

# Graph Derangements\*

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## ABSTRACT

We introduce the notion of a graph derangement, which naturally interpolates between perfect matchings and Hamiltonian cycles. We give a necessary and sufficient condition for the existence of graph derangements on a locally finite graph. This result was first proved by W. T. Tutte in 1953 by applying some deeper results on digraphs. We give a new, simple proof which amounts to a reduction to the (Menger-Egerváry-König-)Hall(-Hall) Theorem on transversals of set systems. We also consider the problem of classifying all cycle types of graph derangements on  $m \times n$  checkerboard graphs. Our presentation *does not* assume any prior knowledge in graph theory or combinatorics: all definitions and proofs of needed theorems are given.

**Keywords:** Graph Derangement Cycle Perfect Matching

## 1. Introduction

### 1.1. The Cockroach Problem

An interesting problem was conveyed to me at the Normal Bar in Athens, GA. The following is a mathematically faithful rendition, although some of the details may be misremembered or mildly embroidered.

Consider a square kitchen floor tiled by  $25 = 5 \times 5$  square tiles in the usual manner. Late one night the house's owner comes down to discover that the floor is crawling with cockroaches: in fact each square tile contains a single cockroach. Displeased, she goes to the kitchen cabinet and pulls out an enormous can of roach spray. The roaches sense what is coming and start skittering. Each roach has enough time to skitter to any adjacent tile. But it will not be so good for two (or more) roaches to skitter to the same tile: that will make an obvious target. Is it possible for the roaches to perform a collective skitter such that each ends up on its own tile?

This is a nice problem to give undergraduates: it is concrete, fun, and far away from what they think they should be doing in a math class.

### 1.2. Solution

Suppose that the tiles are painted black and white with a checkerboard pattern, and that the center square is black, so that there are 13 black squares and 12 white squares.

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Therefore there are 13 roaches who start out on black squares and are seeking a home on only 12 white squares. It is not possible—no more than for pigeons!—for all 13 cockroaches to end up on different white squares.

### 1.3. A Problem for Mathematicians

For a grown mathematician (or even an old hand at mathematical brainteasers), this is not a very challenging problem, since the above parity considerations will quickly leap to mind. Nevertheless there is something about it that encourages further contemplation. There were several other mathematicians at the Normal Bar and they were paying attention too. “What about the  $6 \times 6$  case?” One of them asked. “It reminds me of the Brouwer fixed point theorem,” muttered another<sup>1</sup>.

One natural follow-up is to ask what happens for cockroaches on an  $m \times n$  rectangular grid. The preceding argument works when  $m$  and  $n$  are both odd. On the other hand, if e.g.  $m = n = 2$  it is clearly possible for the cockroaches to skitter, and already there are several different ways. For instance, we could divide the rectangle into two dominos and have the roaches on each domino simply exchanging places. Or we could simply have them proceed in a (counter)clockwise cycle.

Dominos are a good idea in general: if one of  $m$  and  $n$  is even, then an  $m \times n$  rectangular grid may be tiled

<sup>1</sup>Unfortunately, a connection with Brouwer's theorem is *not* achieved here, but see [1].

with dominos, and this gives a way for the roaches to skitter. Just as above though this feels not completely satisfactory and one naturally looks for other skittering patterns: e.g. when  $m = n = 4$ , one can have the roaches in the inner  $2 \times 2$  square skittering clockwise as before and then roaches in the outer ring of the square skittering around in a cycle: isn't that more fun? There are many other skittering patterns as well.

I found considerations like the above to be rather interesting (and I will come back to them later), but for me the real problem was a bit more meta: what is the mathematical structure underlying the Cockroach Problem, and what is the general question being asked about this structure?

Here we translate the Cockroach Problem into graph-theoretic terms. In doing so, we get a graph theoretic problem which in a precise sense interpolates between two famous and classic problems: existence of **perfect matchings** and existence of **Hamiltonian cycles**. On the other hand, the more general problem does not seem to be well known. But it's interesting, and we present it here: it is the existence and classification of **graph derangements**.

