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On a Sufficient and Necessary Condition for Graph Coloring

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ABSTRACT

Using the linear space over the binary field that related to a graph *G*, a sufficient and necessary condition for the chromatic number of *G* is obtained.

KEYWORDS

Vertex Coloring; Chromatic Number; Outer-Kernel Subspace; Plane Graph

1. Introduction

Let $G=(V,E)$ be a graph, where *V* is a set of vertices and *E* is a set of edges of *G*. A vertex coloring of a graph *G* is a coloring to all the vertices of *G* with *p* colors so that no two adjacent vertices have the same color. Such the graph is called p-coloring. The minimal number *p* is called the chromatic number of *G*, and is denoted by $\chi(G)$. The so-called Four Color Problem is that for any plane graph *G*, $\chi(G)\le 4$ [1].

The coloring of a graph *G* is an interesting problem for many people [2]. This is mainly caused by the Four Color Problem [3].

In this paper, putting a graph into a linear space over the binary field $GF(2)$, we obtain the sufficient and necessary condition for the chromatic number of *G*.

And as an application of above result, we give a characterization for a maximal plane graph to be 4-coloring.

2. The Linear Space A_n over $GF(2)$

Now we introduce the linear space over the field $GF(2)$.

Firstly, the field $GF(2)$ contains only two members: $GF(2)=\{0,1\}$, where the addition and multiplication are as usual excepting that $1+1=0$.

Let $V_n=\{a_1,a_2,\cdots,a_n\}$ be the *n* vertices, the all vectors of the linear space A_n are formed of the symbolic expression

$$\sum_{i=1}^{n}\alpha_i a_i,\quad \alpha_i\in GF(2).$$

It has 2^n vectors. The addition of two vectors is defined by

$$\sum_{i=1}^{n}\alpha_i a_i+\sum_{i=1}^{n}\beta_i a_i=\sum_{i=1}^{n}(\alpha_i+\beta_i)a_i .$$

Here, the n vertices $\{a_1,a_2,\cdots,a_n\}$ will serve as the most basic elements of the linear space A_n. They will be as a basis of the linear space A_n. For them the basic assumption is that these *n* vertices are linearly independent in A_n.

According to the addition in $GF(2)$, for any vector u in the linear space A_n, it has

$$u+u=0.$$

Here we denote the zero vector by 0.

For a vector $u=\sum_{i=1}^{n}\alpha_i a_i$, the order of the vector u is defined by

$$|u|=\sum_{i=1}^{n}{}' a_i ,$$

where the $\sum'$ means that the addition is the usual addition in the integer set.

A vector with k order is called a k-order vector, and a vector whose order is even is called an even-order vector.

We now give some structures to the linear space A_n. In other words, we want to "put" a graph into the linear space A_n.

In the linear space A_n, 1-order vectors are vertices of a graph. The edge is expressed as the 2-order vertex, *i.e.* a_i+a_j is the edge $a_i a_j$. So we have two ways to describe a edge: by $a_i a_j$ (in the usual sense), or by a_i+a_j (in the linear space A_n).

In the following, we always discuss a graph in the linear space A_n, it means we express edges with the second form.

All the 2-order vectors in the linear space A_n are the all possible edges with n vertices $\{a_1,a_2,\cdots,a_n\}$, that we denote by E_A:

$$E_A=\{a_i+a_j, i\neq j, i,j\in\{1,\cdots,n\}\}.$$

For a giving graph $G=(V_n,E)$ with n vertices, in the linear space A_n, the elements of the set E are the 2-order vectors of A_n, then the edge set E of G is the subset of the set E_A, $E\subseteq E_A$.

We give two examples here.

1) For the set E_A with all the 2-order vertices in A_n, the graph $K_n=(V_n,E_A)$ is a complete graph, whose any two vertices are adjacent.

2) For a graph $G=(V_n,E)$ with n vertices, the complementary set of E in the set of the 2-order vertices of A_n is $E_A\setminus E$. Then the complementary graph $\hat{G}$ of the graph G is $\hat{G}=(V_n,E_A\setminus E)$.

We now see the addition in A_n. For a path of G with a sequence of edges $a+b_1, b_1+b_2,\cdots,b_k+b$, where the end-points are a and b, the sum of the edges is:

$$(a+b_1)+(b_1+b_2)+\cdots+(b_k+b)=a+b.$$

This expression indicates the relation between the addition in the linear space and the connectivity of a graph. That is why we put the graph into the linear space A_n.

Lemma 1. The sum of even-order vectors is even-order.

This is clear by the property that $a_i+a_j=0$ if and only if $i=j$.

As a special case of the Lemma 1, we have

Lemma 2. Let $a_{i_j}\neq 0,(j=1,2,\cdots,k)$ are the vertices of G, if

$$a_{i_1}+a_{i_2}+\cdots+a_{i_k}=0,$$

then k is even.

