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# New Bounds on Tenacity of Graphs with Small Genus

**Davoud Jelodar[1], Dara Moazzami[2,3,4*]**

[1]Department of Algorithms and Computation, University of Tehran, Tehran, Iran
[2]Department of Engineering Science, College of Engineering, University of Tehran, Tehran, Iran
[3]School of Computer Sciences, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran
[4]Center of Excellence in Geomatics Engineering and Disaster Management, University of Tehran, Tehran, Iran
Email: *dmoazzami@ut.ac.ir

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## Abstract

**A new lower bound on the tenacity $T(G)$ of a graph $G$ in terms of its connectivity $\kappa(G)$ and genus $\gamma(G)$ is obtained. The lower bound and interrelationship involving tenacity and other well-known graphical parameters are considered, and another formulation introduced from which further bounds are derived.**

## Keywords

**Tenacity Parameter, Connectivity, Genus, Planar Graph, Torus**

## 1. Introduction

The concept of graph tenacity was introduced by Cozzens, Moazzami and Stueckle [1] [2], as a measure of network vulnerability and reliability. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. The motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the concept was introduced in 1992. Cozzens *et al.* in [1], introduced two measures of network vulnerability termed the tenacity, $T(G)$, and the Mix-tenacity, $T_m(G)$, of a graph.

The tenacity $T(G)$ of a graph $G$ is defined as

*Corresponding author.

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$$T(G)=\min_{A\subseteq V(G)}\left\{\frac{|A|+m(G-A)}{\omega(G-A)}\right\}$$

where $m(G-A)$ denotes the order (the number of vertices) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $(G-A)$. A set $A\subseteq V(G)$ is said to be a $T$-set of $G$ if $T(G)=\frac{|A|+m(G-A)}{\omega(G-A)}$.

The Mix-tenacity, $T_m(G)$ of a graph $G$ is defined as

$$T_m(G)=\min_{A\subseteq E(G)}\left\{\frac{|A|+m(G-A)}{\omega(G-A)}\right\}.$$

where $m(G-A)$ denotes the order (the number of vertices ) of a largest components of $(G-A)$.

$T(G)$ and $T_m(G)$ turn out to have interesting properties. Following the pioneering work of Cozzens, Moazzami, and Stueckle, [1] [2], several groups of researchers have investigated tenacity, and its related problems.

In [3] and [4] Piazza *et al.* used the Mix-tenacity parameter as Edge-tenacity. This parameter is a combination of cutset $A\subseteq E(G)$ and the number of vertices of the largest component, $m(G-A)$. Also this Parameter didn't seem very satisfactory for Edge-tenacity, Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [5].

The Edge-tenacity $T_e(G)$ of a graph $G$ is defined as

$$T_e(G)=\min_{A\subseteq E(G)}\left\{\frac{|A|+m_e(G-A)}{\omega(G-A)}\right\}$$

where $m_e(G-A)$ denotes the order (the number of edges) of a largest component of $G-A$.

The concept of tenacity of a graph $G$ was introduced in [1] [2], as a useful measure of the "vulnerability" of $G$. In [6], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [1]-[22] they studied more about this new invariant.

All graphs considered are finite, undirected, loopless and without multiple edges. Throughout the paper $G$ will denote a graph with vertex set $V(G)$. Further the minimum degree will be denoted $\delta(G)$, the maximum degree $\Delta(G)$, connectivity $\kappa(G)$, the shortest cycle or girth $g(G)$ and we use $\alpha(G)$ to denote the independence number of $G$.

The genus of a graph is the minimal integer $\gamma$ such that the graph can be drawn without crossing itself on a sphere with $\gamma$ handles. Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing. In topological graph theory there are several definitions of the genus of a group. Arthur T. White introduced the following concept. The genus of a group $G$ is the minimum genus of a (connected, undirected) Cayley graph for $G$. The graph genus problem is NP-complete.

A graph $G$ is toroidal if it can be embedded on the torus. In other words, the graph's vertices can be placed on a torus such that no edges cross. Usually, it is assumed that $G$ is also non-planar.

**Proposition 1** (*a*) *If* $H$ *is a spanning subgraph of* $G$*, then* $T(H)\le T(G)$.

b) $T(G)\le\frac{n-\alpha(G)+1}{\alpha(G)}$, where $n=|V(G)|$.

**Proposition 2** *If* $G$ *is any noncomplete graph,* $T(G-v)\ge T(G)-\frac{1}{2}$.

**Proposition 3** *If* $G$ *is a nonempty graph and* $m$ *is the largest integer such that* $K_{1,m}$ *is an induced subgraph of* $G$*, then* $T(G)\ge\frac{\kappa(G)}{m}$.

