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The Number of Digraphs with Cycles of Length *k*

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ABSTRACT

In this note, we show that the number of digraphs with *n* vertices and with cycles of length *k*, $0 \le k \le n$, is equal to the number of $n \times n$ (0,1)-matrices whose eigenvalues are the collection of copies of the entire *k*th unit roots plus, possibly, 0's. In particular, 1) when $k = 0$, since the digraphs reduce to be acyclic, our result reduces to the main theorem obtained recently in [1] stating that, for each $n = 1, 2, 3, \cdots$, the number of acyclic digraphs is equal to the number of $n \times n$ (0,1)-matrices whose eigenvalues are positive real numbers; and 2) when $k = n$, the digraphs are the Hamiltonian directed cycles and it, therefore, generates another well-known (and trivial) result: the eigenvalues of a Hamiltonian directed cycle with *n* vertices are the *n*th unit roots [2].

KEYWORDS

Acyclic Digraph; Eigenvalue; Power Digraph; (0,1)-Matrix

1. Introduction

A digraph G_n of *n* vertices $\{1, 2, \cdots, n\}$ is a directed graph whose edges are oriented from vertex *i* to vertex *j*, $1 \le i,\ j \le n$. In this note, the digraphs to be considered are with loops or cycles, but parallel edges are forbidden. An *acyclic digraph* is a digraph that has no cycles of any length. Let $D_{n,k}$ be a digraph of n vertices with cycles of length k plus, possibly, an acyclic digraph with $n - mk$ vertices, where m is the number of cycles, and where k is fixed, $0 \le k \le n$. Then it is seen that the cycles in $D_{n,k}$ are *disjoined* (therefore all the cycles that it has are *simple*) and if $k < n$ the digraph is not strongly connected. And, in particular, $D_{n,n}$ is a Hamiltonian directed cycle of size n and $D_{n,0}$ is an acyclic digraph of n vertices, respectively.

The acyclic directed graphs have been considered by several authors in the past decades. A first related result appeared in the literature seems to be the one described in [2]. It says that a digraph G contains no cycle if and only if all eigenvalues of its adjacency matrix are 0. Subsequently, to the best of our knowledge, Robinson [3,4] and Stanley [5] counted the acyclic digraphs independently and showed that if R_n stands for the number of acyclic digraphs of n vertices then

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k} \sim n! \frac{2^{\binom{n}{2}}}{Mp^n},$$

where $p = 1.488\cdots$, and $M = 0.474\cdots$. Later, in [6,7], Bender *et al.* considered the asymptotic number of acyclic digraphs with *q* edges, and subsequently, Gessel counted the acyclic digraphs by their sources and sinks in [8]. Most recently, E. Weisstein of Wolfram Research Inc. calculated the number, M_n, of $n \times n$ $(0,1)$-matrices with real positive eigenvalues and showed that for $n = 1, 2, 3, 4, 5$ the numbers M_n are

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because the numbers were observed to coincide with the first five values of the sequence of the number of acyclic digraphs with n vertices that is obtained by Sloane in [9]. Weisstein conjectured that the two sequences are identical. The conjecture has recently been proven in [1].

Motivated by the above literature, we extend the acyclic digraphs to consider the digraphs $D_{n,k}$ with cycles of length k in this note, where $0 \le k \le n$. Our theorem established in the next section indicates that similar counting theorem holds for more general graphs.

2. The Main Results

Let us first prove the following lemma.

Lemma 1 *Given a positive integer* n, *and a nonnegative integer k with* $0 \le k \le n$, *the eigenvalues of the adjacency matrix* $A = A(D_{n,k})$ *of* $D_{n,k}$ *are copies of the entire kth unit roots plus, possibly,* 0'*s. Conversely, if B is an* $n \times n$ $(0,1)$*-matrix whose eigenvalues are copies of the entire kth unit roots plus, possibly,* 0'*s then its digraph* $D(B)$ *is isomorphic to* $D_{n,k}$, *if ignoring the acyclic parts in the two digraphs.*