Definition. Let A_n be the n-dimensional linear space derived by the graph $G=(V_n,E)$ above, and $R_G\subseteq E_A$ be a set of 2-order vectors. Denote by $\overline{R}_G$ the linear subspace spanned by R_G. If there are no edges of E in $\overline{R}_G$, *i.e.*

$$\overline{R}_G\cap E=\phi \tag{1}$$

then $\overline{R}_G$ is called an outer-kernel subspace of G. And $\overline{R}_G$ is a maximal outer-kernel subspace if the rank of $\overline{R}_G$ is maxima in all the outer-kernel subspace of G.

Now we give some basic properties of a outer-kernel subspace of a graph G.

By definition, $\overline{R}_G$ is a subspace of A_n. Denote the set of all 2-order vectors of $\overline{R}_G$ by $E(\overline{R}_G)$, then $G(\overline{R}_G)=(V_1,E(\overline{R}_G))$ is a subgraph of the complementary graph $\hat{G}$ of G, here V_1 is the 1-order vectors appeared in $E\left(\overline{R}_G\right)$. The subgraph $G\left(\overline{R}_G\right)$ consists of some connected blocks.

Lemma 3. Let $g_1=\left(H,E_1\right)$ be a connected block of $G\left(\overline{R}_G\right)$, and $H=\left\{a_1,a_2,\cdots,a_m\right\}$ be the vertices of g_1, then $m\geq 2$, g_1 is a complete graph K_m in the $G\left(\overline{R}_G\right)$ and

$$B=\operatorname{span}\left\{a_1+a_2,a_2+a_3,\cdots,a_{m-1}+a_m\right\}$$

is the linear subspace of $\overline{R}_G$

This lemma means that every connected block of $G\left(\overline{R}_G\right)$ is a complete graph.

Proof. Because $\overline{R}_G$ is spanned by 2-order vectors, so $m\geq 2$.

Suppose that $a_i+a_j\in E_1, a_j+a_k\in E_1,$ for $\overline{R}_G$ is a linear space , then

$$\left(a_i+a_j\right)+\left(a_j+a_k\right)=a_i+a_k\in\overline{R}_G .$$

Since g_1 is connected block of $G\left(\overline{R}_G\right)$, so $a_i+a_k\in E_1$. On the other hand, if $a_i+a_j\in E_1$, then $a_i,a_j\in H$. Hence all the 2-order vectors formed by the set of vertices H span the linear subspace B of $\overline{R}_G$. Thus the connected block g_1 is a complete graph in $g\left(\overline{R}_G\right)$.

Lemma 4. If $a_1+a_2+\cdots+a_k\in\overline{R}_G$, then k is even and there exists a $i\in\left\{2,\cdots,k\right\}$ such that $a_1+a_i\in\overline{R}_G$.

Proof. By the definition of $\overline{R}_G$, k is even. For $\overline{R}_G$ is spanned by 2-order vectors, so a_1 is in a connected block K_m of $\overline{R}_G$. Thus another vertex $a_i\left(i\neq 1\right)$ of K_m must appear in $a_1+a_2+\cdots+a_k$.

3. The Main Results

The outer-kernel subspace plays an important role in the problem of vertex coloring.

Theorem 1. Let G_n be a graph with *n* vertices, then the sufficient and necessary condition for G_n to be *p*-coloring is that the rank of an outer-kernel subspace $\overline{R}_G$ of G_n is $n-p$.

Proof. First we prove the necessity. Suppose that the graph G_n is p-coloring. Then all the vertices of G_n can be divided into p subsets by the colors:

$$S_1,S_2,\cdots,S_k,Q_{k+1},\cdots,Q_p,\quad k\leq p . \tag{2}$$

That means the vertices with a same color are in the same subset. Because it may have a subset with only one vertex, we denote the one-vertex subsets with different colors by $Q_{k+1},\cdots,Q_p$.

The elements of subset $S_i\left(i=1,2,\cdots,k\right)$ are not less then 2. Denote them by

$$S_i=\left\{a_{i1},a_{i2},\cdots,a_{it_i}\right\},\quad t_i\geq 2 ,$$

then by (2)

$$\sum_{i=1}^{k}t_i+p-k=n . \tag{3}$$

Let

$$E_i=\left\{a_{i1}+a_{i2},a_{i1}+a_{i3},\cdots,a_{i1}+a_{it_i}\right\},\quad i=1,2,\cdots,k ,$$

and

$$R_G=E_1\cup E_2\cup\cdots\cup E_k ,$$

then the vectors of R_G are independent. Hence by (3), the dimension of subspace $\overline{R}_G$ spanned by R_G is

$$\dim\overline{R}_G=\sum_{i=1}^{k}t_i-k=n-p \tag{4}$$

It is clear that $\overline{R}_G\cap E=\phi$.

For the sufficiency, suppose that there exists an outer- kernel subspace $\overline{R}_G$ with condition (1), and the dimension of $\overline{R}_G$ is $n-p$.

We divide the vertices of G into some subsets according to the subspace $\overline{R}_G$. If for two vertices a and b there have

$$a+b\in\overline{R}_G , \tag{5}$$

then we put a and b into a same subset. Like the notation of congruence we denote

$$a\equiv b\left(\operatorname{mod}\overline{R}_G\right).$$

Obviously, if a vertex *a* appears in $\overline{R}_G$, then there has at least another vertex in the same subset with *a*. If a vertex does not appear in $\overline{R}_G$, then this vertex forms a subset by itself, *i.e*. the subset contains only one vertex.

