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Edge-Vertex Dominating Sets and Edge-Vertex Domination Polynomials of Cycles

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Abstract

Let $G = (V, E)$ be a simple graph. A set $S \subseteq E(G)$ is an edge-vertex dominating set of G (or simply an *ev*-dominating set), if for all vertices $v \in V(G)$; there exists an edge $e \in S$ such that e dominates v. Let $D_{ev}(C_n, i)$ denote the family of all *ev*-dominating sets of C_n with cardinality i. Let $d_{ev}(C_n, i) = |D_{ev}(C_n, i)|$. In this paper, we obtain a recursive formula for $d_{ev}(C_n, i)$. Using this recursive formula, we construct the polynomial, $D_{ev}(C_n, x) = \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ev}(C_n, i) x^i$, which we call edge-vertex domination polynomial of C_n (or simply an *ev*-domination polynomial of C_n) and obtain some properties of this polynomial.

Keywords

***ev*-Domination Set, *ev*-Domination Number, *ev*-Domination Polynomials**

1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. A set $S \subseteq V(G)$ is a dominating set of G, if every vertex $V \setminus S$ is adjacent to at least one vertex in S. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = U_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. The domination number of a graph G is defined as the minimum size of a dominating set in G and it is denoted as $\gamma(G)$. A cycle is defined as a closed path, and is denoted by C_n.

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Definition 1.1

For a graph $G = (V, E)$, an edge $e = uv \in E(G)$, *ev*-dominates a vertex $w \in V(G)$ if

1) $u = w$ or $v = w$ (w is incident to e) or

2) uw or vw is an edge in G (w is adjacent to u or v).

Definition 1.2 [1]

A set $S \subseteq E(G)$ is an edge-vertex dominating set of G (or simply an *ev*-dominating set), if for all vertices $v \in V(G)$; there exists an edge $e \in S$ such that e dominates v. The *ev*-domination number of a graph G is defined as the minimum size of an *ev*-dominating set of edges in G and it is denoted as $\gamma_{ev}(G)$.

Definition 1.3

Let $D_{ev}(C_n, i)$ be the family of *ev*-dominating sets of a graph C_n with cardinality i and let $d_{ev}(C_n, i) = |D_{ev}(C_n, i)|$. We call the polynomial $D_{ev}(C_n, x) = \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ev}(C_n, i) x^i$ the *ev*-domination polynomial of the graph C_n.

In the next section, we construct the families of the *ev*-dominating sets of cycles by recursive method. As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also, we denote the set $\{e_1, e_2, \cdots, e_n\}$ by $[e_n]$ and the set $\{1, 2, \cdots, n\}$ by $[n]$, throughout this paper.

2. Edge-Vertex Dominating Sets of Cycles

Let $D_{ev}(C_n, i)$ be the family of *ev*-dominating sets of C_n with cardinality i. We investigate the *ev*-dominating sets of C_n. We need the following lemma to prove our main results in this section.

Lemma 2.1: [2] $\gamma_{ev}(C_n) = \left\lceil \frac{n}{4} \right\rceil$.

By Lemma 2.1 and the definition of *ev*-domination number, one has the following Lemma:

Lemma 2.2: $D_{ev}(C_n, i) = \Phi$ if and only if $i > n$ or $i < \left\lceil \frac{n}{4} \right\rceil$.

Lemma 2.3: If a graph G contains a simple path of length $4k - 1$, then every *ev*-dominating set of G must contain at least k vertices of the path.

Proof: The path has $4k$ vertices. As every edge dominates at most 4 vertices, the $4k$ vertices are covered by at least k edges.

Lemma 2.4. If $Y \in D_{ev}(C_{n-5}, i-1)$, and there exists $x \in [e_n]$ such that $Y \cup \{x\} \in D_{ev}(C_n, i)$ then $Y \in D_{ev}(C_{n-4}, i-1)$.

Proof: Suppose that $Y \notin D_{ev}(C_{n-4}, i-1)$. Since $Y \in D_{ev}(C_{n-5}, i-1)$, Y contains at least one edge labelled e_{n-5}, e_{n-6} or e_{n-7}.

If $e_{n-5} \in Y$, then $Y \in D_{ev}(C_{n-4}, i-1)$, a contradiction. Hence e_{n-6} or $e_{n-7} \in Y$, but then in this case $Y \cup \{x\} \notin D_{ev}(C_n, i)$ for any $x \in [e_n]$, also a contradiction.

Lemma 2.5. [3]

1) If $D_{ev}(C_{n-1}, i-1) = D_{ev}(C_{n-4}, i-1) = \Phi$ then $D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-3}, i-1) = \Phi$.

2) If $D_{ev}(C_{n-1}, i-1) \neq \Phi$, and $D_{ev}(C_{n-4}, i-1) \neq \Phi$ then $D_{ev}(C_{n-2}, i-1) \neq \Phi$ and $D_{ev}(C_{n-3}, i-1) \neq \Phi$.

3) If $D_{ev}(C_{n-1}, i-1) = \Phi$, $D_{ev}(C_{n-2}, i-1) = \Phi$, $D_{ev}(C_{n-3}, i-1) = \Phi$, and $D_{ev}(C_{n-4}, i-1) = \Phi$, then $D_{ev}(C_n, i) = \Phi$.

Proof: 1) Since $D_{ev}(C_{n-1}, i-1) = D_{ev}(C_{n-4}, i-1) = \Phi$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$. In either case, we have $D_{ev}(C_{n-2}, i-1) = \Phi$ and $D_{ev}(C_{n-3}, i-1) = \Phi$.

2) Since $D_{ev}(C_{n-1}, i-1) \neq \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$, by Lemma 2.2, we have $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1$ and $\left\lceil \frac{n-4}{4} \right\rceil \le i-1 \le n-4$. Hence $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2$ and $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 \le n-3$. Therefore,

$D_{ev}(C_{n-2}, i-1) \neq \Phi$ and $D_{ev}(C_{n-3}, i-1) \neq \Phi$.

3) By hypothesis, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$. Therefore, $i > n$ or $i < \left\lceil \frac{n-4}{4} \right\rceil + 1$. Therefore, $i > n$ or $i < \left\lceil \frac{n}{4} \right\rceil$. Therefore, $D_{ev}(C_n, i) = \Phi$.

Lemma 2.6. [4] If $D_{ev}(C_n, i) \neq \Phi$, then

1) $D_{ev}(C_{n-1}, i-1) = D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-3}, i-1) = \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$ if and only if $n = 4k$ and $i = k$ for some $k \in N$.

2) $D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-3}, i-1) = D_{ev}(C_{n-4}, i-1) = \Phi$ and $D_{ev}(C_{n-1}, i-1) \neq \Phi$ if and only if $i = n$.

3) $D_{ev}(C_{n-1}, i-1) = \Phi$, $D_{ev}(C_{n-2}, i-1) \neq \Phi$, $D_{ev}(C_{n-3}, i-1) \neq \Phi$, $D_{ev}(C_{n-4}, i-1) \neq \Phi$, if and only if $n = 4k + 2$ and $i = \left\lceil \frac{4k+2}{4} \right\rceil$ for some $k \in N$.

