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Domination Number of Square of Cartesian Products of Cycles

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Abstract

A set $S \subseteq V(G)$ is a dominating set of G if every vertex of $V(G)-S$ is adjacent to at least one vertex of S. The cardinality of the smallest dominating set of G is called the domination number of G. The square G^2 of a graph G is obtained from G by adding new edges between every two vertices having distance 2 in G. In this paper we study the domination number of square of graphs, find a bound for domination number of square of Cartesian product of cycles, and find the exact value for some of them.

Keywords

Domination Number, Square of a Graph, Cartesian Product

1. Introduction

The usual graph theory notions not herein, refer to [1]. The *neighborhood* of vertex u is denoted by $N(u)=\{v \in V(G): uv \in E(G)\}$ and the *close neighborhood* of vertex u is denoted by $N[u]=N(u)\cup\{u\}$. Let $S \subseteq V(G)$, the *neighborhood* and *closed neighborhood* of S are defined as $N(S)=\bigcup_{u\in S} N(u)$ and $N[S]=\bigcup_{u\in S} N[u]$. If $u \in V(G)$, then $N_k(u)=\{v\in V(G) \mid 1\le d(u,v)\le k\}$. If $S\subseteq V(G)$ and $u\in V(G)$, then $d(u,S)=\min\{d(u,v) \mid v\in S\}$. The *diameter* of G denoted by $diam(G)$ is defined as $diam(G)=\max_{u,v\in V(G)} d_G(u,v)$. A set $S\subseteq V(G)$ is a *dominating set* of G if every vertex of $V(G)-S$ is adjacent to at least one vertex of S. The cardinality of the smallest dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set of cardinality $\gamma(G)$ is called a γ-set of G [2]. A dominating set S is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set. Given any graph G, its *square graph* G^2 is a graph with vertex set $V(G)$ and two vertices are adjacent whenever they are at

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distance 1 or 2 in *G*. For example $C_5^2 = K_5$. A set $S \subseteq V(G)$ is a 2-*distance dominating set* of *G* if $d_G(u,S) = 1$ or 2 for every vertex of $V(G) - S$. The cardinality of the smallest 2-distance dominating set of *G*, denoted by $\gamma^2(G)$, is called 2-*distance domination number* of *G*. Every 2-distance dominating set of *G* is a dominating set of G^2, so $\gamma^2(G) = \gamma(G^2)$. The *Cartesian product* of Graphs *G* and *H* denoted by $G \square H$ is a graph with vertex set $V(G \square H) = V(G) \times V(H)$ and the edge set $E(G \square H) = \{((u,v),(z,w)) : (uz \in E(G) \,\&\, v = w) \text{ or } (u = z \,\&\, vw \in E(H))\}$. The graph $G \square H$ is obtained by locating copies H_i of grpah *H* instead of vertices of *G* and connecting the corresponding vertices of H_i to H_j if vertex v_i is adjacent to v_j in *G*. $G \square H$ is isomorphic to $H \square G$. We denote a cycle with *n* vertices by C_n and a path with *n* vertices by P_n. The bipartite geraph $K_{1,3}$ is named *claw*.

2. Preliminaries Results

Theorem 1. *Let G be a graph. Then*

a) If $uv \in E(G)$, then $\gamma(G - uv) \geq \gamma(G)$.

b) If $uv \in E(G^c)$, then $\gamma(G + uv) \leq \gamma(G)$.

Proof. a) Every dominating set of $G - uv$ is a dominating set of *G* so $\gamma(G - uv) \geq \gamma(G)$.

b) Every dominating set of *G* is a dominating set of $G + uv$ so $\gamma(G + uv) \leq \gamma(G)$.

Theorem 2. [3] *A dominating set S is a minimal dominating set if and only if for each vertex* $u \in S$, *one of the following conditions holds:*

a) *u* is an isolated vertex of *S*.

b) there exist a vertex $v \in V(G) - S$ for which $N(v) \cap S = \{u\}$.