Lemma 5. The vertices from different subset are linear independence on $\overline{R}_G$, *i.e*. if $a_1, a_2, \cdots, a_m$ belong to different subsets respectively, then

$$a_1 + a_2 + \cdots + a_m \not\equiv 0 \left(\bmod \overline{R}_G\right).$$

In fact, if $a_1 + a_2 + \cdots + a_m \in \overline{R}_G$, by Lemma 4, there exists a vertex $a_i \in \{a_2, \cdots, a_m\}$ such that $a_1 + a_i \in \overline{R}_G$. That means a_1, a_i is in the same subset.

Now we go on with the proof of the sufficiency. Suppose that the 2-order vectors $r_1, r_2, \cdots, r_{n-p}$ form a basis of $\overline{R}_G$, and the vertices of the graph *G* are now divided into the disjoint subset $N_1, N_2, \cdots, N_l$ by the method above. Take $b_i \in N_i$, $i = 1, 2, \cdots, l$, then any vertex *a* of *G* must be in some subset N_i and by (5) we have

$$a = b_i + r, \quad r \in \overline{R}_G.$$

So any vertex of G can be expressed by b_i and an element of $\overline{R}_G$. Thus by Lemma 5,

$$b_1, b_2, \cdots, b_l, \quad r_1, r_2, \cdots, r_{n-p}$$

are the basis of linear space A_n. Hence $l = p$.

By the definition of $\overline{R}_G$ and (5) we know that the two vertices in the same subset N_i are non-adjacent. Thus, we can assign one color to the vertices of each subset N_i. So we just need *p* colors for *G*. The graph is p-coloring.

Due to Theorem 1 and the expression (4), we have:

Theorem 2. For a graph *G* with *n* vertices, the sufficient and necessary condition for $\chi(G) = p$ is that the rank of a maximal outer-kernel subspace $\overline{R}_G$ is $n - p$.

4. An Application to Plane Graphs

As an application of Theorem 1, we consider a result of the coloring to the plane graph.

A maximal plane graph is a graph *G* such that for any two non-adjacent vertices *a* and *b* of *G*, *G* added to the edge *ab* makes a non-planar graph. It is clear that all the faces of a maximal plane graph are triangles.

A maximal plane graph is 3-CR-edge coloring if we can color its edges by 3 colors such that the three edges of every its triangle face are coloring by different colors. Later we will see that the CR in the definition is borrowed from the Cauchy-Riemann condition in the complex function theory.

Theorem 3. If a maximal plane graph is 3-CR-edge coloring, then the graph is 4-vertex coloring.

The inverse of the Theorem 3 is true, too. That means if a maximal plane graph is 4-vertex coloring, then the graph is 3-CR-edge coloring.

Proof. We introduce the 2-demensional linear space A_2:

$$A_2 = \{(0,0), (0,1), (1,0), (1,1)\}.$$

Let $G = (V_n, E)$ be 3-CR-edge coloring, and the all edges of *G* can map to the three elements $(0,1), (1,0), (1,1)$ of A_2 by their colors, respectively. That is the mapping *f*

$$f : E \to A_2$$

such that if a, b, c are the vertices of a face of *G*, then

$$f(a+b) + f(b+c) + f(c+a) = 0 \tag{6}$$

For a path of *G* with the end-point *a*, *b* and the sequence of the edges $a + a_{i_1}, a_{i_1} + a_{i_2}, \cdots, a_{i_k} + b$, we define

$$f(a+b) = f\left(a + a_{i_1}\right) + f\left(a_{i_1} + a_{i_2}\right) + \cdots + f\left(a_{i_k} + b\right). \tag{7}$$

By the condition of 3-CR-edge coloring, the extending mapping *f* by (7) is dependent only on the end-point *a* and *b*, and independent on their path.

Let A_n' be the $(n-1)$-dimensional linear subspace spanned by all the 2-order vectors of the space A_n. Then *f* is the homomorphic mapping from subspace A_n' onto the space A_2. The homomorphic kernel $\overline{R}$ consists of such vector *e* of A_n' that satisfies

$$f(e) = 0 \tag{8}$$

Suppose R is the subset of 2-order vectors of A'_n that satisfies (8) and $\overline{R}$ is spanned by *R*. Then by (8) we have

$$\overline{R} \cap E = \phi .$$

Denote the linear independent spanning elements of R by $e_1, e_2, \cdots, e_m$, that is just the basis of $\overline{R}$. And $\dim \overline{R} = m$.

We take $e_\alpha, e_\beta, e_\gamma \in E$ such that

$$f(e_\alpha) = (0,1),\ f(e_\beta) = (1,0),\ f(e_\gamma) = (1,1).$$

Then the linear subspace A'_n is spanning by

$$e_\alpha, e_\beta, e_\gamma, e_1, e_2, \cdots, e_m .$$

Hence $m + 3 \geq n - 1$, and $\dim \overline{R} = m \geq n - 4$. By Theorem 1, the graph *G* is 4-vertex coloring.

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