4) $D_{ev}(C_{n-1}, i-1) \neq \Phi$, $D_{ev}(C_{n-2}, i-1) \neq \Phi$, $D_{ev}(C_{n-3}, i-1) \neq \Phi$ and $D_{ev}(C_{n-4}, i-1) = \Phi$ if and only if $i = n - 2$.

5) $D_{ev}(C_{n-1}, i-1) \neq \Phi$, $D_{ev}(C_{n-2}, i-1) \neq \Phi$, $D_{ev}(C_{n-3}, i-1) = \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$ if and only if $i = n - 1$.

6) $D_{ev}(C_{n-1}, i-1) \neq \Phi$, $D_{ev}(C_{n-2}, i-1) \neq \Phi$, $D_{ev}(C_{n-3}, i-1) \neq \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$ if and only if $\left\lceil \frac{n-1}{4} \right\rceil + 1 \leq i \leq n-3$.

Proof: 1) (⇒) Since $D_{ev}(C_{n-1}, i-1) = D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-3}, i-1) = \Phi$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$. If $i-1 > n-1$, then $i > n$ and by Lemma 2.2, $D_{ev}(C_n, i) = \Phi$, a contradiction.

So

$$i-1 < \left\lceil \frac{n-3}{4} \right\rceil \tag{2.1}$$

and since $D_{ev}(C_{n-4}, i-1) \neq \Phi$, we have

$$\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 \leq n-4 \tag{2.2}$$

From (2.1) and (2.2),

$$\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 < \left\lceil \frac{n-3}{4} \right\rceil \tag{2.3}$$

When n is a multiple of 4, $\left\lceil \frac{n-4}{4} \right\rceil = \frac{n}{4} - 1$ and $\left\lceil \frac{n-3}{4} \right\rceil = \frac{n}{4}$. Therefore, $\frac{n}{4} - 1 \leq i-1 < \frac{n}{4}$. Therefore, $i-1 = \frac{n}{4} - 1$, we get $i = \frac{n}{4}$. Thus, when $n = 4k$, (2.3) holds good and $i = \frac{n}{4} = k$. When $n \neq 4k$, $\left\lceil \frac{n-4}{4} \right\rceil = \left\lceil \frac{n}{4} \right\rceil - 1$ and $\left\lceil \frac{n-3}{4} \right\rceil = \left\lceil \frac{n}{4} \right\rceil - 1$. Therefore, $\left\lceil \frac{n}{4} \right\rceil - 1 \leq i-1 < \left\lceil \frac{n}{4} \right\rceil - 1$, which is not possible.

Hence $n = 4k$ and $i = k$

(⇐) If $n = 4k$ and $i = k$ for some $k \in N$, then by Lemma 2.2, $\left\lceil \frac{n-1}{4} \right\rceil = \left\lceil \frac{4k-1}{4} \right\rceil = k = i > i-1$. Therefore, $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$, which implies $D_{ev}(C_{n-1}, i-1) = \Phi$. Similarly, $D_{ev}(C_{n-2}, i-1) = \Phi$ and $D_{ev}(C_{n-3}, i-1) = \Phi$.

Now $\left\lceil \frac{n-4}{4} \right\rceil = \left\lceil \frac{4k-4}{4} \right\rceil = k-1 = i-1$. Therefore, $\left\lceil \frac{n-4}{4} \right\rceil \le i-1$, which implies $D_{ev}\left(C_{n-4}, i-1\right) \ne \Phi$.

2) (⇒) Since $D_{ev}\left(C_{n-2}, i-1\right) = D_{ev}\left(C_{n-3}, i-1\right) = D_{ev}\left(C_{n-4}, i-1\right) = \Phi$, by Lemma 2.2, $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$. If $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ then by Lemma 2.2, $D_{ev}\left(C_{n-1}, i-1\right) = \Phi$, a contradiction.

So

$$i-1 > n-2 \tag{2.4}$$

Since,

$$D_{ev}\left(C_{n-1}, i-1\right) \ne \Phi, \ \left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1 \tag{2.5}$$

From (2.4) and (2.5), we have $n-1 \ge i-1 > n-2$. Therefore, $i-1 = n-1$. Therefore, $i = n$

(⇐) If $i = n$, then by Lemma 2.2, $\left\lceil \frac{n}{4} \right\rceil \le i \le n$. Therefore, $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1$

Therefore, $D_{ev}\left(C_{n-1}, i-1\right) \ne \Phi$

3) (⇒) Since $D_{ev}\left(C_{n-1}, i-1\right) = \Phi$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$. If $i-1 > n-1$, then $i-1 > n-2 > n-3 > n-4$, by Lemma 2.2, $D_{ev}\left(C_{n-2}, i-1\right) = D_{ev}\left(C_{n-3}, i-1\right) = D_{ev}\left(C_{n-4}, i-1\right) = \Phi$, a contradiction.

Therefore, $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$, which implies,

$$i < \left\lceil \frac{n-1}{4} \right\rceil + 1 \tag{2.6}$$

Since, $D_{ev}\left(C_{n-2}, i-1\right) \ne \Phi$, $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2$.

Hence,

$$\left\lceil \frac{n-2}{4} \right\rceil + 1 \le i \le n-1 \tag{2.7}$$

Similarly,

$$\left\lceil \frac{n-3}{4} \right\rceil + 1 \le i \le n-2 \tag{2.8}$$

and

$$\left\lceil \frac{n-4}{4} \right\rceil + 1 \le i \le n-3 \tag{2.9}$$

From (2.6), (2.7), (2.8) and (2.9),

$$\left\lceil \frac{n-2}{4} \right\rceil + 1 \le i < \left\lceil \frac{n-1}{4} \right\rceil + 1 \tag{2.10}$$

Therefore, (2.10) hold when $k = \frac{n-2}{4}$ or $n = 4k+2$ and $i = k+1 = \left\lceil \frac{4k+2}{4} \right\rceil$, for some $k \in N$. Suppose $n = 4k+2$, then $\left\lceil \frac{n-2}{4} \right\rceil + 1 = k+1$ and $\left\lceil \frac{n-1}{4} \right\rceil + 1 = k+2$. Therefore, from (2.10), we have, $k+1 \le i < k+2$, which implies $i = k+1$. Suppose $n \ne 4k+2$, *i.e.*, $n = 4k, 4k+1, 4k+3$.

Case 1) When $n = 4k$.

From (2.10), we get $\left\lceil \frac{4k-2}{4} \right\rceil + 1 = k+1$ and $\left\lceil \frac{4k-1}{4} \right\rceil + 1 = k+1$. Therefore, $k+1 \le i < k+1$, which is not possible.

Case 2) When $n = 4k+1$. From (2.10), we get $\left\lceil \frac{4k+1-2}{4} \right\rceil + 1 = k+1$ and $\left\lceil \frac{4k+1-1}{4} \right\rceil + 1 = k+1$. Therefore, $k+1 \le i < k+1$, which is not possible.