Theorem 3. [3] *If G is a graph with no isolated vertices and S is a minimal dominating set of G, then* $V(G) - S$ *is a dominating set of G.*

Proof. Let *S* be a *γ*-set of *G*. *S* is a minimal dominating set of *G*. By Theorem 3, $V(G) - S$ is a dominating set of *G* too, so $|S| \leq |V(G) - S|$, so $|S| \leq \frac{n}{2}$.

Theorem 4. [4] *If G is a connected claw free graph, then* $\gamma(G) \leq \left\lceil \frac{n}{3} \right\rceil$.

Theorem 5. [5] *Let G be a graph. Then* $\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$.

Since $\Delta(C_n) = \Delta(P_n) = 2$, by Theorems 4 and 5 we have the following corollary.

Corollary 6. $\gamma(C_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Vizing conjecture

Let *G* and *H* be two graphs. Then $\gamma(G \square H) \geq \gamma(G)\gamma(H)$ [6].

3. Domination Number of Square of Graphs

Theorem 7. *Let S be a dominating set of* G^2. *Then S is a minimal dominating set of* G^2 *if and only if each vertex* $u \in S$ *satisfies at least one of the following conditions*:

a) There exists a vertex $v \in V(G) - S$ for which $N_2(v) \cap S = \{u\}$.

b) $d(u,w) > 2$ for every vertex $w \in S - \{u\}$.

Proof. If $u \in S$ and *u* does't satisfy conditions a) and b), then the set $S - \{u\}$ is a dominating set of G^2 that is contradiction. Conversely, let *S* be a dominating set of G^2 but not minimal. Then there exists a vertex $u \in S$ such that $S - \{u\}$ is a dominating set of G^2, too. So $d(v, S - \{u\}) = 1$ or 2 for every $v \in V(G) - S$; therefore *S* doesn't satisfy in condition a). In addition $d(u, S - \{u\}) = 1$ or 2, so *S* doesn't satisfy in condition b).

Theorem 8. *If* $diam(G) \le 3$, *then* $\gamma(G^2) \le \delta(G)$.

Proof. Let $d(u) = \delta(G)$. Since $diam(G) \le 3$, the set $N(u)$ is a dominating set of G^2. Therefore $\gamma(G^2) \le |N(u)| = d(u) = \delta(G)$.

Theorem 9. *If* $diam(G) \le 4$, *then* $\gamma(G^2) \le \left(\sum_{v \in N(u)} d(v)\right) - d(u)$, *for every* $u \in V(G)$.

Proof. Let u be an arbitrary vertex of G. Let $S(u) = N(N(u)) - \{u\}$. Since $diam(G) \le 4$, $d(v, S(u)) \le 2$, for every $v \in V(G)$. Therefore $S(u)$ is a dominating set of G^2. $|S(u)| \le \sum_{v \in N(u)} (d(v) - 1)$ and $|N(u)| = d(u)$, Hence $\gamma(G^2) \le |S(u)| \le \sum_{v \in N(u)} (d(v) - 1) = \sum_{v \in N(u)} (d(v)) - d(u)$.

Theorem 10. *Let G be a graph. Then* $\gamma\left((G \square K_n)^2\right) = \gamma(G)$.

Proof. Let $V(G) = \{u_1, u_2, \cdots, u_m\}$ and $H_1, H_2, \cdots, H_m$ be the copies of K_n in $G \square K_n$ corresponding to the vertices $u_1, u_2, \cdots, u_m$. Let $S = \{u_{t1}, u_{t2}, \cdots, u_{tk}\}$ be a γ-set of G. Then the set $S' \subseteq V\left((G \square H)^2\right)$ that contains a vertex of each copies $K_{t1}, K_{t2}, \cdots, K_{tk}$ is a γ-set of $(G \square K_n)^2$. Since $|S'| = |S|$, the result holds.

