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Independence Numbers in Trees

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Abstract

The independence number $\alpha(G)$ of a graph *G* is the maximum cardinality among all independent sets of *G*. For any tree *T* of order $n \ge 2$, it is easy to see that $\left\lceil \frac{n}{2} \right\rceil \le \alpha(T) \le n-1$. In addition, if there are duplicated leaves in a tree, then these duplicated leaves are all lying in every maximum independent set. In this paper, we will show that if *T* is a tree of order $n \ge 4$ without duplicated leaves, then $\alpha(T) \le \left\lfloor \frac{2n-1}{3} \right\rfloor$. Moreover, we constructively characterize the extremal trees *T* of order $n \ge 4$, which are without duplicated leaves, achieving these upper bounds.

Keywords

Independent Set, Independence Number, Tree

1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph *G*, we refer to $V(G)$ and $E(G)$ as the vertex set and the edge set, respectively. The cardinality of $V(G)$ is called the *order* of *G*, denoted by $|G|$. The (*open*) *neighborhood* $N_G(x)$ of a vertex *x* is the set of vertices adjacent to *x* in *G*, and the *close neighborhood* $N_G[x]$ is $N_G(x) \cup \{x\}$. A vertex *x* is said to be a *leaf* if $|N_G(x)| = 1$. A vertex *v* of *G* is a *support vertex* if it is adjacent to a leaf in *G*. Two distinct vertices *u* and *v* are called *duplicated* if $N_G(u) = N_G(v)$. Note that *u* and *v* are duplicated vertices in a tree, and then they are both leaves. The *n-path* P_n is the path of order $n \ge 1$. For a subset $A \subseteq V(G)$, the *induced subgraph* induced by *A* is the graph $\langle A \rangle_G$ with vertex set *A* and the edge set $E(\langle A \rangle_G) = \{uv \in E(G) : u \in A \text{ and } v \in A\}$, the *deletion of A from G* is the graph $G - A$ by removing all vertices in *A* and all edges incident to these vertices and the *complement* of *A* is the set $A^c = V(G) \setminus A$. For notation and terminology in graphs we follow [1] in general.

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A set $I \subseteq V(G)$ is an *independent set* of *G* if no two vertices of *I* are adjacent in *G*. The *independence number* $\alpha(G)$ of *G* is the maximum cardinality among all independent sets of *G*. If *I* is an independent set of *G* with cardinality $\alpha(G)$, we call *I* an α-set of *G*. If *I* is an α-set of *G* containing all leaves of *G*, we call *I* an α_L-set of *G*.

The independence problem is to find an α-set in *G*. The problem is known to be NP-hard in many special classes of graphs. Over the past few years, several studies have been made on independence (see [2]-[6]). For any tree *T* of order $n \geq 2$, it is easy to see that $\left\lceil \frac{n}{2} \right\rceil \leq \alpha(T) \leq n-1$. In addition, if there are duplicated leaves in a tree, then these duplicated leaves are all lying in every maximum independent set. In this paper, we will show that if *T* is a tree of order $n \geq 4$ without duplicated leaves, then $\alpha(T) \leq \left\lfloor \frac{2n-1}{3} \right\rfloor$. Moreover, we constructively characterize the extremal trees *T* of order $n \geq 4$, which are without duplicated leaves, achieving these upper bounds.

2. The Upper Bound

In this section, we will show a sharp upper bound on the independence number of a tree *T* without duplicated leaves.

Lemma 1 *If H is an induced subgraph of G*, then $\alpha(H) \leq \alpha(G)$.

Proof. If *S* is an α-set of *H*, then *S* is an independent set of *G*. It follows that $\alpha(H) = |S| \leq \alpha(G)$. □

Lemma 2 ([4]) *If T is a tree of order* $n \geq 1$, *then* $\alpha(T) \geq \left\lceil \frac{n}{2} \right\rceil$.

Lemma 3 ([5]) *If T is a tree of order* $n \geq 3$, *then there exists an* α_L*-set of T.*

Lemma 4 *For an integer* $n \geq 4$, $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil \leq \left\lfloor \frac{2n-1}{3} \right\rfloor$.