Case 3) When $n = 4k+3$. From (2.10), we get $\left\lceil \frac{4k+3-2}{4} \right\rceil + 1 = k+2$ and $\left\lceil \frac{4k+3-1}{4} \right\rceil + 1 = k+2$

Therefore, $k+2 \le i < k+2$, which is not possible. Therefore, $n = 4k+2$

(⇐) If $n = 4k + 2$ and $i = \left\lceil \frac{4k+2}{4} \right\rceil$ for some $k \in N$, and $D_{ev}(C_n, i) \ne \Phi$, then by Lemma 2.2, $\left\lceil \frac{n}{4} \right\rceil \le i \le n$, $\left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{4k+2}{4} \right\rceil = i > i-1$. Therefore, $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$. Therefore, $D_{ev}(C_{n-1}, i-1) = \Phi$.

Also, $\left\lceil \frac{n-2}{4} \right\rceil = k$. Therefore, $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2$ and $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 \le n-3$ and $\left\lceil \frac{n-4}{4} \right\rceil \le i-1 \le n-4$.

Hence $D_{ev}(C_{n-2}, i-1) \ne \Phi$, $D_{ev}(C_{n-3}, i-1) \ne \Phi$, $D_{ev}(C_{n-4}, i-1) \ne \Phi$.

4) (⇒) Since $D_{ev}(C_{n-4}, i-1) = \Phi$, by Lemma 2.2,

$$i-1 > n-4 \quad \text{or} \quad i-1 < \left\lceil \frac{n-4}{4} \right\rceil \tag{2.11}$$

Since $D_{ev}(C_{n-3}, i-1) \ne \Phi$, by Lemma 2.2,

$$\left\lceil \frac{n-3}{4} \right\rceil \le i-1 \le n-3 \tag{2.12}$$

Similarly, $D_{ev}(C_{n-2}, i-1) \ne \Phi$ and $D_{ev}(C_{n-1}, i-1) \ne \Phi$, by Lemma 2.2

$$\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2 \tag{2.13}$$

and

$$\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1 \tag{2.14}$$

From (2.11), we get $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ which is not possible.

Therefore,

$$i-1 > n-4 \Rightarrow i > n-3 \Rightarrow i \ge n-2 \tag{2.15}$$

From (2.12),

$$i-1 \le n-3 \Rightarrow i \le n-2 \tag{2.16}$$

From (2.15) and (2.16), $i = n-2$

(⇐) If $i = n-2$, $i-1 = n-3$ then by Lemma 2.2, $i-1 > n-4$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$. Therefore, $D_{ev}(C_{n-4}, i-1) = \Phi$. Also $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1$, therefore, $D_{ev}(C_{n-1}, i-1) \ne \Phi$; $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2$, therefore, $D_{ev}(C_{n-2}, i-1) \ne \Phi$; and $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 \le n-3$, therefore, $D_{ev}(C_{n-3}, i-1) \ne \Phi$.

5) (⇒) Since $D_{ev}(C_{n-4}, i-1) = \Phi$ by Lemma 2.2,

$$i-1>n-4 \text{ or } i-1<\left\lceil \frac{n-4}{4} \right\rceil \quad (2.17)$$

Since $D_{ev}(C_{n-3}, i-1)=\Phi$ by Lemma 2.2,

$$i-1>n-3 \text{ or } i-1<\left\lceil \frac{n-3}{4} \right\rceil \quad (2.18)$$

Since $D_{ev}(C_{n-2}, i-1)\neq\Phi$ and $D_{ev}(C_{n-1}, i-1)\neq\Phi$, by Lemma 2.2,

$$\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2 \quad (2.19)$$

and

$$\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1 \quad (2.20)$$

From (2.19) and (2.20), we have $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-2$. From (2.18), we have $i-1>n-3$. Therefore, $i-1\ge n-2$. But $i-1\le n-2$. Therefore, $i-1=n-2$. Therefore, $i=n-1$.

($\Leftarrow$) If $i=n-1$, $i-1=n-2$ then by Lemma 2.2, $D_{ev}(C_{n-3}, i-1)=\Phi$ and $i-1>n-4$ therefore, $D_{ev}(C_{n-4}, i-1)=\Phi$ and $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1$, therefore, $D_{ev}(C_{n-1}, i-1)\neq\Phi$ and $\left\lceil \frac{n-2}{4} \right\rceil \le i-2 \le n-2$, therefore, $D_{ev}(C_{n-2}, i-1)\neq\Phi$.

6) ($\Rightarrow$) Since $D_{ev}(C_{n-1}, i-1)\neq\Phi$, $D_{ev}(C_{n-2}, i-1)\neq\Phi$, $D_{ev}(C_{n-3}, i-1)\neq\Phi$, and $D_{ev}(C_{n-4}, i-1)\neq\Phi$, by Lemma 2.2, $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1$, $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2$, $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 \le n-3$, and $\left\lceil \frac{n-4}{4} \right\rceil \le i-1 \le n-4$.

So $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-4$ and hence $\left\lceil \frac{n-1}{4} \right\rceil +1 \le i \le n-3$.

($\Leftarrow$) If $\left\lceil \frac{n-1}{4} \right\rceil +1 \le i \le n-3$, then by Lemma 2.2 we have, $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 \le n-1$, $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 \le n-2$, $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 \le n-3$, $\left\lceil \frac{n-4}{4} \right\rceil \le i-1 \le n-4$.

Therefore, $D_{ev}(C_{n-1}, i-1)\neq\Phi$, $D_{ev}(C_{n-2}, i-1)\neq\Phi$, $D_{ev}(C_{n-3}, i-1)\neq\Phi$, and $D_{ev}(C_{n-4}, i-1)\neq\Phi$.

Theorem 2.7 [5]

For every $n\ge 5$ and $i\ge \left\lceil \frac{n}{4} \right\rceil$,

1) If $D_{ev}(C_{n-1}, i-1)=D_{ev}(C_{n-2}, i-1)=D_{ev}(C_{n-3}, i-1)=\Phi$ and $D_{ev}(C_{n-4}, i-1)\neq\Phi$ then $D_{ev}(C_n, i)=\{\{e_1,e_5,\cdots,e_{n-3}\},\{e_2,e_6,\cdots,e_{n-2}\},\{e_3,e_7,\cdots,e_{n-1}\},\{e_4,e_8,\cdots,e_n\}\}$.

2) If $D_{ev}(C_{n-2}, i-1)=D_{ev}(C_{n-3}, i-1)=D_{ev}(C_{n-4}, i-1)=\Phi$ and $D_{ev}(C_{n-1}, i-1)\neq\Phi$ then $D_{ev}(C_n, i)=\{[e_n]\}$.

