Open Journal of Discrete Mathematics, 2014, 4, 36-43
Published Online April 2014 in SciRes. http://www.scirp.org/journal/ojdm
http://dx.doi.org/10.4236/ojdm.2014.42006

Symmetric Identities from an Invariant in Partition Conjugation and Their Applications in q-Series

Sandy H. L. Chen[1,2]

[1]School of Science, Tianjin Chengjian University, Tianjin, China
[2]Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin, China
Email: chenhuanlin@mail.nankai.edu.cn

Received 21 January 2014; revised 19 February 2014; 17 March 2014

Abstract

For every partition λ and its conjugation λ', there is an important invariant $l_d(\lambda)$, which denotes the number of different parts. That is, $l_d(\lambda)=l_d(\lambda')$. We will derive a series of symmetric q-identities from the invariant in partition conjugation by studying modified Durfee rectangles. The extensive applications of the several symmetric q-identities in q-series [1] will also be discussed. Without too much effort one can obtain much well-known knowledge as well as new formulas by proper substitutions and elementary calculations, such as symmetric identities, mock theta functions, a two-variable reciprocity theorem, identities from Ramanujan's Lost Notebook and so on.

Keywords

Integer Partitions, Conjugation, Invariant, q-Series, Symmetric Identities

1. Definitions and Combinatorial Interpretations

We shall first present some basic definitions and combinatorial interpretations for basic hypergeometric series and integer partition. For simplicity, unless stated otherwise we shall assume that n is a nonnegative integer and $|q|<1$, let

$$(a;q)_\infty=(1-a)(1-aq)\cdots.$$

Definition 1.1 *For any integer n, the q-shifted factorial is defined by*

How to cite this paper: Chen, S.H.L. (2014) Symmetric identities from an Invariant in Partition Conjugation and Their Applications in *q*-Series. *Open Journal of Discrete Mathematics*, **4**, 36-43. http://dx.doi.org/10.4236/ojdm.2014.42006

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Definition 1.2 *A partition* λ *of a positive integer* n *is a finite nonincreasing sequence of integers* $\lambda_1,\lambda_2,\cdots,\lambda_r$, *such that* $\sum_{i=1}^{r}\lambda_i = n$. λ' *is the conjugate partition of* λ. *The largest part, the number of parts, as well as the sum of the parts are denoted by* $a(\lambda)$, $l(\lambda)$ *and* $|\lambda|$, *respectively.*

An effective device for studying partitions is the graphical representation. For a partition λ, its $m\times n$ Durfee rectangle is the maximum rectangle contained in the Ferrers diagram of λ. Conjugation and the several invariants have been used in a variety of ways over the years, see Andrews's encyclopedia [2]. It is worth pointing out that there is a fundamental invariant which despite its simplicity has not received too much attention. This is $l_d(\lambda)$, the number of different parts of λ. For all partitions λ and its conjugation λ',

$$l_d(\lambda) = l_d(\lambda'). \tag{1}$$

In this paper, we shall show how (1) could be used to obtain a series of symmetric identities by studying modified Durfee rectangles. Consider the expansion

$$\frac{1-abq^i}{1-bq^i} = \frac{1-bq^i+bq^i-abq^i}{1-bq^i} = 1+(1-a)\left(bq^i+b^2q^{2i}+b^3q^{3i}+\cdots\right).$$

We interpret this as an expansion involving only one part, namely i, where the power of b records $l(\lambda)$, while that of $(1-a)$ indicates whether the part occurs or not. Thus, we interpret

$$\frac{(abq)_n}{(bq)_n} = \sum_{a(\lambda)\le n}(1-a)^{l_d(\lambda)}b^{l(\lambda)}q^{|\lambda|} \tag{2}$$

as the generating function of partitions λ into parts less than or equal to n, such that the power of b records $l(\lambda)$, while that of $(1-a)$ indicates $l_d(\lambda)$. Then it follows that the three-parameter generating function for all unrestricted partitions λ, namely, the function

$$G(a,b,c;q) = \sum_{\lambda}(1-a)^{l_d(\lambda)}b^{l(\lambda)}c^{a(\lambda)}q^{|\lambda|} = 1+\sum_{n=1}^{\infty}\frac{(1-a)(abq)_{n-1}bc^nq^n}{(bq)_n}. \tag{3}$$