## 2. Graph Derangements and Graph Permutations

### 2.1. Basic Definitions

Let  $G = (V, E)$  be a simple, undirected graph: that is, we are given a set  $V$  of **vertices** and a set  $E$  of **edges**, which are unordered pairs of distinct elements of  $V$ . For  $v_1, v_2 \in V$ , we say that  $v_1$  and  $v_2$  are **adjacent** if  $\{v_1, v_2\} \in E$  and write  $v_1 \sim v_2$ . In other words, for a set  $V$ , to give a graph with vertex set  $V$  is equivalent to giving an anti-reflexive, symmetric binary relation on  $V$ , the **adjacency relation**. A variant formalism is also useful: we may think of a graph as a pair of sets  $(V, E)$  and an **incidence relation** on  $V \times E$ . Namely, for  $x \in V$  and  $e \in E$ ,  $x$  is **incident** to  $e$  if  $x \in e$ , or, less formally, if  $x$  is one of the two vertices comprising the endpoints of  $e$ . If one knows the incidence relation as a subset of  $V \times E$  then one knows in particular for each  $e \in E$  the pair of vertices  $\{v_1, v_2\}$  which are incident to  $e$  and thus one knows the graph  $G$ .

For  $G = (V, E)$  and  $v \in V$ , the **degree** of  $v$  is the number of edges which are incident to  $v$ . A degree zero vertex is **isolated**; a degree one vertex is **pendant**.

A graph is **finite** if its vertex set is finite (and hence its edge set is finite as well). A graph is **locally finite** if every vertex has finite degree.

If  $G = (V, E)$  and  $G' = (V, E')$  are graphs with  $E' \subset E$ , we say  $G'$  is an **edge subgraph** of  $G$ : it has the same underlying vertex set as  $G$  and is obtained from  $G$  by removing some edges. If  $G = (V, E)$  and

$G' = (V', E')$  are finite graphs with  $V' \subset V$  and  $E'$  equal the set of all elements of  $E$  linking two vertices in  $V'$ , we say  $G'$  is an **induced subgraph** of  $G$ .

For a graph  $G = (V, E)$ , a subset  $X \subset V$  is **independent** if for no  $x_1, x_2 \in X$  do we have  $x_1 \sim x_2$ .

Example 2.1: For  $m, n \in \mathbb{Z}^+$ , we define the **checkerboard graph**  $R_{m,n}$ . Its vertex set is  $\{1, \dots, m\} \times \{1, \dots, n\}$ , and we decree that  $(x_1, x_2) \sim (y_1, y_2)$  if  $|x_1 - y_1| + |x_2 - y_2| = 1$ .

Example 2.2: More generally, for  $n \in \mathbb{Z}^+$  and  $m_1, \dots, m_n \in \mathbb{Z}^+$  we may define an  $n$ -dimensional analogue  $R_{m_1, \dots, m_n}$  of  $R_{m,n}$ . Its vertex set is

$\prod_{i=1}^n \{1, \dots, m_i\}$ , and we decree that  $(x_1, \dots, x_m) \sim (y_1, \dots, y_m)$  if  $\sum_{i=1}^m |x_i - y_i| = 1$ .

Example 2.3: For  $m \geq 2, n \geq 1$  we define the **cylinder checkerboard graph**  $C_{m,n}$ . This is a graph with the same vertex set as  $R_{m,n}$  and having edge set consisting of all the edges of  $R_{m,n}$  together with  $(x, n) \sim (x, 1)$  for all  $1 \leq x \leq m$ . We put  $C_m = C_{m,1}$ , the **cycle graph**. The checkerboard graph  $R_{m,n}$  is an edge subgraph of  $C_{m,n}$ .

Example 2.4: For  $m, n \geq 2$  we define the **torus checkerboard graph**  $T_{m,n}$ . Again it has the same vertex set as  $R_{m,n}$  and contains all of the edges of  $R_{m,n}$  together with the following ones: for all  $1 \leq x \leq m$ ,  $(x, n) \sim (x, 1)$  and for all  $1 \leq y \leq n$ ,  $(m, y) \sim (1, y)$ . The cylinder checkerboard graph  $C_{m,n}$  is an edge subgraph of the torus checkerboard graph  $T_{m,n}$ .

For a graph  $G = (V, E)$  and  $x \in V$ , we define the **neighborhood of  $x$**  as  $N_x = \{y \in V \mid x \sim y\}$ . More generally, for any subset  $X \subset V$  we define the **neighborhood of  $X$**  as

$$N(X) = \{y \in V \mid \exists x \in X \text{ such that } x \sim y\} = \bigcup_{x \in X} N_x.$$

Remark 2.5: Although  $x \notin N_x, X$  and  $N(X)$  need not be disjoint. In fact,  $X \cap N(X) = \emptyset$  iff  $X$  is an independent set.

Now the "cockroach skitterings" that we were asking about on  $R_{m,n}$  can be formulated much more generally. Let  $G = (V, E)$  be a graph. A **graph derangement** of  $G$  is an injection  $f : V \rightarrow V$  such that  $f(v) \sim v$  for all  $v \in V$ . Let  $\text{Der}G$  be the set of all graph derangements of  $G$ .