3) If $D_{ev}(C_{n-1}, i-1)=\Phi, D_{ev}(C_{n-2}, i-1)\neq\Phi, D_{ev}(C_{n-3}, i-1)\neq\Phi$ and $D_{ev}(C_{n-4}, i-1)\neq\Phi$ then $D_{ev}(C_n, i)=Y_1\cup Y_2\cup Y_3$, where

$$Y_1=\{\{e_1,e_5,\cdots,e_{n-5},e_{n-1}\},\{e_2,e_6,\cdots,e_{n-4},e_n\},\{e_3,e_7,\cdots,e_{n-3},e_{n-1}\},\{e_4,e_8,\cdots,e_{n-2},e_n\}\}$$

$$Y_2=\left\{ X_2\cup \left\{ \begin{array}{l} \{e_{n-2}\}, \text{ if } e_1\in X_2 / X_2\in D_{ev}(C_{n-3}, i-1) \\ \{e_{n-1}\}, \text{ if } e_2\in X_2 / X_2\in D_{ev}(C_{n-3}, i-1) \\ \{e_n\}, \text{ otherwise} \end{array} \right\} \right\}$$

$$Y_3 = \left\{ X_3 \cup \left\{ \begin{array}{l} \{e_{n-3}\}, \text{ if } e_1 \in X_3 / X_3 \in D_{ev}(C_{n-4}, i-1) \\ \{e_{n-2}\}, \text{ if } e_2 \in X_3 / X_3 \in D_{ev}(C_{n-4}, i-1) \\ \{e_{n-1}\}, \text{ if } e_3 \in X_3 / X_3 \in D_{ev}(C_{n-4}, i-1) \\ \{e_n\}, \text{ otherwise} \end{array} \right\} \right.$$

4) If $D_{ev}(C_{n-3}, i-1) = \Phi$, $D_{ev}(C_{n-2}, i-1) \neq \Phi$, and $D_{ev}(C_{n-1}, i-1) \neq \Phi$ then

$$D_{ev}(C_n, i) = \{[e_n] - \{x\} / x \in [e_n]\}$$

5) If $D_{ev}(C_{n-1}, i-1) \neq \Phi, D_{ev}(C_{n-2}, i-1) \neq \Phi, D_{ev}(C_{n-3}, i-1) \neq \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$ then

$$D_{ev}(C_n, i) = \{\{X_1 \cup \{e_n\} / X_1 \in D_{ev}(C_{n-1}, i-1)\} \cup \{X_2 \cup \{e_{n-1}\} / X_2 \in D_{ev}(C_{n-2}, i-1)\}$$
$$\cup \{X_3 \cup \{e_{n-2}\} / X_3 \in D_{ev}(C_{n-3}, i-1)\} \cup \{X_4 \cup \{e_{n-3}\} / X_4 \in D_{ev}(C_{n-4}, i-1)\}\}$$

Proof:

1) Since, $D_{ev}(C_{n-1}, i-1) = D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-3}, i-1) = \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$, by Lemma 2.6 (i) $n = 4k$, and $i = k$ for some $k \in N$. The sets $\{e_1, e_5, \cdots, e_{n-3}\}, \{e_2, e_6, \cdots, e_{n-2}\}, \{e_3, e_7, \cdots, e_{n-1}\}, \{e_4, e_8, \cdots, e_n\}$ have $\frac{n}{4}$ elements and each one covers all vertices. Also, no other sets of cardinality $\frac{n}{4}$ covers all vertices. Therefore, the collection of *ev*-dominating sets of cardinality $\frac{n}{4}$ is

$$\{\{e_1, e_5, \cdots, e_{n-3}\}, \{e_2, e_6, \cdots, e_{n-2}\}, \{e_3, e_7, \cdots, e_{n-1}\}, \{e_4, e_8, \cdots, e_n\}\}$$

Hence, $D_{ev}(C_n, i) = \{\{e_1, e_5, \cdots, e_{n-3}\}, \{e_2, e_6, \cdots, e_{n-2}\}, \{e_3, e_7, \cdots, e_{n-1}\}, \{e_4, e_8, \cdots, e_n\}\}$.

2) We have $D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-3}, i-1) = D_{ev}(C_{n-4}, i-1) = \Phi$ and $D_{ev}(C_{n-1}, i-1) \neq \Phi$. By Lemma 2.6 (2), we have $i = n$. So, $D_{ev}(C_n, i) = \{[e_n]\}$.

3) We have $D_{ev}(C_{n-1}, i-1) = \Phi, D_{ev}(C_{n-2}, i-1) \neq \Phi, D_{ev}(C_{n-3}, i-1) \neq \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$, by Lemma 2.6 (3), $n = 4k+2$ and $i = \left\lceil \frac{4k+2}{4} \right\rceil = k+1$, for some $k \in N$.

Let $Y_1 = \{\{e_1, e_5, \cdots, e_{4k-3}, e_{4k+1}\}, \{e_2, e_6, \cdots, e_{4k-2}, e_{4k+2}\}, \{e_3, e_7, \cdots, e_{4k-1}, e_{4k+1}\}, \{e_4, e_8, \cdots, e_{4k}, e_{4k+2}\}\}$

$$Y_2 = \left\{ X_2 \cup \left\{ \begin{array}{l} \{e_{4k}\}, \text{ if } e_1 \in X_2 / X_2 \in D_{ev}(C_{4k-1}, k) \\ \{e_{4k+1}\}, \text{ if } e_2 \in X_2 / X_2 \in D_{ev}(C_{4k-1}, k) \\ \{e_{4k+2}\}, \text{ otherwise} \end{array} \right\} \right.$$

$$Y_3 = \left\{ X_3 \cup \left\{ \begin{array}{l} \{e_{4k-1}\}, \text{ if } e_1 \in X_3 / X_3 \in D_{ev}(C_{4k-2}, k) \\ \{e_{4k}\}, \text{ if } e_2 \in X_3 / X_3 \in D_{ev}(C_{4k-2}, k) \\ \{e_{4k+1}\}, \text{ if } e_3 \in X_3 / X_3 \in D_{ev}(C_{4k-2}, k) \\ \{e_{4k+2}\}, \text{ otherwise} \end{array} \right\} \right.$$

We shall prove that $D_{ev}(C_{4k+2}, k+1) = Y_1 \cup Y_2 \cup Y_3$. It is clear that $Y_1 \subseteq D_{ev}(C_{4k+2}, k+1)$, $Y_2 \subseteq D_{ev}(C_{4k+2}, k+1)$, and $Y_3 \subseteq D_{ev}(C_{4k+2}, k+1)$. Therefore, $Y_1 \cup Y_2 \cup Y_3 \subseteq D_{ev}(C_{4k+2}, k+1)$.

Conversely, Let $Y \in D_{ev}(C_{4k+2}, k+1)$. Suppose, Y is of the form $\{e_1, e_5, \cdots, e_{4k-3}, e_{4k+1}\}$, $\{e_2, e_6, \cdots, e_{4k-2}, e_{4k+2}\}, \{e_3, e_7, \cdots, e_{4k-1}, e_{4k+1}\}, \{e_4, e_8, \cdots, e_{4k}, e_{4k+2}\}$ then $Y \in Y_1 \subseteq D_{ev}(C_{4k}, k)$. Now suppose,

$Y \notin Y_1$ and Y is of the form $\left\{ X_2 \cup \begin{cases} \{e_{4k}\}, \text{ if } e_1 \in X_2 / X_2 \in D_{ev}(C_{4k-1},k) \\ \{e_{4k+1}\}, \text{ if } e_2 \in X_2 / X_2 \in D_{ev}(C_{4k-1},k) \\ \{e_{4k+2}\}, \text{ otherwise} \end{cases} \right\}$ then $Y \in Y_2 \subseteq D_{ev}(C_{4k-1},k)$.