**Corollary 1** *a) If* $G$ *is noncomplete and claw-free then* $T(G) \geq \frac{\kappa(G)}{2}$.

*b) If* $G$ *is a nontrivial tree then* $T(G) \geq \frac{1}{\Delta(G)}$.

*c) If* $G$ *is r-regular and r-connected then* $T(G) \geq 1$.

The following well-known results on genus will be used.

**Proposition 4** *If* $G$ *is a connected graph of genus* $\gamma$, *connectivity* $\kappa$, *girth* $g$, *having* $p$ *vertices,* $q$ *edges and* $r$ *regions, then*

a) $q \leq \frac{g(p+2\gamma-2)}{g-2}, \kappa \leq \frac{2g+\left(1+\frac{2\gamma}{p}-\frac{2}{p}\right)}{g-2}$

b) $\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ [23]

c) $\gamma(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil, p \geq 3$ [24]

## 2. Lower Bound

In this section we establish lower bounds on the tenacity of a graph in terms of its connectivity and genus.

We begin by presenting a theorem due to Schmeichel and Bloom.

**Theorem 2.1** (*Schmeichel and Bloom* [25]) *Let G be a graph with genus* $\gamma$. *If* $G$ *has connectivity* $\kappa$, *with* $\kappa \geq 3$, *then*

$$\omega(G-X) \leq \frac{2}{\kappa-2}\left(|X|-2+2\gamma\right)$$

for all $X \subset V(G)$ with $|X| \geq \kappa$.

It is now to drive the bounds on the tenacity that we seek.

**Theorem 2.2** *If* $G$ *is a connected graph of genus* $\gamma$ *and connectivity* $\kappa$, *then*

a) $T(G) > \kappa/2 - 1$, if $\gamma = 0$ or $1$, and

b) $T(G) \geq \frac{(\kappa+1)(\kappa-2)}{2(\kappa-2+2\gamma)}$, if $\gamma \geq 2$.

*Proof.* First, note that the inequalities hold trivially if $\kappa = 1$ or $2$. So suppose $\kappa \geq 3$.

First, suppose that $\gamma = 0$. Let $S$ be a $T$-set. Then since $|S| \geq \kappa$, by Theorem 2.1 we have

$$\omega(G-S) = \omega \leq \frac{2}{\kappa-2}\left(|S|-2+2\gamma\right)$$

So $|S| \geq \frac{\omega(\kappa-2)}{2} + 2$. Since $|S| \geq \kappa$, $T = \frac{|S|+m(G-S)}{\omega(G-S)} \geq \frac{\kappa+1}{\omega}$ and hence $\omega \geq \frac{\kappa+1}{T}$. Therefore,

$$T \geq \left(\frac{\kappa-2}{2}\right) + \frac{3}{\omega} > \frac{\kappa}{2} - 1,$$

if $\gamma = 1$, we have $|S| \geq \frac{\omega(\kappa-2)}{2}$ then

$$T \geq \left(\frac{\kappa-2}{2}\right) + \frac{1}{\omega} > \frac{\kappa}{2} - 1,$$

and part (a) is proved.

So suppose $\gamma \geq 2$. Again, let $S$ be a $T$-set in $G$. Then

$$\omega(G-S) = \omega \leq \frac{2}{\kappa-2}\left(|S|-2+2\gamma\right),$$

and thus

$$|S| \geq \frac{\omega(\kappa-2)}{2} - (2\gamma-2),$$

and $\omega \geq \dfrac{\kappa+1}{T}$, so

$$T \geq \frac{\kappa-2}{2} - \frac{2\gamma-3}{\omega} \geq \frac{\kappa-2}{2} - \frac{2\gamma-3}{\frac{\kappa+1}{T}} \geq \frac{(\kappa+1)(\kappa-2)}{2(\kappa-2+2\gamma)},$$

the result follows.

The above bounds is illustrated by a subset of the complete bipartite graph. Let $\kappa = 3$ and $\gamma$ be integer such that $4\gamma$ is a multiple of $\kappa - 2$ and $4\gamma \geq (\kappa-2)^2$. Then $K_{\kappa, 2+4\gamma/(\kappa-2)}$ has connectivity $\kappa$, genus $\gamma$ and tenacity $\dfrac{(\kappa+1)(\kappa-2)}{2(\kappa-2+2\gamma)}$.