Example 2.6: If  $G = K_n$  is the complete graph on the vertex set  $[n] = \{1, \dots, n\}$ , then a graph permutation of  $G$  is nothing else than a permutation of  $[n]$ . A derangement of  $K_n$  is a derangement in the usual sense, i.e., a fixed-point free permutation. Derangements exist iff  $n > 1$  and the number of them is asymptotic to  $\frac{n!}{e}$  as  $n \rightarrow \infty$ .

It is natural to also consider a slightly more general definition.

For a graph  $G=(V,E)$ , a **graph permutation** of  $G$  is an injection  $f:V \rightarrow V$  such that for all  $v \in V$ , either  $f(v) \sim v$  or  $f(v)=v$ . Let  $\text{Perm}G$  be the set of all graph permutations of  $G$ . Thus a graph derangement is precisely a graph permutation which is fixed point free: for all  $v \in V$ ,  $f(v) \neq v$ .

**Lemma 1** Let  $G=(V,E)$  be a graph.

- 1) If  $G$  has an isolated vertex,  $\text{Der}G = \emptyset$ .
- 2) If  $G$  has two pendant vertices adjacent to a common vertex,  $\text{Der}G = \emptyset$ .

*Proof.* Left to the reader as an exercise to get comfortable with the definitions.  $\square$

Remark 2.7: If  $G'$  is an edge subgraph of  $G$ , any graph derangement (resp. graph permutation)  $\sigma$  of  $G'$  is also a graph derangement (resp. graph permutation) of  $G$ .

### 2.2. Cycles and Surjectivity

Given a graph  $G$ , we wish not only to decide whether  $\text{Der}G$  is nonempty but also to study its structure. The collection of all derangements on a given graph is likely to be a very complicated object: consider for instance

$\text{Der}K_n$ , which has size asymptotic to  $\frac{n!}{e}$ . Just as in the

case of ordinary permutations and derangements, it seems interesting to study the possible **cycle types** of graph derangements and graph permutations on a given graph  $G$ . Let us give careful definitions of these.

First, let  $V$  be a set and  $f:V \rightarrow V$  an function. For  $m \in \mathbb{Z}^+$ , let  $f^m = f \circ \dots \circ f$  be the  $m$ th iterate of  $f$ . We introduce a relation  $\approx$  on  $V$  as follows: for  $x, y \in V$ ,  $x \approx y$  iff there are  $m, n \in \mathbb{Z}^+$  such that  $f^m x = f^n y$ . This is an equivalence relation on  $V$ : the reflexivity and the symmetry are immediate, and as for the transitivity: if  $x, y, z \in V$  are such that  $x \approx y$  and  $y \approx z$  then there are  $a, b, c, d \in \mathbb{Z}^+$  with  $f^a x = f^b y$  and  $f^c y = f^d z$ , and then

$$\begin{aligned} f^{a+c} x &= f^c \circ f^a x = f^c \circ f^b y = f^{b+c} y \\ &= f^b \circ f^c y = f^b \circ f^d z = f^{b+d} z. \end{aligned}$$

Now suppose  $f$  is injective: we now call the  $\approx$ -equivalence classes **cycles**. Let  $x \in V$ , and denote the cycle containing  $x$  by  $C_x$ . Then:

- $C_x$  is **finite** iff  $x$  lies in the image of  $f$  and there is  $m \in \mathbb{Z}^+$  such that  $f^m x = x$ .
- $C_x$  is **singly infinite** iff it is infinite and there are  $y \in V$ ,  $m \in \mathbb{Z}^+$  such that  $f^m y = x$  and  $y$  is not in the image of  $f$ .
- $C_x$  is **doubly infinite** iff it is infinite and every  $y \in C_x$  lies in the image of  $f$ .

These cases are mutually exclusive and exhaustive, so  $f$  is surjective iff there are no singly infinite cycles. Suppose  $G=(V,E)$  is a graph and  $f \in \text{Perm}G$ . We

can define the **cycle type** of  $f$  as a map from the set of possible cycle types into the class of cardinal numbers. When  $V$  is finite, this amounts to a partition of  $\#V$  in the usual sense: e.g. the cycle type of roaches skittering counterclockwise on a  $5 \times 5$  grid is  $(1,8,16)$ . A graph permutation is a graph derangement if it has no 1-cycles. We will say a graph derangement is **matchless** if it has no 2-cycles.

**Proposition 2** Suppose that a graph  $G$  admits a graph derangement. Then  $G$  admits a surjective graph derangement.