Now suppose, $Y \notin Y_1$, $Y \notin Y_2$ and $Y \in D_{ev}(C_{4k+2},k+1)$. We split $D_{ev}(C_{4k-2},k)$ as four parts. If $X_3 \in D_{ev}(C_{4k-2},k)$ with $e_1 \in X_3$ then $X_3 \cup \{e_{4k-1}\} \in Y_3$ and $X_3 \cup \{e_{n-3}\} \notin Y_2$ and $\notin Y_1$. If $X_3 \in D_{ev}(C_{4k-2},k)$ with $e_2 \in X_3$ then $X_3 \cup \{e_{4k}\} \in Y_3$ and $X_3 \cup \{e_{4k}\} \notin Y_2$ and $\notin Y_1$. If $X_3 \in D_{ev}(C_{4k-2},k)$ with $e_3 \in X_3$ then $X_3 \cup \{e_{4k+1}\} \in Y_3$ and $X_3 \cup \{e_{4k+1}\} \notin Y_2$ and $\notin Y_1$. If $X_3 \in D_{ev}(C_{4k-2},k)$ with $e_4 \in X_3$ then $X_3 \cup \{e_{4k+2}\} \in Y_3$ and $X_3 \cup \{e_{4k+2}\} \notin Y_2$ and $\notin Y_1$. In this case Y is of the form $X_3 \cup \begin{cases} \{e_{4k-1}\}, \text{ if } e_1 \in X_3 / X_3 \in D_{ev}(C_{4k-2},k) \\ \{e_{4k}\}, \text{ if } e_2 \in X_3 / X_3 \in D_{ev}(C_{4k-2},k) \\ \{e_{4k+1}\}, \text{ if } e_3 \in X_3 / X_3 \in D_{ev}(C_{4k-2},k) \\ \{e_{4k+2}\}, \text{ otherwise} \end{cases}$ then $Y \in Y_3 \in D_{ev}(C_{4k-2},k)$. Therefore, $D_{ev}(C_{4k+2},k+1) \subseteq Y_1 \cup Y_2 \cup Y_3$. Thus, we have proved that

$$D_{ev}(C_n,i) = Y_1 \cup Y_2 \cup \cup Y_3$$

where $Y_1 = \{\{e_1,e_5,\cdots,e_{n-5},e_{n-1}\},\{e_2,e_6,\cdots,e_{n-4},e_n\},\{e_3,e_7,\cdots,e_{n-3},e_{n-1}\},\{e_4,e_8,\cdots,e_{n-2},e_n\}\}$

$$Y_2 = \left\{ X_2 \cup \begin{cases} \{e_{n-2}\}, \text{ if } e_1 \in X_2 / X_2 \in D_{ev}(C_{n-3},i-1) \\ \{e_{n-1}\}, \text{ if } e_2 \in X_2 / X_2 \in D_{ev}(C_{n-3},i-1) \\ \{e_n\}, \text{ otherwise} \end{cases} \right\}$$

$$Y_3 = \left\{ X_3 \cup \begin{cases} \{e_{n-3}\}, \text{ if } e_1 \in X_3 / X_3 \in D_{ev}(C_{n-4},i-1) \\ \{e_{n-2}\}, \text{ if } e_2 \in X_3 / X_3 \in D_{ev}(C_{n-4},i-1) \\ \{e_{n-1}\}, \text{ if } e_3 \in X_3 / X_3 \in D_{ev}(C_{n-4},i-1) \\ \{e_n\}, \text{ otherwise} \end{cases} \right\}.$$

4) If $D_{ev}(C_{n-3},i-1) = \Phi, D_{ev}(C_{n-2},i-1) \neq \Phi$, and $D_{ev}(C_{n-1},i-1) \neq \Phi$, by Lemma 3.6 (iv) $i = n-1$. Therefore $D_{ev}(C_n,i) = \{[e_n] - \{x\} / x \in [e_n]\}$.

5) $D_{ev}(C_{n-1},i-1) \neq \Phi, D_{ev}(C_{n-2},i-1) \neq \Phi, D_{ev}(C_{n-3},i-1) \neq \Phi$ and $D_{ev}(C_{n-4},i-1) \neq \Phi$.

Clearly, $$\begin{aligned} &\{\{X_1 \cup \{e_n\} / X_1 \in D_{ev}(C_{n-1},i-1)\} \cup \{X_2 \cup \{e_{n-1}\} / X_2 \in D_{ev}(C_{n-2},i-1)\} \\ &\cup \{X_3 \cup \{e_{n-2}\} / X_3 \in D_{ev}(C_{n-3},i-1)\} \cup \{X_4 \cup \{e_{n-3}\} / X_4 \in D_{ev}(C_{n-4},i-1)\}\} \subseteq D_{ev}(C_n,i). \end{aligned}$$

Conversely, let $Y \in D_{ev}(C_n,i)$. Then e_n or e_{n-1} or e_{n-2} or $e_{n-3} \in Y$. If $e_n \in Y$, then we can write $Y = X_1 \cup \{e_n\}$, for some $X_1 \in D_{ev}(C_{n-1},i-1)$. If $e_{n-1} \in Y$ and $e_n \notin Y$ then we can write $Y = X_2 \cup \{e_{n-1}\}$, for some $X_2 \in D_{ev}(C_{n-2},i-1)$. If $e_{n-2} \in Y$ and $e_n \notin Y, e_{n-1} \notin Y$ then we can write $Y = X_3 \cup \{e_{n-2}\}$, for some $X_3 \in D_{ev}(C_{n-3},i-1)$. If $e_{n-3} \in Y$, $e_n \notin Y, e_{n-1} \notin Y$, $e_{n-3} \notin Y$ then we can write $Y = X_4 \cup \{e_{n-3}\}$, for some $X_4 \in D_{ev}(C_{n-1},i-1)$.

Therefore we proved that

$$\begin{aligned} D_{ev}(C_n,i) \subseteq &\{\{X_1 \cup \{e_n\} / X_1 \in D_{ev}(C_{n-1},i-1)\} \cup \{X_2 \cup \{e_{n-1}\} / X_2 \in D_{ev}(C_{n-2},i-1)\} \\ &\cup \{X_3 \cup \{e_{n-2}\} / X_3 \in D_{ev}(C_{n-3},i-1)\} \cup \{X_4 \cup \{e_{n-3}\} / X_4 \in D_{ev}(C_{n-4},i-1)\}\}. \end{aligned}$$

Hence,
$$D_{ev}(C_n,i)=\{\{X_1\cup\{e_n\}/X_1\in D_{ev}(C_{n-1},i-1)\}\cup\{X_2\cup\{e_{n-1}\}/X_2\in D_{ev}(C_{n-2},i-1)\}$$
$$\cup\{X_3\cup\{e_{n-2}\}/X_3\in D_{ev}(C_{n-3},i-1)\}\cup\{X_4\cup\{e_{n-3}\}/X_4\in D_{ev}(C_{n-4},i-1)\}\}$$

3. Edge-Vertex Domination Polynomials of Cycles

Let $D_{ev}(C_n,x)=\sum_{i=\left\lceil \frac{n}{4}\right\rceil}^{n} d_{ev}(C_n,i)x^i$ be the *ev*-domination polynomial of a cycle C_n. In this section, we derive the expression for $D_{ev}(C_n,x)$.

