges of $K_{2,2}$.

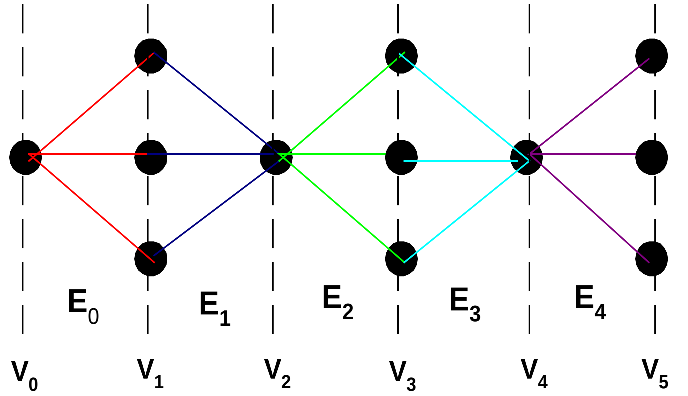

Figure 2. $\mathbb{P}_6(K_{1,3})$, the path of 6 sets of vertices and 5 edges of $K_{1,3}$.

Consider $s \ge 0$ paths of length $k \ge 1$, all attached to the same vertex (root vertex). This tree will be called $T(s,k)$. Clearly, $T(s,1)$ is the star with s edges and $T(1,k)$ is the path with k edges. Define L_s as a set of all leaves of $T(s,k)$ *i.e.* $L_s = \{v : v \text{ is a leaf in } T(s,k)\}$ and $|L\_\{s\}| = s$.

In the following section, we will compute $N(n,G)$ where $G = \mathbb{P}_{d+1}(F)$ such that $F = K_{1,1}$ as in theorem 8 and $F = T(3,3) \bigcup^{L_3} K_1$ as in theorem 11.

2. Mutually Orthogonal Graph-Path Squares

The following result was shown in [7]. Here we present another technique for the proof.

Theorem 8. Let q be a prime number. Then $N(q, \mathbb{P}_{q+1}(K_{1,1})) = q$.

Proof. Let G_{ij} be a subgraph of $K_{q,q}$ with q edges; for fixed $j \in \mathbb{Z}_q$ and $0 \le i \le q-1$, define the q half-starter vectors as follows, $v(G_{ij}) = (j, i+j-1, 2i+j-2^2, \cdots, (-2)(i+2)+j, (-1)(i+1)+j)$; our task is to prove the orthogonality of those q half-stater vectors in mutually. Let us define the half starter vector $v(G_{ij})$ as $v_k(G_{ij}) = k(i-k)+j$ for all $k \in \mathbb{Z}_q$. Then for all two different elements $k,l \in \mathbb{Z}_q$, we have $\{v_k(G_{ij}) - v_l(G_{ij}) = k(i-k) - l(i-l) = (k-l)(i-(k+l))\}$, then v_k, v_l are mutually orthogonal half-starter vectors of graphs G_{kj} and G_{lj} of $K_{q,q}$ respectively iff $(i-(k+l),q) = 1$. It remains to prove the isomorphism of $v_k(G_{ij})$ half starter graphs G_{ij} of $K_{q,q}$ for all $k \in \mathbb{Z}_q$. Let $v_k(G_{ij}) = k(i-k)+j = v_k(G_{is}) = k(i-k)+s$, and therefore $j = s$. Furthermore, if $v_k(G_{ij}) = k(i-k)+j = v_l(G_{ij}) = l(i-l)+j$, then $(k-l)(i-(k+l)) = 0$, since $i-(k+l) \ne 0$ (orthogonality of v_k and v_l), and therefore $k = l$. Moreover, for any $i, j \in \mathbb{Z}_q$ the ij^{th} graph isomorphic to G_{ij} has the edges: $E(G_{ij}) = \{(-i^2+j)_0 (i(1-i)+j)_1\}$. ■

An immediate consequence of the Theorem 8 and Conjecture 1 is the following result.

Example 9. The three mutually orthogonal decompositions (MOD) of $K_{3,3}$ by P_4 given in **Figure 3** are associated with the three mutually orthogonal P_4-squares as in Equation (1):

$$\mathcal{L}_0 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}. \tag{1}$$

Note that, every row in **Figure 3** represents edge decompositions of $K_{3,3}$ by P_4.

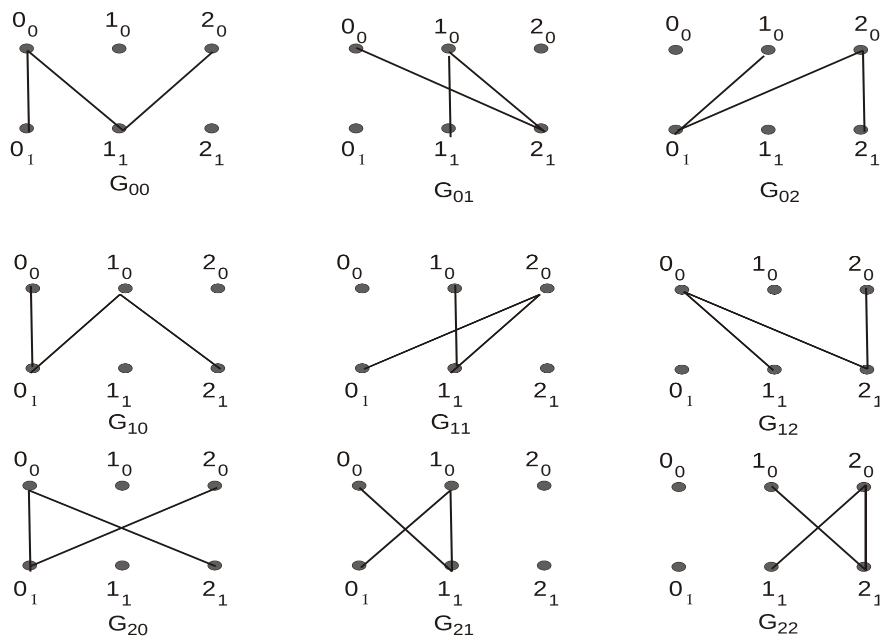

Figure 3. 3MOD of $K_{3,3}$ by P_4.