*Proof.* It is sufficient to show that any graph derangement can be modified to yield a graph derangement with no singly infinite cycles, and for that it suffices to consider one singly infinite cycle, which may be viewed as the derangement  $n \mapsto n+1$  on the graph  $\mathbb{Z}^+$  with  $n \sim n+1$  for all  $n \in \mathbb{Z}^+$ . This derangement can be decomposed into an infinite union of 2-cycles:

$$1 \leftrightarrow 2, 3 \leftrightarrow 4, \dots, 2n-1 \leftrightarrow 2n, \dots \square$$

### 2.3. Disconnected Graphs

**Proposition 3** Let  $G$  be a graph with components  $\{G_i\}_{i \in I}$ , and let  $f \in \text{Perm}G$ .

- 1) For all  $i \in I$ ,  $f(G_i) = G_i$ .
- 2) Conversely, given graph permutations  $f_i$  on each  $G_i$ ,  $f = \bigcup_{i \in I} f_i : G \rightarrow G$  is a graph permutation. Moreover  $f \in \text{Der}G \iff f_i \in \text{Der}G_i$  for all  $i \in I$ .

The proof is immediate. Thus we may as well restrict attention to connected graphs.

### 2.4. Bipartite Graphs

A **bipartition** of a graph  $G=(V,E)$  is a partition  $V = V_1 \sqcup V_2$  of the vertex set such that each  $V_i$  is an independent set. A graph is **bipartite** if it admits at least one bipartition.

For  $k \in \mathbb{Z}^+$ , a **k-coloring** of a graph  $G=(V,E)$  is a map  $C:V \rightarrow \{1, \dots, k\}$  such that for all  $x, y \in V$ ,  $x \sim y \implies C(x) \neq C(y)$ . There is a bijective correspondence between 2-colorings of  $G$  and bipartitions of  $G$ : given a 2-coloring  $C$  we define  $V_i = \{x \in V \mid C(x) = i\}$ , and given a bipartition we define  $C(x) = i$  if  $x \in V_i$ . Thus a graph is bipartite iff it admits a 2-coloring.

Remark 2.8: For a graph  $G=(V,E)$ , a map  $C:V \rightarrow \{0,1\}$  is a 2-coloring of  $G$  iff its restriction to each connected component  $G_i$  is a 2-coloring of  $G_i$ . It follows that a graph is bipartite iff all of its connected components are bipartite.

Remark 2.9: Any subgraph  $G'$  of a bipartite graph  $G$  is bipartite. Indeed, any 2-coloring of  $G$  restricts to a 2-coloring of  $G'$ .

Example 2.10: The cycle graph  $C_m$  is bipartite iff  $m$  is even.

**Corollary 4** *Let  $G$  be a bipartite graph, and let  $\sigma \in \text{Der}G$ . Then every finite cycle of  $\sigma$  has even length.*

*Proof.* Since a subgraph of a bipartite graph is bipartite, a bipartite graph cannot admit a cycle of odd finite degree<sup>2</sup>.  $\square$

Example 2.11:

1) The  $n$ -dimensional checkerboard graph  $R_{m_1, \dots, m_n}$  admits a graph derangement iff  $m_1, \dots, m_n$  are not all odd.

2) The cylinder checkerboard graphs  $C_{m,n}$  all admit graph derangements. Hence, by Remark 2.7, so do the torus checkerboard graphs  $T_{m,n}$ .

3) For odd  $n$ , the square checkerboard graph  $R_{n,n}$  admits graph permutation with a single fixed point. For instance, by dividing the square into concentric rings we get a graph permutation with cycle type

$$\left\{ 1, 8, 16, \dots, 8 \left( \frac{n-1}{2} \right) \right\}.$$

### 3. Existence Theorems

#### 3.1. Halls' Theorems

The main tool in all of our Existence Theorems is a truly basic result of combinatorial theory. There are several (in fact, notoriously many) equivalent versions, but for our purposes it will be helpful to single out two different formulations.

**Theorem 5 (Halls' Theorem: Transversal Form)** *Let  $V$  be a set and  $\mathcal{S} = \{S_i\}_{i \in I}$  be an indexed family of finite subsets of  $V$ . The following are equivalent:*

1) (Hall Condition) *For every finite subfamily  $J \subset I$ ,  $\#J \leq \#\bigcup_{i \in J} S_i$ .*

2)  $(V, I)$  *admits a transversal: a subset  $X \subset V$  and a bijection  $f : X \rightarrow I$  such that for all  $x \in X$ ,  $x \in S_{f(x)}$ .*

We will deduce Theorem 5 from the following reformulation.