Theorem 3.1 [6]

1) If $D_{ev}(C_n,i)$ is the family of *ev*-dominating sets with cardinality *i* of C_n, then

$$d_{ev}(C_n,i)=d_{ev}(C_{n-1},i-1)+d_{ev}(C_{n-2},i-1)+d_{ev}(C_{n-3},i-1)+d_{ev}(C_{n-4},i-1)$$

where $d_{ev}(C_n,i)=|D_{ev}(C_n,i)|$.

2) For every $n\ge 5$,

$$D_{ev}(C_n,x)=x[D_{ev}(C_{n-1},x)+D_{ev}(C_{n-2},x)+D_{ev}(C_{n-3},x)+D_{ev}(C_{n-4},x)]$$

with the initial values

$$D_{ev}(C_1,x)=x,$$
$$D_{ev}(C_2,x)=2x^2+x,$$
$$D_{ev}(C_3,x)=3x^3+3x^2+x,$$
$$D_{ev}(C_4,x)=4x^4+6x^3+4x^2+x.$$

Proof:

1) Using (1), (2), (3), (4) and (5) of Theorem 2.7, we prove (1) part.

Suppose, $D_{ev}(C_{n-1},i-1)=D_{ev}(C_{n-2},i-1)=D_{ev}(C_{n-3},i-1)=\Phi$ and $D_{ev}(C_{n-4},i-1)\ne\Phi$ then, $D_{ev}(C_n,i)=\{\{e_1,e_5,\cdots,e_{n-3}\},\{e_2,e_6,\cdots,e_{n-2}\},\{e_3,e_7,\cdots,e_{n-1}\},\{e_4,e_8,\cdots,e_n\}\}$.

Therefore, $|D_{ev}(C_n,i)|=|\{\{e_1,e_5,\cdots,e_{n-3}\},\{e_2,e_6,\cdots,e_{n-2}\},\{e_3,e_7,\cdots,e_{n-1}\},\{e_4,e_8,\cdots,e_n\}\}|=4$. In this case $|D_{ev}(C_{n-4},i-1)|=|\{\{e_1,e_5,\cdots,e_{n-3}\},\{e_2,e_6,\cdots,e_{n-2}\},\{e_3,e_7,\cdots,e_{n-1}\},\{e_4,e_8,\cdots,e_n\}\}|=4$ and $|D_{ev}(C_{n-1},i-1)|=|D_{ev}(C_{n-2},i-1)|=|D_{ev}(C_{n-3},i-1)|=0$. Therefore, in this case the theorem holds.

Suppose, $D_{ev}(C_{n-2},i-1)=D_{ev}(C_{n-3},i-1)=D(C_{n-4},i-1)=\Phi$ and $D_{ev}(C_{n-1},i-1)\ne\Phi$, then $D_{ev}(C_n,i)=\{[e_n]\}$ Therefore, $|D_{ev}(C_n,i)|=|\{[e_n]\}|=1$. In this case $D_{ev}(C_{n-1},i-1)=\{[e_{n-1}]\}$. Therefore, $|D_{ev}(C_{n-1},i-1)|=|\{[e_{n-1}]\}|=1$ and $|D_{ev}(C_{n-2},i-1)|=|D_{ev}(C_{n-3},i-1)|=|D_{ev}(C_{n-4},i-1)|=0$. Therefore, in this case the theorem holds.

Suppose, $D_{ev}(C_{n-1},i-1)=\Phi, D_{ev}(C_{n-2},i-1)\ne\Phi, D(C_{n-3}^2,i-1)\ne\Phi$ and $D(C_{n-4}^2,i-1)\ne\Phi$. In this case,

$$D_{ev}(C_n,i)=Y_1\cup Y_2\cup Y_3$$

where

$$Y_1=\{\{e_1,e_5,\cdots,e_{n-5},e_{n-1}\},\{e_2,e_6,\cdots,e_{n-4},e_n\},\{e_3,e_7,\cdots,e_{n-3},e_{n-1}\},\{e_4,e_8,\cdots,e_{n-2},e_n\}\}$$

$$Y_2=\left\{X_2\cup\left\{\begin{array}{l}\{e_{n-2}\},\text{ if }1\in X_2/X_2\in D_{ev}(C_{n-2},i-1)\\ \{e_{n-1}\},\text{ if }2\in X_2/X_2\in D_{ev}(C_{n-2},i-1)\\ \{e_n\},\text{ otherwise}\end{array}\right\}\right.$$

$$Y_3 = \begin{cases} X_3 \cup \begin{Bmatrix} \{e_{n-3}\}, \text{ if } 1 \in X_3 / X_3 \in D_{ev}(C_{n-3}, i-1) \\ \{e_{n-2}\}, \text{ if } 2 \in X_3 / X_3 \in D_{ev}(C_{n-3}, i-1) \\ \{e_{n-1}\}, \text{ if } 3 \in X_3 / X_3 \in D_{ev}(C_{n-3}, i-1) \\ \{e_n\}, \text{ otherwise} \end{Bmatrix} \end{cases}$$

Therefore, $|D_{ev}(C_n, i)| = 4 + |D_{ev}(C_{n-3}, i-1)| + |D_{ev}(C_{n-4}, i-1)|$. Also, $|D_{ev}(C_{n-1}, i-1)| = 0$ and $|D_{ev}(C_{n-2}, i-1)| = 4$. Therefore, $|D_{ev}(C_n, i)| = |D_{ev}(C_{n-1}, i-1)| + |D_{ev}(C_{n-3}, i-1)| + |D_{ev}(C_{n-4}, i-1)|$ and $|D_{ev}(C_{n-1}, i-1)| = 0$. Therefore, in this case the theorem holds.

Suppose,

$$D_{ev}(C_{n-1}, i-1) \neq \Phi,\ D_{ev}(C_{n-2}, i-1) \neq \Phi \text{ and } D_{ev}(C_{n-3}, i-1) = \Phi$$

Then we have $D_{ev}(C_n, i) = \{[e_n] - \{x\} / x \in [e_n]\}$.

Therefore, $|D_{ev}(C_n, i)| = n$. In this case, $|D_{ev}(C_{n-1}, i-1)| = n-1$, $|D_{ev}(C_{n-2}, i-1)| = 1$ and $|D_{ev}(C_{n-3}, i-1)| = 0$. Therefore,

$$|D_{ev}(C_n, i)| = |D_{ev}(C_{n-1}, i-1)| + |D_{ev}(C_{n-2}, i-1)| + |D_{ev}(C_{n-3}, i-1)| = n - 1 + 1 + 0 = n.$$

Therefore, in this case the theorem holds.