We consider all partitions λ for which $a(\lambda)=n$. This accounts for the term c^nq^n in (3). Since λ contains n as a part, we have the factors $(1-a)$ and b in the numerator. The part n may repeat, which is given by $\frac{1}{1-bq^n}$. The repetition of n will not contribute to $l_d(\lambda)$ and so there is no further power of $(1-a)$ contributed by the part n. The part $1,2,\cdots,n-1$ could repeat and their contribution to the generating function is given by the term

$$\frac{(abq)_{n-1}}{(bq)_{n-1}}.$$

Formula (3) follows.

2. Symmetric Expressions for $G(a,b,c;q)$

In this section, we give several symmetric expansions for $G(a,b,c;q)$ via modified Durfee rectangles analysis of partition.

Theorem 2.1

$$G(a,b,c;q) = 1+\frac{bcq(1-a)}{1-cq}+\sum_{n=1}^{\infty}\frac{(1-a)b^{n+1}c^nq^{n^2+n}(abq)_{n-1}(acq)_n\left(1-abcq^{2n+1}\right)}{(bq)_n(cq)_{n+1}}.$$

Proof. For every partitions λ and $n\ge 1$, the Ferrers graph contains a largest $n\times(n+1)$ Durfee rectangle with side n horizontally by $n+1$ vertically. Then to the right of the Durfee rectangle, we have a partition λ_a

which has at most $n+1$ parts, or equivalently, λ_a', the conjugate of λ_a, with largest part $\le n+1$. Below the Durfee rectangle we have a partition λ_b whose parts $\le n$. We now divide our consideration into four cases.

1) $l(\lambda_a)=n+1, a(\lambda_b)\le n-1$

We consider the contribution of the partition λ_a to $G(a,b,c;q)$ by utilizing its conjugate λ_a'.

$$\sum_{\lambda_a}(1-a)^{l_d(\lambda_a)}c^{a(\lambda_a)}q^{|\lambda_a|}=\sum_{\lambda_a'}(1-a)^{l_d(\lambda_a')}c^{l(\lambda_a')}q^{|\lambda_a'|}=\frac{(1-a)cq^{n+1}}{1-cq^{n+1}}\frac{(acq)_n}{(cq)_n}=\frac{(1-a)cq^{n+1}(acq)_n}{(cq)_{n+1}}.$$

The factor cq^{n+1} in the numerator arises from the column of length $n+1$ lying to the right of the Durfee rectangle. Because λ_a contributes nothing to $l(\lambda)$, we omit b.

The contribution of the partition λ_b to $G(a,b,c;q)$ is

$$\sum_{\lambda_b}(1-a)^{l_d(\lambda_b)}b^{a(\lambda_b)}q^{|\lambda_b|}=\frac{(abq)_{n-1}}{(bq)_{n-1}}.$$

Note that the parameter c is absent because the partition λ_b has no contribution to $a(\lambda)$.

Meanwhile, the contribution of the modified Durfee rectangle to $G(a,b,c;q)$ is

$$b^{n+1}c^nq^{n(n+1)}.$$

Thus, we derive the generating function of every unrestricted partitions λ:

$$\frac{(1-a)b^{n+1}c^{n+1}q^{n^2+2n+1}(abq)_{n-1}(acq)_n}{(bq)_{n-1}(cq)_{n+1}}.$$