Proof. It is straightforward to check that $\left\lceil \frac{n}{2} \right\rceil \leq \left\lfloor \frac{2n-1}{3} \right\rfloor$ for $n \geq 4$. Let $P_n : v_1 - v_2 - \cdots - v_n$. Since P_n is a tree of order $n \geq 4$, by Lemma 2, we have that $\alpha(P_n) \geq \left\lceil \frac{n}{2} \right\rceil$. Suppose that there exists an independent set *I* of P_n with $|I| \geq \left\lceil \frac{n}{2} \right\rceil + 1$, then there exists *i*, $1 \leq i \leq n-1$, such that $v_i \in I$ and $v_{i+1} \in I$. This is a contradiction, therefore we obtain that $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$. □

Theorem 1 If *T is a tree of order* $n \geq 4$ *without duplicated leaves, then* $\alpha(T) \leq \left\lfloor \frac{2n-1}{3} \right\rfloor$.

Proof. We prove it by induction on $n \geq 4$. By Lemma 4 and *T* is a tree without duplicated leaves, it's true for all $n \leq 6$. For all $n \geq 7$ we assume that the assertion is true for all $n' < n$. Suppose that *T* is a tree of order $n \geq 7$ without duplicated leaves and *x* is a leaf lying on a longest path of *T*. Let $y \in N_T(x)$. Since *T* has no duplicated leaves, this implies that $d_T(y) = 2$, say $N_T(y) = \{x, z\}$. Let $T' = T - N_T[x]$, then *T'* is a tree of order $n-2$. For the case in which *T'* has no duplicated leaves, by induction hypothesis, we have that $\alpha(T') \leq \left\lfloor \frac{2(n-2)-1}{3} \right\rfloor = \left\lfloor \frac{2n-5}{3} \right\rfloor$. Since an α-set of *T'*, together with $\{x\}$, form an α-set of *T*. Therefore we obtain that $\alpha(T) = \alpha(T') + 1 \leq \left\lfloor \frac{2n-5}{3} \right\rfloor + 1 = \left\lfloor \frac{2n-2}{3} \right\rfloor \leq \left\lfloor \frac{2n-1}{3} \right\rfloor$. For the other case in which *T'* has duplicated leaves *z* and z', then $T'' = T' - \{z\}$ is a tree of order $n-3 \geq 4$ without duplicated leaves. By induction hypothesis, we have that $\alpha(T'') \leq \left\lfloor \frac{2(n-3)-1}{3} \right\rfloor = \left\lfloor \frac{2n-7}{3} \right\rfloor$. Since an α_L-set of T'', together with $\{x, z\}$, form an α-set of *T*. Therefore, we obtain that $\alpha(T)=\alpha(T'')+2\le\left\lfloor\frac{2n-7}{3}\right\rfloor+2=\left\lfloor\frac{2n-1}{3}\right\rfloor$. Hence we conclude that $\alpha(T)\le\left\lfloor\frac{2n-1}{3}\right\rfloor$. □

Note that the result in Theorem 1 is sharp and some such *T* are illustrated below.

3. Extremal Trees

Let $\mathcal{T}(n)$ be the class of all trees *T* of order $n\ge 4$ without duplicated leaves such that $\alpha(T)=\left\lfloor\frac{2n-1}{3}\right\rfloor$. We will constructively characterize these extremal trees. Let $L(T)$ and $U(T)$, respectively, denote the collections of all leaves and all support vertices of *T*. First, we define four operations on a tree *T* of order $n\ge 4$ as follows, where *I* is an α_L-set of *T*.

Operation O1. Join a vertex $u\in I$ of T to a vertex v_1 of P_1 such that $I_{O1}=(I-\{u\})\cup\{v_1\}$, where $|T|=n\equiv 2 \pmod 3$.

Operation O2. Join a vertex $u\in I^c\setminus U(T)$ of T to a vertex v_1 of P_1 such that $I_{O2}=I\cup\{v_1\}$, where $|T|=n\equiv 0,1 \pmod 3$.