Suppose, $D_{ev}(C_{n-1}, i-1) \neq \Phi,\ D_{ev}(C_{n-2}, i-1) \neq \Phi,\ D(C_{n-3}^2, i-1) \neq \Phi$ and $D_{ev}(C_{n-4}, i-1) \neq \Phi$. In this case, we have

$$\begin{aligned} D_{ev}(C_n, i) = \{&\{X_1 \cup \{e_n\} / X_1 \in D_{ev}(C_{n-1}, i-1)\} \cup \{X_2 \cup \{e_{n-1}\} / X_2 \in D_{ev}(C_{n-2}, i-1)\} \\ &\cup \{X_3 \cup \{e_{n-2}\} / X_3 \in D_{ev}(C_{n-3}, i-1)\} \cup \{X_4 \cup \{e_{n-3}\} / X_4 \in D_{ev}(C_{n-4}, i-1)\}\}. \end{aligned}$$

Therefore,

$$|D_{ev}(C_n, i)| = |D_{ev}(C_{n-1}, i-1)| + |D_{ev}(C_{n-2}, i-1)| + |D_{ev}(C_{n-3}, i-1)| + |D_{ev}(C_{n-4}, i-1)|.$$

Hence,

$$d_{ev}(C_n, i) = d_{ev}(C_{n-1}, i-1) + d_{ev}(C_{n-2}, i-1) + d_{ev}(C_{n-3}, i-1) + d_{ev}(C_{n-4}, i-1).$$

$$d_{ev}(C_n, i)x^i = d_{ev}(C_{n-1}, i-1)x^i + d_{ev}(C_{n-2}, i-1)x^i + d_{ev}(C_{n-3}, i-1)x^i + d_{ev}(C_{n-4}, i-1)x^i$$

$$\sum d_{ev}(C_n, i)x^i = \sum d_{ev}(C_{n-1}, i-1)x^i + \sum d_{ev}(C_{n-2}, i-1)x^i + \sum d_{ev}(C_{n-3}, i-1)x^i + \sum d_{ev}(C_{n-4}, i-1)x^i$$

$$\begin{aligned} \sum d_{ev}(C_n, i)x^i &= x\sum d_{ev}(C_{n-1}, i-1)x^{i-1} + x\sum d_{ev}(C_{n-2}, i-1)x^{i-1} \\ &\quad + x\sum d_{ev}(C_{n-3}, i-1)x^{i-1} + x\sum d_{ev}(C_{n-4}, i-1)x^{i-1} \end{aligned}$$

$$\sum d_{ev}(C_n, i)x^i = x\left[\sum d_{ev}(C_{n-1}, i-1)x^{i-1} + \sum d_{ev}(C_{n-2}, i-1)x^{i-1} + \sum d_{ev}(C_{n-3}, i-1)x^{i-1} + \sum d_{ev}(C_{n-4}, i-1)x^{i-1}\right]$$

$$D_{ev}(C_n, x) = x\left[D_{ev}(C_{n-1}, x) + D_{ev}(C_{n-2}, x) + D_{ev}(C_{n-3}, x) + D_{ev}(C_{n-4}, x)\right].$$

with the initial values

$$D_{ev}(C_1, x) = x$$

$$D_{ev}(C_2, x) = 2x^2 + x$$

$$D_{ev}(C_3, x) = 3x^3 + 3x^2 + x$$

$$D_{ev}(C_4, x) = 4x^4 + 6x^3 + 4x^2 + x$$

We obtain $d_{ev}(C_n, i)$ for $1 \le n \le 16$ as shown in **Table 1**.

In the following Theorem, we obtain some properties of $d_{ev}(C_n, i)$.

Theorem 3.2

The following properties hold for the coefficients of $D_{ev}(C_n, x)$;

1) $d_{ev}(C_{4n}, n) = 4$, for every $n \in$ N.

2) $d_{ev}(C_n, n) = 1$, for every $n \in$ N.

3) $d_{ev}(C_n, n-1) = n$, for every $n \ge 2$.

4) $d_{ev}(C_n, n-2) = nC_2 = \frac{n(n-1)}{2}$, for every $n \ge 3$.

5) $d_{ev}(C_n, n-3) = nC_3 = \frac{n(n-1)(n-2)}{6}$, for every $n \ge 4$.

6) $d_{ev}(C_n, n-4) = nC_4 - n = \frac{n(n-1)(n-2)(n-3)}{4!} - n$, for every $n \ge 5$.

7) $d_{ev}(C_{4n-1}, n) = 4n - 1$, for every $n \in$ N.

Proof:

1) Since $D_{ev}(C_{4n}, n) = \{\{e_1, e_5, \cdots, e_{n-3}\}, \{e_2, e_6, \cdots, e_{n-2}\}, \{e_3, e_7, \cdots, e_{n-1}\}, \{e_4, e_8, \cdots, e_n\}\}$, we have $d_{ev}(C_{4n}, n) = 4$.

2) Since $D_{ev}(C_n, n) = \{[e_n]\}$, we have the result $d_{ev}(C_n, i) = 1$ for every $n \in$ N.

3) Since $D_{ev}(C_n, n-1) = \{[e_n] - \{x\} / x \in [e_n]\}$, we have $d_{ev}(C_n, n-1) = n$ for $n \ge 2$.

4) By induction on n. The result is true for $n = 3$. L.H.S. $= d_{ev}(C_3, 1) = 3$ (from **Table 1**) R.H.S. $= \frac{3 \times 2}{2} = 3$. Therefore, the result is true for n = 3. Now suppose that the result is true for all numbers less than 'n' and we prove it for n. By Theorem 3.1

Table 1. $d_{ev}(C_n, i)$, the number of *ev*-dominating set of C_n with cardinality i.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n																
1	1															
2	2	1														
3	3	3	1													
4	4	6	4	1												
5	0	10	10	5	1											
6	0	9	20	15	6	1										
7	0	7	28	35	21	7	1									
8	0	4	32	62	56	28	8	1								
9	0	0	30	90	117	84	36	9	1							
10	0	0	20	110	202	200	120	45	10	1						
11	0	0	11	110	297	396	319	165	55	11	1					
12	0	0	4	93	372	672	708	483	220	66	12	1				
13	0	0	0	65	403	988	1352	1183	702	286	78	13	1			
14	0	0	0	35	378	1274	2256	2499	1876	987	364	91	14	1		
15	0	0	0	15	303	1450	3330	4635	4330	2853	1350	455	105	15	1	
16	0	0	0	4	208	1456	4384	7646	8800	7128	4192	1804	560	120	16	1

$$d_{ev}(C_n,n-2)=d_{ev}(C_{n-1},n-3)+d_{ev}(C_{n-2},n-3)+d_{ev}(C_{n-3},n-3)+d_{ev}(C_{n-4},n-3)$$
$$=\frac{(n-1)(n-2)}{2}+(n-1)=\frac{1}{2}\left[(n-1)(n-2)+2(n-1)\right]=\frac{1}{2}\left[(n-1)(n-2+2)\right]=\frac{1}{2}\left[(n-1)n\right]$$

5) By induction on n, the result is true for $n=4$. L.H.S. $=d_{ev}(C_4,1)=4$ (from **Table 1**).