2) $l(\lambda_a)\le n, a(\lambda_b)=n.$

The generating function of every unrestricted partitions λ:

$$\frac{(1-a)b^{n+2}c^nq^{n^2+2n}(abq)_{n-1}(acq)_n}{(bq)_n(cq)_n}.$$

3) $l(\lambda_a)\le n, a(\lambda_b)\le n-1.$

The generating function of every unrestricted partitions λ:

$$\frac{(1-a)b^{n+1}c^nq^{n^2+n}(abq)_{n-1}(acq)_n}{(bq)_{n-1}(cq)_n}.$$

4) $l(\lambda_a)=n+1, a(\lambda_b)=n.$

The generating function of every unrestricted partitions λ:

$$\frac{(1-a)^2b^{n+2}c^{n+1}q^{n^2+3n+1}(abq)_{n-1}(acq)_n}{(bq)_n(cq)_{n+1}}.$$

Summing these four generating functions for $n\ge1$, we get an expression for $G(a,b,c;q)$:

$$G(a,b,c;q)=1+\frac{bcq(1-a)}{1-cq}+\sum_{n=1}^{\infty}\frac{(1-a)b^{n+1}c^nq^{n^2+n}(abq)_{n-1}(acq)_n(1-abcq^{2n+1})}{(bq)_n(cq)_{n+1}}.$$

Remark 2.2 *Under partition conjugation,* $l(\lambda)$ *and* $a(\lambda)$ *are interchanged, it follows that* b *and* c *are symmetric in* $G(a,b,c;q)$.

Theorem 2.3 *From formula* (3) *and the symmetry of* b *and* c, *we have*

$$G(a,b,c;q)=1+\sum_{n=1}^{\infty}\frac{(1-a)(abq)_{n-1}bc^nq^n}{(bq)_n} \tag{4}$$

$$=1+\sum_{n=1}^{\infty}\frac{(1-a)(acq)_{n-1}cb^nq^n}{(cq)_n} \tag{5}$$

Theorem 2.4 *From Theorem* 2.1 *and the symmetry of* b *and* c, *we have*

$$G(a,b,c;q)=1+\frac{bcq(1-a)}{1-cq}+\sum_{n=1}^{\infty}\frac{(1-a)b^{n+1}c^{n}q^{n^2+n}(abq)_{n-1}(acq)_{n}(1-abcq^{2n+1})}{(bq)_{n}(cq)_{n+1}} \quad (6)$$

$$=1+\frac{bcq(1-a)}{1-bq}+\sum_{n=1}^{\infty}\frac{(1-a)b^{n}c^{n+1}q^{n^2+n}(abq)_{n}(acq)_{n-1}(1-abcq^{2n+1})}{(bq)_{n+1}(cq)_{n}} \quad (7)$$

$$=1+\sum_{n=1}^{\infty}\frac{(1-a)b^{n}c^{n}q^{n^2}(abq)_{n-1}(acq)_{n-1}(1-abcq^{2n})}{(bq)_{n}(cq)_{n}}, \quad (8)$$

where (8) results from the $n\times n$ Durfee square analysis.

3. The Applications of the Symmetric Identities in *q*-Series

In this section, we shall explore the extensive applications of formulas (4) to (8) in q-series. Without too much effort one can obtain much well-know knowledge as well as new formulas by proper substitutions and elementary calculations. It will be overly clear that the list of nice application is sheer endless.

3.1. Symmetric Identities

From (4) and (5), we get the following beautiful symmetric identity.

Corollary 3.1

$$\sum_{n=0}^{\infty}\frac{(abq)_n}{(bq)_{n+1}}(cq)^n=\sum_{n=0}^{\infty}\frac{(acq)_n}{(cq)_{n+1}}(bq)^n. \quad (9)$$

Taking $a=q/abcd$, $b=ac/q$, $c=bc/q$ in (9), we derive the following identity, from which Liu [3] proved an identity of Andrews.

Corollary 3.2

$$(1-bc)\sum_{n=0}^{\infty}\frac{(q/bd)_n}{(acq)_n}(bc)^n=(1-ac)\sum_{n=0}^{\infty}\frac{(q/ad)_n}{(bcq)_n}(ac)^n. \quad (10)$$

Setting $q\to q^2$ and then taking $a=aq/b$, $c=t$ in (4) and (5), we have

Corollary 3.3

$$\sum_{n=0}^{\infty}\frac{(aq^3;q^2)_n}{(bq^2;q^2)_{n+1}}bt^{n+1}q^{2n+2}=\sum_{n=0}^{\infty}\frac{\left(\frac{atq^3}{b};q^2\right)_n}{(tq^2;q^2)_{n+1}}tb^{n+1}q^{2n+2}, \quad (11)$$

which was first stated and proved by N. J. Fine [4]. Andrews derived it combinatorially from the consideration of partitions without repeated odd parts in [5].