**Theorem 6 (Halls' Theorem: Marriage Form)** *Let  $G = (V_1, V_2, E)$  be a bipartitioned graph in which every  $v \in V_1$  has finite degree. The following are equivalent:*

1) (Cockroach Condition) *For every finite subset of  $V_1$ ,  $\#V_1 \leq \#N(V_1)$ .*

2) *There is a semiperfect matching, that is, an injection  $\iota : V_1 \rightarrow V_2$  such that for all  $x \in V_1$ ,  $x \sim \iota(x)$ .*

*Proof.* We follow [2]. Step 1: Suppose  $V_1$  is finite. We go by induction on  $\#V_1$ . The case  $\#V_1 = 1$  is trivial. Now suppose that  $\#V_1 = n > 1$  and that the result holds for all bipartitioned graphs with first vertex set of cardinality smaller than  $n$ . It will be notationally convenient to suppose that  $V_1 = \{1, \dots, n\}$ , and we do so.

<sup>2</sup>Conversely, a graph with no cycles of odd degree is bipartite, a famous result of D. König.

Case 1: Suppose that for all  $1 \leq k < n$ , every  $k$ -element subset of  $V_1$  has at least  $k+1$  neighbors. Then we may match  $n$  to any element of  $V_2$  and semiperfectly match  $\{1, \dots, n-1\}$  into the remaining elements of  $V_2$  by induction.

Case 2: Otherwise, for some  $k$ ,  $1 \leq k < n$ , there is a  $k$ -element subset  $X \subset V_1$  such that  $\#N(X) = k$ . The subset  $X$  may be semiperfectly matched into  $V_2$  by induction, say via  $\iota_1 : X \rightarrow V_2$ , so it suffices to show that the Hall Condition still holds on the induced bipartitioned subgraph on  $(V_1 \setminus X, V_2 \setminus \iota_1(X))$ . Indeed, if not, then for some  $h$ ,  $1 \leq h \leq n-k$ , there would be an  $h$ -element subset  $Y \subset V_1 \setminus X$  having fewer than  $h$  neighbors in  $V_2 \setminus \iota_1(X)$ , but then

$$\begin{aligned} \#N(X \cup Y) &= \#(N(X) \cup N(Y)) \leq \#N(X) + \#N(Y) \\ &< k + h = \#(X \cup Y). \end{aligned}$$

Step 2: Suppose  $V_1$  is infinite. For  $x \in V_1$ , endow  $N_x$  with the discrete topology; endow  $N = \prod_{x \in V_1} N_x$  with the product topology. Each  $N_x$  is finite hence compact, so  $N$  is compact by Tychonoff's Theorem. For any finite subset  $X \subset V_1$ , let

$$H_X = \{n = \{n_i\} \in N \mid n_x \neq n_y, \forall x \neq y \in X\}.$$

Then  $H_X$  is closed in  $N$  and is nonempty by Step 1. Since  $G$  is compact, there is  $n \in \bigcap_X H_X$ , and any such  $n$  is a semiperfect matching.  $\square$

Remark 3.1: Theorems 5 and 6 are equivalent results:

Assume Theorem 5. In the setting of Theorem 6, take  $V = V_2$ ,  $I = V_1$  and  $\mathcal{S} = \{N_x\}_{x \in I}$ . The local finiteness of the graph means each element of  $\mathcal{S}$  is finite, and the assumed Cockroach Condition is precisely the Hall Condition, so by Theorem 5 there is  $X \subset V_2$  and a bijection  $f : X \rightarrow I$  such that  $x \in X \Rightarrow f(x) \in N_x$ . Let  $\iota = f^{-1} : V_1 \rightarrow X$ . Then for  $y \in V_1$ , let  $y = \iota(x)$ , so  $x = f(y) \sim y = \iota(x)$ .

Assume Theorem 6. In the setting of Theorem 5 take  $V_1 = I$ ,  $V_2 = V$ , and  $E$  the set of pairs  $(i, x)$  such that  $x \in S_i$ . Since each  $S_i$  is finite, the graph is locally finite, and the assumed Hall Condition is precisely the Cockroach Condition, so by Theorem 6 there is a semiperfect matching  $\iota : I \rightarrow V$ . Let  $X = f(I)$ , and let  $f : X \rightarrow I$  be the inverse function. For  $x \in X$ , if  $i = f(x)$ ,  $x = \iota(i)$ , so  $x \in S_i = S_{f(x)}$ .

Remark 3.2: Theorem 5 was first proved for finite  $I$  by Philip Hall [3]. Eventually it was realized that equivalent or stronger versions of P. Hall's Theorem had been proven earlier by Menger [4], Egerváry [5] and König [6]. The matrimonial interpretation was introduced some years later by Halmos and Vaughan [2]. Nevertheless, with typical disregard for history the most common name for the finite form of either Theorem 5 or 13 is *Hall's Marriage Theorem*.