Theorem 11. *For every* $n \ge 3$, $\gamma(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$.

Proof. The graphs C_3^2 and C_4^2 are complete graphs, therefore $\gamma(C_3^2) = \gamma(C_4^2) = 1$. So the result holds for C_3^2 and C_4^2. Let $C_n = u_1 u_2 \cdots u_n u_1$, $n \ge 5$. Since $\Delta(C_n^2) = 4$, by the Theorem 5 we have $\gamma(C_n^2) \ge \left\lceil \frac{n}{5} \right\rceil$. On the other hand by **Figure 1** the set $S = \left\{u_{5k+1} : k = 0, 1, \cdots, \left\lceil \frac{n}{5} \right\rceil - 1\right\}$ is a dominating set of size $\left\lceil \frac{n}{5} \right\rceil$ for C_n^2. So $\gamma(C_n^2) \le \left\lceil \frac{n}{5} \right\rceil$, therefore $\gamma(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$.

Theorem 12. *For every* $n \ge 1$, $\gamma(P_n^2) = \left\lceil \frac{n}{5} \right\rceil$.

Proof. $\gamma(P_1^2) = \gamma(P_2^2) = \gamma(P_3^2) = \gamma(P_4^2) = 1$, and the result holds for these graphs. Let $P_n = u_1 u_2 \cdots u_n$, $n \ge 5$. Since $\Delta(P_n^2) = 4$, by Theorem 5 we have $\gamma(P_n^2) \ge \left\lceil \frac{n}{5} \right\rceil$. By **Figure 2** the set:

$$S = \begin{cases} \left\{u_{5k+3} : k = 0, 1, \cdots, \left\lfloor \frac{n}{5} \right\rfloor - 1\right\} & \text{if } n \equiv 0 \pmod 5 \\ \left\{u_{5k+3} : k = 0, 1, \cdots, \left\lfloor \frac{n}{5} \right\rfloor - 1\right\} \cup \{u_n\} & \text{if } n \equiv 1, 2, 3 \pmod 5 \\ \left\{u_{5k+3} : k = 0, 1, \cdots, \left\lfloor \frac{n}{5} \right\rfloor - 1\right\} \cup \{u_{n-1}\} & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

is a dominating set of size $\left\lceil \frac{n}{5} \right\rceil$ for P_n^2, so $\gamma(P_n^2) \le \left\lceil \frac{n}{5} \right\rceil$; therefore $\gamma(P_n^2) = \left\lceil \frac{n}{5} \right\rceil$.

Theorem 13. *For every* $m, n \ge 1$, $\gamma\left((P_m \square P_n)^2\right) \ge \left\lceil \frac{mn}{13} \right\rceil$, *and for every* $m, n \ge 3$, $\gamma\left((C_m \square C_n)^2\right) \ge \left\lceil \frac{mn}{13} \right\rceil$.

Proof. The graphs $P_m \square P_n$ and $C_m \square C_n$ have mn vertices and every vertex u dominates at least 13 vertices in $(P_m \square P_n)^2$ and $(C_m \square C_n)^2$ (**Figure 3**), so the result holds.

By Theorem 13, $\gamma\left((P_m \square P_n)^2\right)$ or $\gamma\left((C_m \square C_n)^2\right)$ equals the minimum number of diamonds like **Figure 4**. we can cover all the vertices of $P_m \square P_n$ or $C_m \square C_n$.

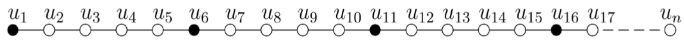

Figure 1. A dominating set of C_n^2.

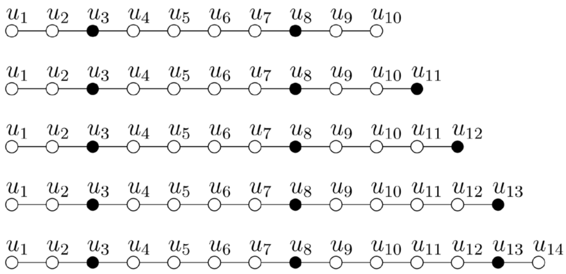

Figure 2. A dominating set of P_n^2.