## 2.1. Planar Graphs and the Lower Bound of Tenacity

We next investigate the bounds provided above if $G$ is a planar or toroidal graph. To this end we require the definition of a *Kleetope*, $\tau(G)$, of an embedding $G$ of a graph. If $G$ is a graph embedded with regions $R_1, R_2, \cdots, R_r$, then $\tau(G)$ is the graph obtained from $G$ by, for $1 \leq i \leq r$, inserting a vertex $v_i$ into the interior of $R_i$ and joining $v_i$ to each vertex on the boundary of $R_i$. Note that the embedding of $G$ extends naturally to an embedding of $\tau(G)$. In particular, if $G$ is a plane graph then so is $\tau(G)$. *Kleetopes* are sometimes used as examples of graphs with maximum independence number for given genus and connectivity (see [26]).

The bound in Theorem 2.2a is not sharp for $\kappa = 1$ and $\gamma = 0$. But the following examples show that the bound is suitable for $\gamma = 0$ and all possible values of $2 \leq \kappa \leq 5$. Furthermore, such examples can be obtained with the maximum girth allowed for such connectivity. Note that by proposition 4a, if $g$ is the girth,

$$\begin{cases} \kappa < \dfrac{2g}{g-2} & \text{for } \gamma = 0, \\ \kappa \leq \dfrac{2g}{g-2} & \text{for } \gamma = 1. \end{cases}$$

Indeed we can always obtain any girth from 3 up to the maximum allowed. This is often done by taking the example with maximum girth and adding an edge incident with a vertex in the T-set to create the desired short cycle.

**Example 1**

a) For $\kappa = 2$ the girth can be arbitrarily large. For $n \geq 3$ consider the graph $G_n$ obtained by taking $n$ disjoint copies of the path $P_n$ on $n$ vertices and identifying the corresponding ends into two vertices. This is a planar graph with tenacity $T(G_n) \leq \dfrac{2+n}{n} \to 1$ as $n \to +\infty$ and girth $g \to +\infty$.

b) For $\kappa = 3$ the girth is at most $5$. A generalized Herschel graph $H_n (n \geq 1)$ is defined as follows. Form a cyclic chain of 4-cycles by taking $n$ disjoint 4-cycles $a_i b_i c_i d_i a_i$, $1 \leq i \leq n$, and identifying $c_i$ and $a_{i+1}$ (including $c_n$ and $a_1$). Then introduce vertices $b$ and $d$ and make $b$ adjacent to each $b_i$ and make $d$ adjacent to each $d_i$. The result is a 3-connected planar graph of girth 4, (see **Figure 1**). Now, let $G_n$ be obtained by replacing each of the $b_i$ and $d_i$ by dodecahedron as follows. To make notation simpler we explain how to replace a generic node $x$ of degree 3 with a dodecahedron $D$. Suppose the outer cycle of $D$ is $v_1v_2v_3v_4v_5v_1$ in clockwise order and the neighbors of $x$ are $y_1, y_2$ and $y_3$ in clockwise order. Then replace $x$ and its incident edges by $D$ and the edges $v_1y_1, v_2y_2$ and $v_4y_3$. The resulting graph $G_n$ is 3-connected (recall that the dodecahedron is 3-connected), planar, and has girth 5.

Furthermore, for $S = \{b, d, a_1 = c_n, a_2 = c_1, \cdots, a_n = c_{n-1}\}$,

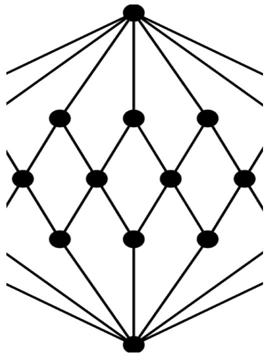

**Figure 1.** Part of the 3-connected planar graph with $g=4$.

$$T(G_n) \le \frac{|S|+m(G_n-S)}{\omega(G_n-S)} = \frac{(n+2)+5}{2n} \xrightarrow[+\infty]{n} \frac{1}{2}^+,$$

while $\dfrac{\kappa(G_n)}{2}-1=\dfrac{1}{2}$.

c) If $\kappa=4$ then $g=3$. Let $L_n$ be a ladder graph with two rails and $n$ rungs between them. Rails be $P_{1n}$ with vertices $a_1,a_2,\cdots,a_n$ and $P_{2n}$ with vertices $b_1,b_2,\cdots,b_n$. Now make $G_n$, introduce vertices $a$ and $b$ and make a adjacent to each $a_i$ and b adjacent to each $b_i$. $G_n$ is a planar graph with $\kappa=4$ and $g=3$, (see **Figure 2**). For $S=\{a,b,a_1,b_2,a_3,b_4,\cdots\}$,

$$T(G_n) \le \frac{|S|+m(G_n-S)}{\omega(G_n-S)} = \frac{n+3}{n} \xrightarrow[+\infty]{n} 1^+,$$

whereas $\dfrac{\kappa(G_n)}{2}-1=1$.

d) if $\kappa=5$ then $g=3$. For positive integer $n$ the graph $R_n$ is defined inductively as follows: $C_{2n}$ is a $2n$ vertices cycle with $a_1a_2\cdots a_{2n}$ in clockwise order and $C_n$ is a $n$ vertices cycle with $b_1b_2\cdots b_n$ in clockwise order.