Remark 3.3: The generalization to arbitrary index sets was given by Marshall Hall Jr. [7], whence “Halls’ Theorem” (*i.e.*, the theorem of more than one Hall).

Remark 3.4: M. Hall Jr.’s argument used Zorn’s Lemma, which is equivalent to the Axiom of Choice (AC). The proof supplied above uses Tychonoff’s Theorem, which is also equivalent to (AC) [8]. However, all of our spaces are Hausdorff. By examining the proof of Tychonoff’s Theorem using ultrafilters [9], one sees that when the spaces are Hausdorff, by the uniqueness of limits one does not need (AC) but only that every filter can be extended to an ultrafilter (UL). In turn, (UL) is equivalent to the fact that every Boolean ring has a prime ideal (BPIT). (BPIT) is known to be *weaker* than (AC), hence Halls’ Theorem cannot imply (AC).

**Question 1** Does Halls’ Theorem imply the Boolean Prime Ideal Theorem?

Remark 3.5: The use of compactness of a product of finite, discrete spaces is a clue for the *cognoscenti* that it should be possible to find a nontopological proof using the Compactness Theorem from model theory. The reader who is interested and knowledgeable about such things will enjoy doing so. The Compactness Theorem (and also the Completeness Theorem) is known to be equivalent to (BPIT).

Example 3.6 ([10], pp. 288-289): Let  $V_1$  be the set of non-negative integers and let  $V_2$  be the set of positive integers. For all *positive* integers  $x \in V_1$ , we decree that  $x$  is adjacent to the corresponding positive integer in  $V_2$  and to no other elements of  $V_2$ . However, we decree that  $0 \in V_1$  is adjacent to *every* element of  $V_2$ . It is clear that there is no semiperfect matching, because if we match  $0$  to any  $n \in V_2$ , then the corresponding element  $n \in V_1$  cannot be matched. But the Cockroach Condition holds: for a finite subset  $X \subset V_1$ , if  $0 \notin X$  then  $\#N(V_1) = \#V_1$ , whereas if  $0 \in X$  then  $N(V_1) = V_2$ .

### 3.2. The First Existence Theorem

**Theorem 7** Consider the following conditions on a graph  $G = (V, E)$ :

- (D)  $DerG \neq \emptyset$ .
- (H) For all subsets  $X \subset V$ ,  $\#X \leq \#N(X)$ .
- (H') For all finite subsets  $X \subset V$ ,  $\#X \leq \#N(X)$ .

- 1) Then (D)  $\Rightarrow$  (H)  $\Rightarrow$  (H').
- 2) If  $G$  is locally finite, then (H')  $\Rightarrow$  (D) and thus (D)  $\Leftrightarrow$  (H)  $\Leftrightarrow$  (H').

*Proof.* a) (D)  $\Rightarrow$  (H): If  $\sigma \in DerG$  and  $X \subset V$ , then  $\sigma: V \rightarrow V$  is an injection with  $\sigma(X) \subset N(X)$ . Thus  $\#X \leq \#N(X)$ . (H)  $\Rightarrow$  (H') is immediate.

3) (H')  $\Rightarrow$  (D) if  $G$  is locally finite: for each  $x \in V$ , let  $S_x = N_x$ , and let  $I = \{S_x\}_{x \in V}$ . Since  $G$  is locally finite,  $I$  is an indexed family of finite subsets of  $V$ . By assumption, for any finite subfamily  $J \subset I$ ,

$$\#J \leq \#N(J) = \#\bigcup_{x \in J} N_x = \#\bigcup_{i \in J} S_i :$$

this is the Hall Condition. Thus by Theorem 5, there is  $X \subset V$  and a bijection  $f: X \rightarrow V$  such that for all  $x \in X$ ,  $x \in N_{f(x)}$ , *i.e.*,  $x \sim f(x)$ . Let  $\sigma = f^{-1}: V \rightarrow X$ . Then for all  $y \in V$ ,  $\sigma(y) \sim f(\sigma(y)) = y$ , so  $\sigma \in DerG$ .

Remark 3.7: Example 3.3 shows (H) need not imply (D) without the assumption of local finiteness. The graph with vertex set  $\mathbb{R}$  and such that every  $x \in \mathbb{R}$  is adjacent to every integer  $n > x$  satisfies (H') but not (H).