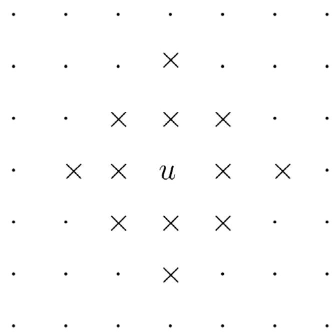

Figure 3. Dominated vertices by u in $(P_m \square P_n)^2$ and $(C_m \square C_n)^2$.

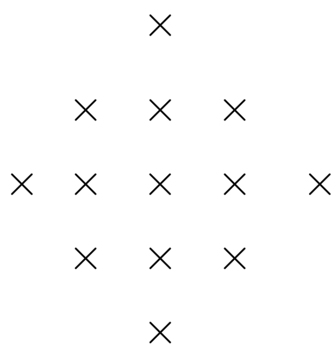

Figure 4. Dominated vertices by one vertex in $(P_m \square P_n)^2$ and $(C_m \square C_n)^2$.

In this paper we use *short display* or *s.d* to show the graphs $P_m \square P_n$ and $C_m \square C_n$ for simplicity; it means that we don't draw the edges of these graphs and draw only their vertices.

Theorem 14. *For every* $k,t \geq 1$, $\gamma\left(\left(C_{13k} \square C_{13t}\right)^2\right) = 13kt$.

Proof. By Theorem 13 we have $\gamma\left(\left(C_{13} \square C_{13}\right)^2\right) \geq 13$. In **Figure 5** that is *s.d* of $C_{13} \square C_{13}$. It is determined by a γ-set of size 13 for $\left(C_{13} \square C_{13}\right)^2$. Therefore $\gamma\left(\left(C_{13} \square C_{13}\right)^2\right) \leq 13$; hence $\gamma\left(\left(C_{13} \square C_{13}\right)^2\right) = 13$.

We can obtain *s.d* of $C_{13k} \square C_{13t}$ with dominating set of size $13kt$ for $\left(C_{13k} \square C_{13t}\right)^2$ by locating kt copies of **Figure 5** in k rows and t columns. Hence $\gamma\left(\left(C_{13k} \square C_{13t}\right)^2\right) \leq 13kt$. By Theorem 13 we have

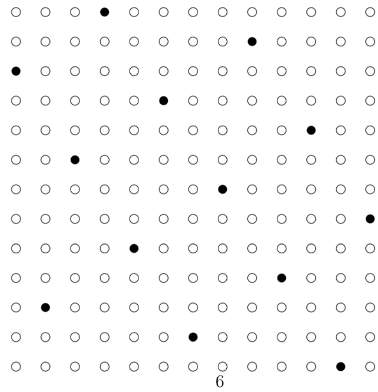

Figure 5. A dominating set of size 13 for $(C_{13}\square C_{13})^2$.

$\gamma\left((C_{13k}\square C_{13t})^2\right) \geq 13kt$, so $\gamma\left((C_{13k}\square C_{13t})^2\right) = 13kt$

Theorem 15. $\gamma\left((C_3\square C_n)^2\right) = \left\lceil \frac{n}{3} \right\rceil$, *for every* $n \geq 3$.