Operation O3. Join a vertex *u* of T to a leaf v_2 of P_2 (say $P_2: v_1-v_2$) such that $I_{O3}=I\cup\{v_1\}$, where $|T|=n\equiv 1,2 \pmod 3$.

Operation O4. Join a vertex $u\in I^c$ of *T* to a leaf v_3 of P_3 (say $P_3: v_1-v_2-v_3$) such that $I_{O4}=I\cup\{v_1,v_3\}$.

Lemma 5 *Suppose that* $T\in\mathcal{T}(n)$ *for* $n\ge 4$. *If I is an* α_L*-set of T, then*

$$|I^c\setminus U(T)|\le\begin{cases}0, & \text{if } n\equiv 2 \pmod 3,\\ 1, & \text{if } n\equiv 1 \pmod 3,\\ 2, & \text{if } n\equiv 0 \pmod 3.\end{cases}$$

Proof. It's true for all $n\le 6$. So we assume that $n\ge 7$. Since *I* is an α_L-set of *T*, this implies that $U(T)\subseteq I^c$. By Theorem 1, we have that

$$|I|=\begin{cases}\left\lfloor\frac{2(3k+2)-1}{3}\right\rfloor=2k+1, & \text{if } n=3k+2,\\ \left\lfloor\frac{2(3k+1)-1}{3}\right\rfloor=2k, & \text{if } n=3k+1,\\ \left\lfloor\frac{2(3k)-1}{3}\right\rfloor=2k-1, & \text{if } n=3k.\end{cases}$$

Hence we obtain that $|I^c|=n-|I|=k+1$. Let $B=I-L(T)=\{z_1,z_2,\cdots,z_b\}$. Note that $N_T(z_i)\subseteq I^c$ and $|N_T(z_i)|\ge 2$ for every *i*. In addition, $|N_T(z_i)\cap N_T(z_j)|\le 1$, these imply that $|I^c|\ge\left|\bigcup_{i=1}^{b}N_T(z_i)\right|\ge b+1$. Thus we obtain that $|B|=b\le|I^c|-1=k$. It follows that

$$\begin{aligned}|I^c\setminus U(T)|&=|I^c|-|U(T)|=|I^c|-|L(T)|\\&=|I^c|-(|I|-b)\le n-2|I|+k\\&=\begin{cases}(3k+2)-2(2k+1)+k=0, & \text{if } n=3k+2,\\ (3k+1)-2(2k)+k=1, & \text{if } n=3k+1,\\ (3k)-2(2k-1)+k=2, & \text{if } n=3k.\end{cases}\end{aligned}$$

This completes the proof. □

Lemma 6 *Let* $T\in\mathcal{T}(n)$ *be a tree of order* $n\equiv 2 \pmod 3$ *with an* α_L*-set I. Suppose that T′ is obtained from T by* Operation O1, *then* $T'\in\mathcal{T}(n+1)$ *is a tree of order* $n+1$ *and* I_{O1} *is an* α_L*-set of T′.*

Proof. Suppose that $T\in\mathcal{T}(n)$ is a tree of order $n\equiv 2$ (mod 3) with an α_L-set *I*, by Lemma 5, then

$I^c = U(T)$. Let T' be the tree obtained from T by Operation O1. Since $u \in I$, this implies that u is not a support vertex of T and T' is a tree of order $n+1$ without duplicated leaves. On the other hand, I_{O1} is an independent set of T' with $L(T') \subseteq I_{O1}$ such that $\left\lfloor \frac{2(n+1)-1}{3} \right\rfloor \ge \alpha(T') \ge |I_{O1}| = |I| = \left\lfloor \frac{2n-1}{3} \right\rfloor = \left\lfloor \frac{2(n+1)-1}{3} \right\rfloor$, where $n \equiv 2 \pmod 3$. Hence $\alpha(T') = \left\lfloor \frac{2(n+1)-1}{3} \right\rfloor$. In conclusion, $T' \in \mathscr{T}(n+1)$ is a tree of order $n+1$ with an α_L-set I_{O1}. □