3.2. Mock Theta Functions

In his famous last letter to Hardy [6], Ramanujan introduced 17 mock theta functions without giving an explicit definition, among which, one third order mock theta function is as follows

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q)_n^2}. \quad (12)$$

In 1966, Andrews [7] defined the following generalization of $f(q)$

$$f(\alpha;q)=\sum_{n=0}^{\infty}\frac{\alpha^n q^{n^2-n}}{(-q)_n(-\alpha)_n}. \quad (13)$$

Moreover, Watson [8] added three functions to the list of Ramanujan's third order mock theta functions and the following identity is just one of them

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}.$$

By proper substitutions in Theorem 2.3 and 2.4, we obtain much simpler expressions for the above mock theta functions. Through the specializations $a=0,b=-1$ and $c=-\alpha q^{-1}$ in (4) and (8), we derive a simpler transformation formula for $f(\alpha;q)$:

Corollary 3.4

$$\sum_{n=0}^{\infty}\frac{\alpha^n q^{n^2-n}}{(-q)_n(-\alpha)_n}=2-\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{(-q)_n}. \tag{14}$$

Taking $\alpha=q$ in (14), a representation for $f(q)$ follows, with the powers diminished

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q)_n^2}=2-\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(-q)_n}. \tag{15}$$

Fine [1] first derived (15) by applying some transformation formulas and Liu [[9], Theorem 3.7] proved it combinatorially by an application of involution. Changing q to q^2 and then putting $a=0,b=q,c=1/q$ in (4) and (6), we get a new expression for $\omega(q)$, with the powers diminished:

Corollary 3.5

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{q^n}{(q;q^2)_{n+1}}. \tag{16}$$

3.3. A Two-Variable Reciprocity Theorem

Taking $a=-1/bc, b=-a$ and then letting $c\to 0$ in (4), we have

$$\rho(a,b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}.$$

In his lost notebook [10], Ramanujan offers a beautiful reciprocity theorem

$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty}. \tag{17}$$

After the same substitutions in (5) and (7), respectively, we get

Corollary 3.6

$$\rho(a,b)=1+\frac{1}{b}\sum_{n=0}^{\infty}\left(-\frac{1}{b}\right)_n(-aq)^n \tag{18}$$

$$=1+\sum_{n=0}^{\infty}\frac{(-1/b)_n a^{2n}b^{-n-1}q^{3n(n+1)/2}\left(1-aq^{2n+1}/b\right)}{(-aq)_{n+1}}. \tag{19}$$

Formula (18) is a slightly simpler representation of $\rho(a,b)$. From (19) and the above reciprocity theorem (17), we get the following two variable generalization of the Quintuple Product Identity [[11], Theorem 3.1] without any proof:

Corollary 3.7 A Two-Variable Generalization of the Quintuple Product Identity

For $a,b\neq q^{-n}$, $1\le n<\infty$,

$$\left(\frac{1}{a}-\frac{1}{b}\right)\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(aq)_\infty(bq)_\infty}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n(1/a)_n a^{-n-1}b^{2n}q^{3n(n+1)/2}\left(1-bq^{2n+1}/a\right)}{(bq)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n(1/b)_n a^{2n}b^{-n-1}q^{3n(n+1)/2}\left(1-aq^{2n+1}/b\right)}{(aq)_{n+1}}. \quad (20)$$

3.4. Identities from Ramanujan's Lost Notebook

By special substitutions, we could go through a series of important Entries in Ramanujan's Lost Notebook [12]. We take several of them as examples, for their combinatorial proofs, see [13]. The function

$$\phi(a)=\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(bq)_n},$$

is defined by Ramanujan. Setting $a=-a/bc, b=b,$ and then letting $c\to 0$, then (4) and (6) can be reduced to