Proof. Since $C_3 = K_3$, by Theorem 10 and Corollary 6 we have

$$\gamma\left((C_3\square C_n)^2\right) = \gamma\left((K_3\square C_n)^2\right) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$$

Theorem 16. $\gamma\left((C_4\square C_n)^2\right) = \left\lceil \frac{4n}{13} \right\rceil$, $n = 4,5,6,7$, *and*

$$\gamma\left((C_4\square C_{6k+t})^2\right) \leq \begin{cases} 2k & \text{if } t = 0, \\ 2k+1 & \text{if } t = 1, \\ 2(k+1) & \text{if } t = 2,3,4,5. \end{cases}$$

Proof. By Theorem 13 we have $\gamma\left((C_4\square C_n)^2\right) \geq \left\lceil \frac{4n}{13} \right\rceil$. In **Figure 6** it is determined by a dominating set of size $\left\lceil \frac{4n}{13} \right\rceil$ for $(C_4\square C_n)^2$, $n = 4,5,6,7$, so for these graphs we have $\gamma\left((C_4\square C_n)^2\right) = \left\lceil \frac{4n}{13} \right\rceil$.

In **Figure 6**, the seventh column of *s.d* of $C_4\square C_7$ (from left to right) is similar to the first column of *s.d* of $C_4\square P_1$, $C_4\square P_2$ and $C_4\square C_n$, $n = 3,4,5,6,7$. By setting *s.d* of k graphs $C_4\square C_7$ and one *s.d* of $C_4\square P_2$ or $C_4\square C_3$ or $C_4\square C_4$ or $C_4\square C_5$ consecutively from left to right such that the first column of every *s.d* of graph locates on the last column of *s.d* of the previous graph, we can obtain a *s.d* of $C_4\square C_{6k+t}$ with a dominating set of size $2(k+1)$ for $(C_4\square C_{6k+t})^2$, $t = 2,3,4,5$. By the same setting for *s.d* of k graphs $C_4\square C_7$ we can obtain a *s.d* of $C_4\square C_{6k+1}$ with a dominating set of size $2k+1$ for $(C_4\square C_{6k+1})^2$. Also by the same setting for *s.d* of $k-1$ graphs $C_4\square C_7$ and one *s.d* of $C_4\square C_6$ we can obtain a *s.d* of $C_4\square C_{6k}$ with a dominating set of size $2k$ for $(C_4\square C_{6k})^2$.

Theorem 17. $\gamma\left((C_5\square C_n)^2\right) = \left\lceil \frac{5n}{13} \right\rceil$, $n = 3,4,6$, and

$$\gamma\left((C_5\square C_{6k+t})^2\right) \leq \begin{cases} 3k & \text{if } t = 0, \\ 3k+1 & \text{if } t = 1, \\ 3k+2 & \text{if } t = 2,3,4, \\ 3(k+1) & \text{if } t = 5. \end{cases}$$

Proof. By Theorem 13 we have $\gamma\left((C_5\square C_n)^2\right) \geq \left\lceil \frac{5n}{13} \right\rceil$. In **Figure 7** it is determined by a dominating set for

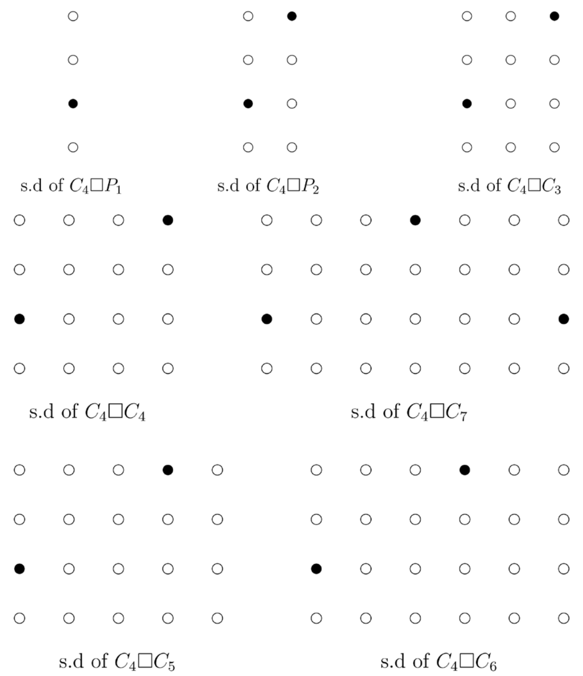

Figure 6. A dominating set for $(C_4\square C_n)^2$, $n = 1,2,\cdots,7$.