R.H.S. $=d_{ev}(C_4,1)=\frac{4.3.2}{6}=4$. Therefore, the result is true for $n=4$. Now suppose the result is true for all natural numbers less than n. By Theorem 3.1,

$$d_{ev}(C_n,n-3)=d_{ev}(C_{n-1},n-4)+d_{ev}(C_{n-2},n-4)+d_{ev}(C_{n-3},n-4)+d_{ev}(C_{n-4},n-4)$$
$$=\frac{(n-1)(n-2)(n-3)}{6}+\frac{(n-2)(n-3)}{2}+(n-2)$$
$$=\frac{1}{6}\left[(n-1)(n-2)(n-3)+3(n-2)(n-3)+6(n-2)\right]$$
$$=\frac{(n-2)}{6}\left[(n-1)(n-3)+3(n-3)+6\right]$$
$$=\frac{(n-2)}{6}\left[(n-1)(n-3)+3(n-1)\right]$$
$$=\frac{(n-2)(n-1)}{6}\left[n-3+3\right]=\frac{(n-2)(n-1)n}{6}$$

6) By induction on n, the result is true for $n = 5$. L.H.S. $=d_{ev}(C_5,1)=0$ (from **Table 1**)

$$\text{R.H.S.}=\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}-n=\frac{5\times4\times3\times2}{1\times2\times3\times4}-5=0$$

Therefore the result is true for $n=5$.

Now suppose that the result is true for all natural numbers less than n and we prove it for n. By Theorem 3.1,

$$d_{ev}(C_n,n-4)=d_{ev}(C_{n-1},n-5)+d_{ev}(C_{n-2},n-5)+d_{ev}(C_{n-3},n-5)+d_{ev}(C_{n-4},n-5)$$
$$=\frac{(n-1)(n-2)(n-3)(n-4)}{24}-(n-1)+\frac{(n-2)(n-3)(n-4)}{6}+\frac{(n-3)(n-4)}{2}+(n-4)$$
$$=\frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4)+4(n-2)(n-3)(n-4)+12(n-3)(n-4)+24(n-4)+24\right]-n$$
$$=\frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4)+4(n-2)(n-3)(n-4)+12(n-3)(n-4)+24(n-4+1)\right]-n$$
$$=\frac{1}{24}\left[(n-1)(n-2)(n-3)(n-4)+4(n-2)(n-3)(n-4)+12(n-3)(n-4)+24(n-3)\right]-n$$
$$=\frac{(n-3)}{24}\left[(n-1)(n-2)(n-4)+4(n-2)(n-4)+12(n-4+2)\right]-n$$
$$=\frac{(n-3)(n-2)}{24}\left[(n-1)(n-4)+4(n-4)+12\right]-n$$
$$=\frac{(n-3)(n-2)}{24}\left[(n-1)(n-4)+4(n-4+3)\right]-n$$
$$=\frac{(n-1)(n-2)(n-3)}{24}\left[n-4+4\right]-n$$
$$=\frac{n(n-1)(n-2)(n-3)}{24}-n$$

7) From the table it is true.

Theorem 3.3

1) $\sum_{i=n}^{4n} d_{ev}(C_i, n) = 4\sum_{i=n}^{4n-4} d_{ev}(C_i, n-1),\ n \geq 2.$

2) For every $j \geq \left\lceil \frac{n}{4} \right\rceil$, $d_{ev}(C_{n+1}, j+1) - d_{ev}(C_n, j+1) = d_{ev}(C_n, j) - d_{ev}(C_{n-4}, j)$.

3) If $S_n = \sum_{i=\left\lceil \frac{n}{4} \right\rceil}^{n} d_{ev}(C_n, i)$, then for every $n \geq 5$, $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$ with initial values $S_1 = 1$, $S_2 = 3$, $S_3 = 7$, and $S_4 = 15$.

Proof:

1) We prove by induction on n.

First suppose that $n = 2$ then,

$$\sum_{i=2}^{8} d_{ev}(C_i, 2) = 4\sum_{i=2}^{4} d_{ev}(C_i, 1) = 40$$

$$\begin{aligned}\sum_{i=k}^{4k} d_{ev}(C_i, k) &= \sum_{i=k}^{4k} d_{ev}(C_{i-1}, k-1) + \sum_{i=k}^{4k} d_{ev}(C_{i-2}, k-1) \\ &\quad + \sum_{i=k}^{4k} d_{ev}(C_{i-3}, k-1) + \sum_{i=k}^{4k} d_{ev}(C_{i-4}, k-1) \\ &= 4\sum_{i=k-1}^{4(k-1)} d_{ev}(C_{i-1}, k-2) + 4\sum_{i=k-1}^{4(k-1)} d_{ev}(C_{i-2}, k-2) \\ &\quad + 4\sum_{i=k-1}^{4(k-1)} d_{ev}(C_{i-3}, k-2) + 4\sum_{i=k-1}^{4(k-1)} d_{ev}(C_{i-4}, k-2) \\ &= 4\sum_{i=k-1}^{4k-4} d_{ev}(C_i, k-1)\end{aligned}$$

We have the result.

2) By Theorem 3.1, we have

$$\begin{aligned}d_{ev}(C_{n+1}, j+1) - d_{ev}(C_n, j+1) &= d_{ev}(C_n, j) + d_{ev}(C_{n-1}, j) + d_{ev}(C_{n-2}, j) + d_{ev}(C_{n-3}, j) \\ &\quad - d_{ev}(C_{n-1}, j) - d_{ev}(C_{n-2}, j) - d_{ev}(C_{n-3}, j) - d_{ev}(C_{n-4}, j)\end{aligned}$$

$$d_{ev}(C_{n+1}, j+1) - d_{ev}(C_n, j+1) = d_{ev}(C_n, j) - d_{ev}(C_{n-4}, j)$$

Therefore, we have the result.

3) By Theorem 3.1, we have

$$S_n = \sum_{i=\left\lceil \frac{n}{4} \right\rceil}^{n} d_{ev}(C_n, i)$$

$$\begin{aligned}S_n &= \sum_{i=\left\lceil \frac{n}{4} \right\rceil}^{n} \left[d_{ev}(C_{n-1}, i-1) + d_{ev}(C_{n-2}, i-1) + d_{ev}(C_{n-3}, i-1) + d_{ev}(C_{n-4}, i-1) \right] \\ &= \sum_{i=\left\lceil \frac{n}{4} \right\rceil-1}^{n-1} d_{ev}(C_{n-1}, i) + \sum_{i=\left\lceil \frac{n}{4} \right\rceil-1}^{n-1} d_{ev}(C_{n-2}, i) + \sum_{i=\left\lceil \frac{n}{4} \right\rceil-1}^{n-1} d_{ev}(C_{n-3}, i) + \sum_{i=\left\lceil \frac{n}{4} \right\rceil-1}^{n-1} d_{ev}(C_{n-4}, i)\end{aligned}$$

$$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}.$$

4. Concluding Remarks

In [7], the domination polynomial of cycle was studied and obtained the very important property, $d(C_n, i) = d(C_{n-1}, i-1) + d(C_{n-2}, i-1) + d(C_{n-3}, i-1)$. It is interesting that we have derived an analogues relation for the edge-vertex domination of cycles of the form, $d_{ev}(C_n, i) = d_{ev}(C_{n-1}, i-1) + d_{ev}(C_{n-2}, i-1) + d_{ev}(C_{n-3}, i-1) + d_{ev}(C_{n-4}, i-1)$. One can characterise the roots of the polynomial $D_{ev}(C_n, x)$ and identify whether they are real or complex. Another interesting character to be investigated is whether $D_{ev}(C_n, x)$ is log-concave or not.

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