Corollary 3.8 (Entry 9.2.2)

$$\phi(a)=\sum_{n=0}^{\infty}\frac{(-aq/b)_n a^n b^n q^{n(3n+1)/2}\left(1+aq^{2n+1}\right)}{(bq)_n}.$$

The same substitutions in (4) and (8), we have

Corollary 3.9 (Entry 9.2.3)

$$\phi(a)=\sum_{n=0}^{\infty}\frac{(-aq/b)_{n-1} a^n b^{n-1} q^{n(3n-1)/2}\left(1+aq^{2n}\right)}{(bq)_n}.$$

Putting $a=b, b=-a$ and $c=a$, and then setting $b\to 0$ in (4) and (8), we have

Corollary 3.10 (Entry 9.2.4)

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} q^{n^2}}{\left(a^2q^2;q^2\right)_n}=1-a\sum_{n=1}^{\infty}\frac{a^n q^n}{(-aq)_n}.$$

For the above identity, it is interesting to note that the terms in a and q on the right side are the same as those on the left side, but with the powers diminished. In (4) and (6), we replace q by q^2 and take $a=b, b=aq$ and $c=-a/q$, and then set $b\to 0$, the Entry 9.2.5 in Ramanujan's Lost Notebook [12] follows:

Corollary 3.11 (Entry 9.2.5)

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^n}{\left(aq;q^2\right)_{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} q^{2n^2+2n}}{\left(a^2q^2;q^4\right)_{n+1}}.$$

Berndt and Yee [13] proved the above two corollaries combinatorially by accounting for partitions into distinct parts. Replacing q by q^2 and taking $a=q, b=a$ and $c=a/q$ in (4) and (6), then we get the following Entry. Berndt and Yee [13] derived it by employing 2-modular partitions.

Corollary 3.12 (Entry 9.3.1)

$$\sum_{n=0}^{\infty}\frac{\left(aq;q^2\right)_n (aq)^n}{\left(aq^2;q^2\right)_n}=\sum_{n=0}^{\infty}a^{2n}q^{2n^2+n}\left(1+aq^{2n+1}\right)=\sum_{n=0}^{\infty}a^n q^{n(n+1)/2}.$$

In (4) and (6), we take $a=-a/c, b=-a$, and then set $c\to 0$, Entry 9.4.1 follows:

Corollary 3.13 (Entry 9.4.1)

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(-aq;q)_n}=\sum_{n=0}^{\infty}a^{3n}q^{n(3n+1)/2}\left(1-a^2q^{2n+1}\right). \quad (21)$$

This identity was derived from Franklin involution by Berndt and Yee [13] and was also got from two entries by Warnaar [14], where analytic methods were employed.

3.5. Further Consequences

Corollary 3.14

$$\sum_{n=0}^{\infty}\frac{q^n}{\left(-q;q^2\right)_{n+1}}=\sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}\left(1+q^{4n+2}\right). \tag{22}$$

Proof. Taking $q\to q^2$, and then letting $a=0, b=-1/q$ and $c=1/q$ in (4) and (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^n}{\left(-q;q^2\right)_{n+1}}&=-\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2-2n}}{\left(q^2;q^4\right)_n}\\ &=1-\sum_{n=0}^{\infty}(-1)^n q^{6n^2-4n}\left(1-q^{8n}\right)\quad(\text{by})\quad=\sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}\left(1+q^{4n+2}\right).\end{aligned} \tag{23}$$

Identity (22) is a false theta series identity. Results like these were studied by L. J. Rogers [15], however, the elegant result appears to have escaped him. Andrews [16] proved identity (22) by using three transformation formulas and showed that (22) implied a partition identity like that deduced from Euler's Pentagonal Number Theorem ([2], p. 10).