**Lemma 8** Let  $G$  be a locally finite graph which violates the cockroach condition: there is a finite subset  $X \subset V(G)$  such that  $\#X > \#N(X)$ . Then there is an independent subset  $Y \subset X$  such that  $\#Y > \#N(Y)$ .

*Proof.* Let  $Y \subset X$  be the subset of all vertices which are not adjacent to any element of  $X$ , so  $Y$  is an independent set. Put  $m_1 = \#Y$ ,  $m_2 = \#(X \setminus Y)$ , and  $n = m_1 + m_2 = \#X$ . By hypothesis,  $\#N(X) < n = m_1 + m_2$ ; since  $X \setminus Y \subset N(X) \setminus N(Y)$ , we find  $\#N(Y) < m_1 + m_2 - m_2 < m_1 = \#Y$ .

Combining Proposition 2, Theorem 7 and Lemma 8 we deduce the following result.

**Theorem 9** (First Existence Theorem)

For a locally finite graph, the following are equivalent:

- 1) For every finite independent set  $X$  in  $G$ ,  $\#X \leq \#N(X)$ .
- 2)  $G$  admits a surjective graph derangement.

Remark 3.8: Theorem 9 was first proved by W.T. Tutte ([11], 7.1). Most of Tutte’s paper is concerned with related—but deeper—results on digraphs. The result which is our Theorem 9 appears at the end of the paper and is proven by passage to an auxiliary digraph and reduction to previous results. Perhaps because this was not the main focus of [11], Theorem 9 seems not to be well known. In particular, our observation that one need only apply Theorem 5 appears to be new.

### 3.3. Bipartite Existence Theorems

A **matching** on a graph  $V = (G, E)$  is a subset  $\mathcal{M} \subset E$  such that no two edges in  $\mathcal{M}$  share a common vertex. A matching  $\mathcal{M}$  is **perfect** if every vertex of  $G$  is incident to exactly one edge in  $\mathcal{M}$ .

A graph permutation is **dyadic** if all of its cycles have length at most 2.

**Proposition 10** Let  $G$  be a graph.

- 1) Matchings of  $G$  correspond bijectively to dyadic graph permutations.
- 2) Under this bijection perfect matchings correspond to dyadic graph derangements.

*Proof.* 1) Let  $\mathcal{M} \subset E$  be a matching. We define  $\sigma_{\mathcal{M}} \in SymV$  as follows: if  $x$  incident to the edge

$e = \{x, y\}$ , we put  $\sigma x = y$ . Otherwise we put  $\sigma x = x$ . This is well-defined since by definition every vertex is incident to at most one edge and gives rise to a dyadic graph permutation. Conversely, to any dyadic graph permutation  $\sigma \in \text{Sym} V$ , let  $X \subset V$  be the subset of vertices which are not fixed by  $\sigma$  and put  $\mathcal{M}_\sigma = \bigcup_{x \in X} \{x, \sigma x\}$ . Then  $\mathcal{M} \mapsto \sigma_{\mathcal{M}}$  and  $\mathcal{M} \mapsto \mathcal{M}_\sigma$  are mutually inverse.

2) A matching  $\mathcal{M}$  is perfect iff every vertex  $x \in V$  is incident to an edge of  $\mathcal{M}$  iff the permutation  $\sigma_{\mathcal{M}}$  is fixed-point free.  $\square$

Let  $G = (V_1, V_2, E)$  be a bipartitioned graph. A **semi-perfect matching** on  $G$  is a matching  $\mathcal{M} \subset E$  such that every vertex of  $V_1$  is incident to exactly one element of  $\mathcal{M}$ . Thus a subset  $\mathcal{M} \subset G$  is a perfect matching on  $(V_1 \cup V_2, E)$  iff it is a semiperfect matching on both  $(V_1, V_2, E)$  and on  $(V_2, V_1, E)$ .

A **semiderangement** of  $G = (V_1, V_2, E)$  is an injective function  $t: V_1 \rightarrow V_2$  such that  $x \sim t(x)$  for all  $x \in V_1$ .

But in fact we have defined the same thing twice: a semiderangement of a bipartitioned graph is nothing else than a semiperfect matching.

**Theorem 11 (Semiderangement Existence Theorem)** Consider the following conditions on a bipartitioned graph  $G = (V_1, V_2, E)$ :

(SD) There is an injection  $t: V_1 \rightarrow V_2$  such that for all  $x \in V_1$ ,  $x \sim t(x)$ .

(H) For every finite subset  $J \subset V_1$ ,  $\#J \leq \#N(J)$ .

Then (SD)  $\Rightarrow$  (H), and if  $G$  is locally finite, (H)  $\Rightarrow$  (SD).