Proof. Assume that $D_{n,k}$ has m cycles of length k. We show that the eigenvalues of A are m copies of the entire kth unit roots plus $n-mk$ 0's. Since relabeling the vertices of a graph does not change the eigenvalues of its adjacency matrix, and since the m cycles of length k are disjoined, we may number the vertices consistently with the partial order so that A has the upper block-triangular as follows:

$$A = \begin{pmatrix} A_1 & * & * & \cdots & * & * & * \\ 0 & A_2 & * & \cdots & * & * & * \\ & & \cdots & & & & \\ 0 & 0 & \cdots & A_m & * & * & * \\ 0 & 0 & \cdots & 0 & 0 & * & * \\ & & \cdots & & & 0 & * \\ 0 & 0 & \cdots & & & 0 & 0 \end{pmatrix}$$

where each A_i, $i = 1, 2, \cdots, m$, is the adjacency matrix of a (directed simple) cycles of length k, and where '*' is either a block matrix of 0, 1, or 1, or 0's. From linear algebra, it can be easily proven that, for any i ($\le m$), the characteristic polynomial of A_i is $|\lambda I - A_i| = \lambda^k - 1$. So the eigenvalues of A_i are the kth unit roots. Since the eigenvalues of A are collection of the eigenvalues of these A_i and $n-mk$ 0's, its eigenvalues are m copies of the entire kth unit roots plus $n-mk$ zeroes.

Conversely, if B is an $n \times n$ $(0,1)$-matrix whose eigenvalues are m copies of the entire kth unit roots plus $n-mk$ 0's, then its graph $D(B)$ is a digraph and, for any $i(\le n)$, the ith eigenvalues of B^k, $\lambda_i(B^k)$, is either 1 or 0. We now consider the power digraphs of $D(B)$ with adjacency matrix B. Since for all $l = 1, 2, 3, \cdots$ \

$$trace(B^{lk}) = \sum_{i=1}^{n} \lambda_i^{lk} = mk,$$

the number of closed walks of length l in the kth power graph $D(B)^k$ of $D(B)$ is mk. Since the eigenvalues of B^k are either 1 or 0, the diagonal elements of B^k must be 1 or 0. In fact, from Perron-Frobenius theory (e. g., [10], p. 28, (1,6) Corollary (a)), we have $B^k(i,i) \le \rho(B^k) = 1$, $\rho(B^k)$ is the largest eigenvalue of B^k, which implies $B^k(i,i) = 1$ or 0, and B^k has exactly mk 1's on its diagonal. Thus, counting all the closed walks in the kth power graph $D(B)^k$ we conclude that $D(B)$ is a digraph with m disjoined cycles of length k plus an acyclic graph with $n-mk$ vertices. Putting the thing back to B implies that B is the adjacency matrix of a digraph with m cycles of length k plus, possibly, an acyclic digraph with $n-mk$ vertices. The proof is complete.

For $m = 0, 1, 2, \cdots$, counting the number of the digraphs and the $(0,1)$-matrices in the above lemma, we immediately have the following

Theorem 1 *For each* $n = 1, 2, 3, \cdots$, *and nonnegative integer k,* $0 \le k \le n$, *the number of digraphs* $D_{n,k}$ *with cycles of length* k *is equal to the number of* $n \times n$ $(0,1)$*-matrices whose eigenvalues are the collection of copies of the entire kth unit roots plus, possibly,* 0'*s.*

Note that when $k = 0$, because of the one to one corresponding, this leads to an alternative proof of the above conjecture (the main theorem of [1]). That is, for each $n = 1, 2, 3, \cdots$, the number of acyclic digraphs $D_{n,0}$

is equal to the number of $n\times n$ $(0,1)$-matrices whose eigenvalues are 0's --- from linear algebra, which is equivalent to saying that, for each $n=1,2,3,\cdots$, the number of acyclic digraphs $D_{n,0}$ is equal to the number of $n\times n$ $(0,1)$-matrices whose n eigenvalues are equal to 1 --- and from [1], which is also equivalent to saying that, for each $n=1,2,3,\cdots$, the number of acyclic digraphs $D_{n,0}$ is equal to the number of $n\times n$ $(0,1)$-matrices whose eigenvalues are positive real numbers.

Corollary 1 *If a digraph D has cycles of lengths k_i, $i=1,2,\cdots,t$, and the cycles are piecewise disjoined then the eigenvalues of its adjacency matrix A are collection of the entire k_ith unit roots, $i=1,2,\cdots,t$, plus* 0'*s. And vice versa.*

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