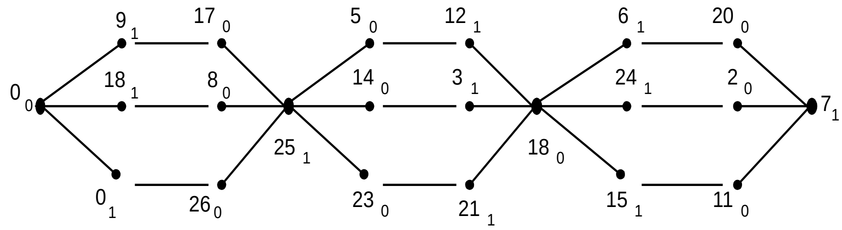

Figure 4. $\mathbb{P}_4\left(T(3,3)\bigcup^{L_3} K_1\right)$, the path of 4 vertices and 3 edges of $T(3,3)\bigcup^{L_3} K_1$.

The following result is a generalization of the Theorem 8.

Theorem 10. Let n be a prime power such that $n = q^x$ with integer power $x \ge 1$ of a prime number q and G be a subgraph of $K_{n,n}$. Then $N(n,G) \ge q$.

Proof. For fixed $j \in \mathbb{Z}_n, 0 \le i \le q-1$ and $G_{ij} \cong G$, define the q half-starter vectors as follows,

$$v(G_{ij}) = \left(j, i+j-1, 2i+j-2^2, \cdots, (-2)(i+2)+j, (-1)(i+1)+j\right). \tag{2}$$

Our task is to prove the orthogonality of those q half-starter vectors in mutually. Let us define the half starter vector $v(G_{ij})$ as $v_k(G_{ij}) = k(i-k)+j$ for all $k \in \mathbb{Z}_q$. Then for all two different elements $k,l \in \mathbb{Z}_q$, we have $\left\{v_k(G_{ij}) - v_l(G_{ij}) = k(i-k) - l(i-l) = (k-l)(i-(k+l))\right\}$, and then v_k, v_l are mutually orthogonal half-stater vectors of graphs G_{kj} and G_{lj} of $K_{n,n}$ respectively iff $(i-(k+l), n) = 1$. It remains to prove the isomorphism of $v_k(G_{ij})$ half starter graphs G_{ij} of $K_{n,n}$ for all $k \in \mathbb{Z}_q$. Let $v_k(G_{ij}) = k(i-k)+j = v_k(G_{is}) = k(i-k)+s$, and therefore $j = s$. Furthermore, if $v_k(G_{ij}) = k(i-k)+j = v_l(G_{ij}) = l(i-l)+j$, then $(k-l)(i-(k+l)) = 0$, since $i-(k+l) \ne 0$ (orthogonality of v_k and v_l), and therefore $k = l$. Moreover, for any $0 \le i \le q-1, j \in \mathbb{Z}_n$ the ij^{th} graph G_{ij} isomorphic to G has the edges:

$$E(G_{ij}) = \left\{(n-i^2+j)_0 \, (n-i(i-1)+j)_1\right\}. \tag{3}$$

■

Note that, in the special case $x = 1$ the Theorem 10 proved El-Shanawany conjecture; also, in the case $q = 2, x \ne 1$, and $G \ne 2K_2$, Theorem 10 constructed an orthogonal double cover of $K_{n,n}$ by G.

Furthermore, we can construct the following result using Theorem 10 in case $x > 1$ and $q = 3$.

Theorem 11. Let $x \ge 2$ be a positive integer such that $n = 3^x$ and $\mathbb{P}_{3^{x-2}+1}\left(T(3,3)\bigcup^{L_3} K_1\right)$ be a subgraph of $K_{3^x,3^x}$. Then $N\left(3^x, \mathbb{P}_{3^{x-2}+1}\left(T(3,3)\bigcup^{L_3} K_1\right)\right) \ge 3$.

Proof. The result follows from the vector in Equation (2) and its edges in Equation (3) with $G = \mathbb{P}_{d+1}(F), F = T(3,3)\bigcup^{L_3} K_1$ such that $|E(G)| = 3^x = |E(F)|d = 9d$, imply that $d = 3^{x-2}$ which define the number of graphs isomorphic to F. As a direct application of Theorem 11; see **Figure 4**. ■

Conjecture 12. $N\left(q^x, \mathbb{P}_{q+1}(K_{1,1})\right) = q^x$ if q is a prime number with an integer power $x \ge 1$.

Conjecture 13. $N\left(q^x, \mathbb{P}_{\frac{q^x}{sk}+1}\left(T(3,3)\bigcup^{L_3} K_1\right)\right) \ge q$ if q is a prime number with an integer power $x \ge 1$ and s,k are positive integers.

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