Lemma 7 *Let $T \in \mathscr{T}(n)$ be a tree of order $n \equiv 0,1 (\mathrm{mod}\, 3)$ with an α_L-set I such that $|I^c - U(T)| \ge 1$. If T' is obtained from T by* Operation O2, *then $T' \in \mathscr{T}(n+1)$ is a tree of order $n+1$ and I_{O2} is an α_L-set of T'.*

Proof. Note that such a tree T exists, as, for instance, the tree in **Figure 1** is as desired. If $T \in \mathscr{T}(n)$ is a tree of order $n \equiv 0,1 \pmod 3$ with an α_L-set I such that $|I^c - U(T)| \ge 1$. Let T' be the tree obtained from T by Operation O2. Since u is not a support vertex of T, this implies that T' is a tree of order $n+1$ without duplicated leaves. And I_{O2} is an independent set of T' with $L(T') \subseteq I_{O2}$ such that $\left\lfloor \frac{2(n+1)-1}{3} \right\rfloor \ge \alpha(T') \ge |I_{O2}| = |I| + 1 = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1 = \left\lfloor \frac{2(n+1)-1}{3} \right\rfloor$, where $n \equiv 0,1 \pmod 3$. Hence $\alpha(T') = \left\lfloor \frac{2(n+1)-1}{3} \right\rfloor$. In conclusion, $T' \in \mathscr{T}(n+1)$ is a tree of order $n+1$ with an α_L-set I_{O2}. □

Lemma 8 *Let $T \in \mathscr{T}(n)$ be a tree of order $n \equiv 1,2 (\mathrm{mod}\, 3)$ with an α_L-set I. If T' is obtained from T by* Operation O3, *then $T' \in \mathscr{T}(n+2)$ is a tree of order $n+2$ and I_{O3} is an α_L-set of T'.*

Proof. Note that T' is a tree of order $n + 2$ without duplicated leaves. And I_{O3} is an independent set of T' with $L(T') \subseteq I_{O3}$ such that $\left\lfloor \frac{2(n+2)-1}{3} \right\rfloor \ge \alpha(T') \ge |I_{O3}| = |I| + 1 = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1 = \left\lfloor \frac{2(n+2)-1}{3} \right\rfloor$, where $n \equiv 1,2 \pmod 3$. Hence $\alpha(T') = \left\lfloor \frac{2(n+2)-1}{3} \right\rfloor$. In conclusion, $T' \in \mathscr{T}(n+2)$ is a tree of order $n+2$ with an α_L-set I_{O3}. □

Lemma 9 *Let $T \in \mathscr{T}(n)$ be a tree of order $n \ge 4$ with an α_L-set I. If T' is obtained from T by Operation* O4, *then $T' \in \mathscr{T}(n+3)$ is a tree of order $n+3$ and I_{O4} is an α_L-set of T'.*

Proof. Note that T' is a tree of order $n+3$ without duplicated leaves. And I_{O4} is an independent set of T' with $L(T') \subseteq I_{O4}$ such that $\left\lfloor \frac{2(n+3)-1}{3} \right\rfloor \ge \alpha(T') \ge |I_{O4}| = |I| + 2 = \left\lfloor \frac{2n-1}{3} \right\rfloor + 2 = \left\lfloor \frac{2(n+3)-1}{3} \right\rfloor$. Hence $\alpha(T') = \left\lfloor \frac{2(n+3)-1}{3} \right\rfloor$. In conclusion, $T' \in \mathscr{T}(n+3)$ is a tree of order $n+3$ with an α_L-set I_{O4}. □

Let $\mathscr{C}$ be the class of all trees obtained from P_4 or P_5 by a finite sequence of Operations O1-O4. Suppose that $\mathscr{T} = \bigcup_{n \ge 4} \mathscr{T}(n)$, we will show that $\mathscr{T} = \mathscr{C}$.