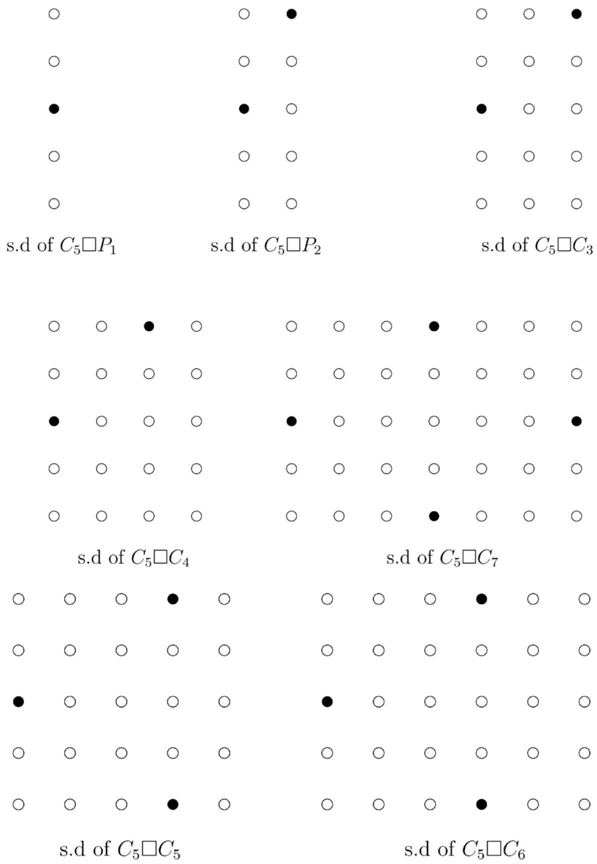

Figure 7. A dominating set for $(C_5\square C_n)^2$, $n = 1,2,\cdots,7$.

$(C_5\square P_1)^2$, $(C_5\square P_2)^2$ and $(C_5\square C_n)^2$, $n=3,4,5,6,7$.

By **Figure 7** we have $\gamma\left((C_5\square C_n)^2\right)\le\left\lceil\frac{5n}{13}\right\rceil$, $n=3,4,6$. So for these graphs equality holds.

In **Figure 7**, the seventh column of *s.d* of $C_5\square C_7$ (from left to right) is similar to the first column of *s.d* of $C_5\square P_1$, $C_5\square P_2$ and $C_5\square C_n$, $n=3,4,5,6,7$. By setting *s.d* of *k* graphs $C_5\square C_7$ and one *s.d* of $C_5\square P_2$ or $C_5\square C_3$ or $C_5\square C_4$ consecutively from left to right such that the first column of every *s.d* of graph locates on the last column of the previous *s.d* of graph, we can obtain a *s.d* of $C_5\square C_{6k+t}$ with a dominating set of size $3k+2$ for $(C_5\square C_{6k+t})^2$, $t=2,3,4$. By the same setting for *s.d* of *k* graphs $C_5\square C_7$ we can obtain a *s.d* of $C_5\square C_{6k+1}$ with a dominating set of size $3k+1$ for $(C_5\square C_{6k+1})^2$ and by the same setting for *s.d* of *k* graphs $C_5\square C_7$ and one *s.d* of $C_5\square C_5$ we can obtain a *s.d* of $C_5\square C_{6k+5}$ with a dominating set of size $3(k+1)$ for $(C_5\square C_{6k+5})^2$. Also by the same setting for *s.d* of $k-1$ graphs $C_5\square C_7$ and one *s.d* of $C_5\square C_6$ we can obtain a *s.d* of $C_5\square C_{6k}$ with a dominating set of size $3k$ for $(C_5\square C_{6k})^2$.

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