Make two edges between $b_i$ and vertices $a_{2i-1}$ and $a_{2i}$, then introduce $C=\bigcup_{i=1}^n c_i$ and $D=\bigcup_{i=1}^n d_i$, make edges $c_ib_i, c_ia_{2i-1}, c_ia_{2i}, d_ib_i, d_ia_{2i}, d_ib_{i+1}$ and $d_ia_{2i+1}$ (note that $1\le i\le n$), in **Figure 3** you can see a $R_4$ graph with empty cycle vertices as set of $C$ and empty rectangle vertices as set of $D$. Suppose that $S$ is cut set and $S=R_n-(C\cup D)$, then

$$T(R_n) \le \frac{|S|+m(R_n-S)}{\omega(R_n-S)} = \frac{3n+1}{2n} \xrightarrow[+\infty]{n} \frac{3}{2}^+,$$

whereas $\dfrac{\kappa(R_n)}{2}-1=3/2$.

## 2.2. Toroidal Graphs

We next consider toroidal graphs in more depth. For $3\le\kappa\le 6$ we provide graphs with $T\ge\dfrac{\kappa}{2}-1$ and maximum girth.

**Example 2** (*a*) *For $\gamma=1$ and $\kappa=2$, the family graphs described in Example* 1(*a*) *for planar graphs shows that 2-connected graphs can have tenacity arbitrarily close to* 1. *(Examples specifically with genus* 1 *can be obtained by adding two edges to $G_n$, for $n\ge 4$).*

b) For $\kappa=4$ then $g\le 4$. The graph $H_n=C_4\times C_n$, for $n$ an even integer has genus 1, connectivity 4,

$T \geq 1$ (since, for example, its bipartite and hamiltonian) and girth 4.

c) If $\kappa = 5$ then $g = 3$. Consider the following graph $W_n$ where every region is a pentagon: Let $i = 0,1,\cdots,n-1, V(W_n) = \{a_i, b_i, c_i, d_i, e_i, f_i\}$ and $E(W_n) = \{a_i a_{i+1}, a_i b_i, a_i c_i, b_i d_i, d_i b_{i+1}, c_i e_i, e_i c_{i+1}, d_i f_i, e_i f_i, f_i f_{i+1}\}$ where addition is taken modulo $n$. The graph $W_6$ is shown in **Figure 4**.

We note that $W_n$ is toroidal with a pentagonal embedding. Let $H_n = \tau(W_n)$. Then $\gamma(H_n) = 1$, $\kappa(H_n) = 5$,

$$T(H_n) \geq \frac{p(W_n)}{r(W_n)} = \frac{3}{2} = \frac{\kappa}{2} - 1 .$$

d) If $\kappa = 6$ then $g = 3$. Consider the cubic bipartite "honeycomb" graph $W_n$ on $12n$ vertices where every region is a hexagon. Then $H_n = \tau(W_n)$, satisfies $\gamma(H_n) = 1, \kappa(H_n) = 6$ and

$$T(H_n) \geq \frac{p(W_n)}{r(W_n)} = 2 = \frac{\kappa}{2} - 1 .$$

e) If $\kappa = 3$ then $g \leq 6$. Consider any (3-connected) bipartite graph $H$ which has partite sets $A$ and $B$ where every vertex in $A$ has degree 3 and every vertex in $B$ has degree 6 and is embedded in the torus with every region a quadrilateral. For example, $K_{3,6}$. Such an $H_n$ can also be obtained by modifying the honeycomb graph $W_n$, depicted in **Figure 5** as follows: If the bipartite sets for $W_n$, are $A$ and $B$, then add in each region a new vertex and join it to the three vertices of $A$ on the boundary of the region; the new vertices are added to $B$. Now form $G_n$ by taking $H_n$, and replacing every vertex of degree 3 by a