Theorem 2 *T is in $\mathscr{C}$ if and only if T is in $\mathscr{T}$.*

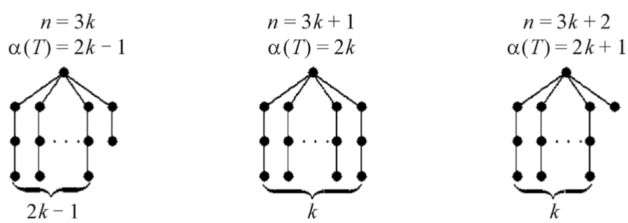

Figure 1. The trees T with $\alpha(T) = \left\lfloor \frac{2n-1}{3} \right\rfloor$.

Proof. If *T* is in $\mathscr{C}$, by Lemmas 6, 7, 8 and 9, then *T* is in $\mathscr{T}$. Now, we want to show the converse by contradiction. Suppose to the contrary that there exists a tree $T \in \mathscr{T}$ and $T \notin \mathscr{C}$ such that $|T|$ is as small as possible. We can see that $|T| \geq 7$. Let $P: x-y-z-\cdots$ be a longest path of *T*. Then $N_T(y)=\{x,z\}$ and $T' = T - N_T[x]$ is a tree of order $n' = n-2$. We consider two cases.

Case 1. *T′* has no duplicated leaves.

For an α_L-set *I* of *T*, $I' = I - \{x\}$ is an independent set of *T′*, this implies that $\alpha(T') \geq |I'| = \alpha(T)-1$. By Theorem 1, we have that $\left\lfloor \frac{2n-1}{3} \right\rfloor - 1 = \left\lfloor \frac{2n-4}{3} \right\rfloor = \alpha(T) - 1 \leq \alpha(T') \leq \left\lfloor \frac{2(n-2)-1}{3} \right\rfloor = \left\lfloor \frac{2n-5}{3} \right\rfloor \leq \left\lfloor \frac{2n-4}{3} \right\rfloor$. Then $\alpha(T') = \left\lfloor \frac{2(n-2)-1}{3} \right\rfloor = \left\lfloor \frac{2n-1}{3} \right\rfloor - 1$ and $n \equiv 0,1$ (mod 3). This follows that $T' \in \mathscr{T}(n')$, where $n' = n-2 \equiv 1,2$ (mod 3), by hypothesis, $T' \in \mathscr{C}$. Note that *T* can be obtained from T' by Operation O3, this implies that $T \in \mathscr{C}$, which is a contradiction.

Case 2. *T′* has duplicated leaves *z* and *z′*.

Let $T'' = T' - \{z\}$. Then T'' is a tree of order $n-3$. Since *z′* is a leaf of *T*, this implies that *z* and *z′* are in every α_L-set of *T*. For an α_L-set *I* of *T*, $I'' = I - \{x,z\}$ is an independent set of T'', thus $\alpha(T'') \geq |I''| = \alpha(T)-2$. By Theorem 1, we have that $\left\lfloor \frac{2n-1}{3} \right\rfloor - 2 = \left\lfloor \frac{2n-7}{3} \right\rfloor = \alpha(T) - 2 \leq \alpha(T'') \leq \left\lfloor \frac{2(n-3)-1}{3} \right\rfloor = \left\lfloor \frac{2n-7}{3} \right\rfloor$. Then $\alpha(T'') = \left\lfloor \frac{2(n-3)-1}{3} \right\rfloor$. This follows that $T'' \in \mathscr{T}(n'')$, where $n'' = n-3$, by hypothesis, $T'' \in \mathscr{C}$. Note that *T* can be obtained from T'' by Operation O4, this implies that $T \in \mathscr{C}$, which is a contradiction.

By Cases 1 and 2, we conclude that *T* is in $\mathscr{T}$, then *T* is in $\mathscr{C}$. □

Now, we obtain the main theorem in this paper.

Theorem 3 *Suppose that T is a tree of order* $n \geq 4$ *without duplicated leaves, then* $\alpha(T) \leq \left\lfloor \frac{2n-1}{3} \right\rfloor$. *Furthermore, the equality holds if and only if* $T \in \mathscr{C}$.

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