Taking $a=0, b=-1$ and $c=1$ in (4) and (5), we generalize the not at all deep but elegant identity:

Corollary 3.15

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q)_n}=2-\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(q;q)_n}=2-\frac{1}{(-q;q)_\infty}. \tag{24}$$

Taking $a=-a, b=\frac{1}{\sqrt{a}}, c=\frac{1}{\sqrt{a}}$, and then setting $a\to\infty$, (4) and (8) can be reduced to the famous Gauss triangle series

Corollary 3.16

$$1+\sum_{n=1}^{\infty}q^{2n^2-n}\left(1+q^{2n}\right)=\sum_{n=-\infty}^{\infty}q^{2n^2-n}=\sum_{n=0}^{\infty}q^{n(n+1)/2} \tag{25}$$

References

[1] Gasper, G. and Rahman, M. (2004) Basic Hypergeometric Series. 2nd Edition, Cambridge University Press, Cambridge.

[2] Andrews, G.E. (1976) The Theory of Partitions, Encyclopedia of Math, and Its Applications. Addison-Wesley Publishing Co., Boston.

[3] Liu, Z.G. (2003) Some Operator Identities and Q-Series Transformation Formulas. *Discrete Mathematics*, **265**, 119-139. http://dx.doi.org/10.1016/S0012-365X(02)00626-X.

[4] Fine, N.J. (1988) Basic Hypergeometric Series and Applications. Mathematical Surveys and Monographs, 1988. http://dx.doi.org/10.1090/surv/027

[5] Andrews, G.E. (1972) Two Theorems of Gauss and Allied Identities Proved Arithmetically. *Pacific Journal of Mathematics*, **41**, 563-578. http://dx.doi.org/10.2140/pjm.1972.41.563.

[6] Berndt, B.C. and Rankin, R.A. (1995) Ramanujan: Letters and Commentary. American Mathematical Society, Providence, London Mathematical Society, London.

[7] Andrews, G.E. (1966) On Basic Hypergeometric Series, Mock Theta Functions, and Partitions (I). *The Quarterly Journal of Mathematics*, **17**, 64-80. http://dx.doi.org/10.1093/qmath/17.1.64.

[8] Watson, G.N. (1936) The Final Problem: An Account of the Mock Theta Functions. *Journal of the London Mathematical Society*, **11**, 55-80. http://dx.doi.org/10.1112/jlms/s1-11.1.55

[9] Liu, X.C. (2012) On Flushed Partitions and Concave Compositions. *European Journal of Combinatorics*, **33**, 663-678. http://dx.doi.org/10.1016/j.ejc.2011.12.004

[10] Ramanujan, S. (1988) The Lost Notebook and Other Unpublished Paper. Springer-Verlag, Berlin.

[11] Berndt, B.C., Chan, S.H., Yeap, B.P. and Yee, A.J. (2007) A Reciprocity Theorem for Certain Q-Series Found in Ramanujan's Lost Notebook. *The Ramanujan Journal*, **13**, 27-37. http://dx.doi.org/10.1007/s11139-006-0241-5

[12] Andrews, G.E. and Berndt, B.C. (2005) Ramanujan's Lost Notebook, Part I. Springer, New York.

[13] Berndt, B.C. and Yee, A.J. (2003) Combinatorial Proofs of Identities in Ramanujan's Lost Notebook Associated with the Rogers-Fine Identity and False Theta Functions. *Annals of Combinatorics*, **7**, 409-423. http://dx.doi.org/10.1007/s00026-003-0194-y.

[14] Warnaar, S.O. (2003) Partial Theta Functions. I. Beyond the Lost Notebook. *Proceedings of the London Mathematical Society*, **87**, 363-395. http://dx.doi.org/10.1112/S002461150201403X.

[15] Rogers, L.J. (1917) On Two Theorems of Combinatory Analysis and Some Allied Identities. *Proceedings of the London Mathematical Society*, **16**, 316-336.

[16] Andrews, G.E. (1979) An Introduction to Ramanujan's Lost Notebook, *The American Mathematical Monthly*, **86**, 89-108. http://dx.doi.org/10.2307/2321943