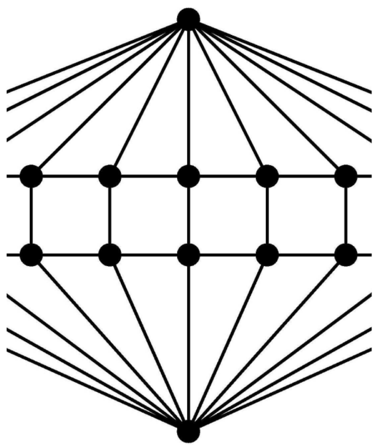

**Figure 2.** Part of planar graph with $g = 3$ and $\kappa = 4$.

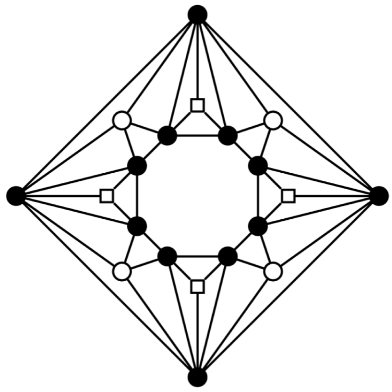

**Figure 3.** A graph $R_4$ with $\kappa = 5$ and $g = 3$.

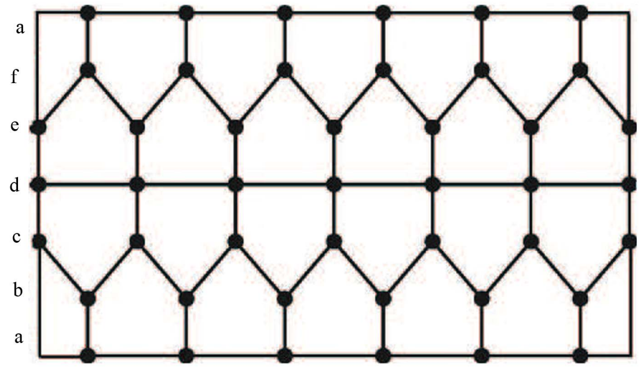


**Figure 4.** Pentagonal embedding graph.

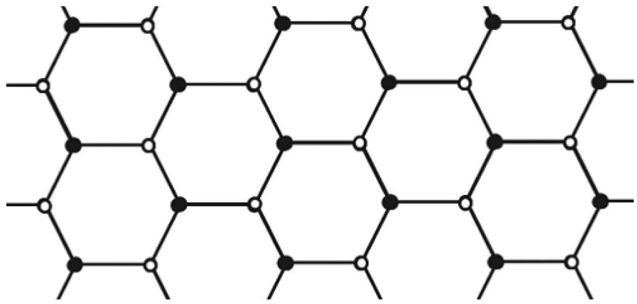

**Figure 5.** A honeycomp graph.

dodecahedron as described in Example 1(b). The resulting graph $H_n$ satisfies $\gamma(H_n)=1$, $\kappa(H_n)=3$ and $T(H_n)\le\frac{1}{2}=\frac{\kappa}{2}-1$.

The graph $G_n$ constructed in Example 2(e) has girth $g=5$. The lower bound given in Theorem 2.2(b) cannot be obtained if $\gamma=1,\kappa=3$ and $g=6$ as is shown next.

**Lemma 2.3** *If* $G$ *is a graph with* $\gamma(G)=1,\kappa(G)=3$ *and* $g(G)=6$, *then* $T(G)\ge 1$.

*Proof.* Let $G$ be a toroidal graph satisfying the hypothesis of the lemma. Then Euler's formula (or Proposition 4(a)) shows that the graph is 3-regular. So by Corollary 1(c) the tenacity is at least 1.

## 3. Conclusions

The sharpness of the bound $T(G)\ge\frac{(\kappa+1)(\kappa-2)}{2(\kappa-2+2\gamma)}$, if $\gamma\ge 2$ is illustrated by a subset of the complete bipartite graph. Let $\kappa\ge 3$ and $\gamma$ be integer such that $4\gamma$ is a multiple of $\kappa-2$ and $4\gamma\ge(\kappa-2)^2$. Then $K_{\kappa,2+4\gamma/(\kappa-2)}$ has connectivity $\kappa$, genus $\gamma$ and tenacity $\frac{(\kappa+1)(\kappa-2)}{2(\kappa-2+2\gamma)}$. So the bound in Theorem 2.2(b) is attained by an infinite class of graphs, all of girth 4.

The bound in Theorem 2.2a is not sharp for $\kappa=1$ and $\gamma=0$. But the example 1 showed that the bound is sharp for $\gamma=0$ and all possible values of $2\le\kappa\le 5$.

For Toroidal graphs when $3\le\kappa\le 6$ we introduced graphs with $T\ge\frac{\kappa}{2}-1$ and maximum girth.

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