*Proof.* This is precisely Theorem 6 stated in the language of semiderangements.  $\square$

**Theorem 12 (Königs' Theorem)** Let  $G = (V_1, V_2, E)$  be a bipartitioned graph, which need not be locally finite. Suppose there is a semiderangement  $t_1: V_1 \rightarrow V_2$  of  $(V_1, V_2, E)$  and a semiderangement  $t_2: V_2 \rightarrow V_1$  of  $(V_2, V_1, E)$ . Then  $G$  admits a perfect matching, i.e., a bijection  $f: V_1 \rightarrow V_2$  such that  $x \sim f(x)$  for all  $x \in V_1$ .

*Proof.* Let  $V = V_1 \amalg V_2$ . Then  $t = t_1 \amalg t_2: V \rightarrow V$  is a graph derangement. The injection  $t$  partitions  $V$  into finite cycles, doubly infinite cycles, and singly infinite cycles. For each cycle  $C$  which is finite or doubly infinite,  $t_1: C \cap V_1 \rightarrow C \cap V_2$  and  $t_2: C \cap V_2 \rightarrow C \cap V_1$  are bijections. Thus if there are no singly infinite cycles, taking  $f = t$  we are done. If  $C$  is a singly infinite cycle, it has an initial vertex  $x_1$ . If  $x_1 \in V_1$ , then  $t_1: C \cap V_1 \rightarrow C \cap V_2$  is surjective; the problem occurs if  $x_1 \in V_2$ : then  $x_1$  does not lie in the image of  $t_1$ . We have  $x_2 = t_2(x_1) \in V_1$ ,  $x_3 = t_1(x_2) \in V_2$ , and so forth. So we can repair matters by defining  $t_1$  on  $x_2, x_4, \dots$  by  $x_2 \mapsto x_1$ ,  $x_4 \mapsto x_3$ , and so forth. Doing this on every singly infinite cycle with initial vertex lying in  $V_2$  a

bijection  $f: V_1 \rightarrow V_2$ . Moreover, since  $t_2$  is a semi-derangement and  $t(x_{2n-1}) = x_{2n}$  for all  $n \in \mathbb{Z}^+$ , we have  $f(x_{2n}) = x_{2n-1} \sim x_{2n}$ , so  $f$  is a bijection.

Remark 3.9: Suppose  $V_1$  and  $V_2$  are sets and  $t_1: V_1 \rightarrow V_2$  and  $t_2: V_2 \rightarrow V_1$  are injections between them. If we apply Theorem 12 to the bipartitioned graph on  $(V_1, V_2)$  in which  $x \in V_1 \sim y \in V_2 \Leftarrow t_1(x) = y$  or  $t_2(y) = x$ , we get a bijection  $f: V_1 \rightarrow V_2$ : this is the celebrated **Cantor-Bernstein Theorem**. As a proof of Cantor-Bernstein, this argument was given by Gyula ("Julius") König [12] and remains to this day one of the standard proofs. His son Dénes König explicitly made the connection to matching in infinite graphs in his seminal text [13].

**Theorem 13 (Second Existence Theorem)** Let  $G = (V_1, V_2, E)$  be a locally finite bipartitioned graph. The following are equivalent:

- 1)  $G$  admits a perfect matching.
- 2)  $G$  admits a dyadic graph derangement.
- 3)  $G$  admits a graph derangement.
- 4) For every subset  $J \subset V$ ,  $\#J \leq \#N(J)$ .

*Proof.* 1)  $\Leftrightarrow$  2) by Proposition 10.

2)  $\Rightarrow$  3) is immediate.

3)  $\Rightarrow$  4) is the same easy argument we have already seen.

4)  $\Rightarrow$  1): By Theorem 11, we have semiderangements  $t_1: V_1 \rightarrow V_2$  and  $t_2: V_2 \rightarrow V_1$ . By Theorem 12 this gives a perfect matching.  $\square$

Remark 3.10: The equivalence 1)  $\Leftrightarrow$  4) above is due to R. Rado [14].

### 3.4. An Equivalence

Theorems 9 and 13 are "equivalent" in the sense that they were proved using equivalent formulations of Halls' Theorem (together with, in the case of Theorem 13, a Cantor-Bernstein argument). In this section we will show their equivalence in a stronger sense: each can be rapidly deduced from the other.

Assume Theorem 9, and let  $G = (V_1, V_2, E)$  be a locally finite bipartitioned graph satisfying the Cockroach Condition: for all finite independent subsets  $J \subset V_1 \amalg V_2$ ,  $\#NJ \geq \#J$ . Then Theorem 9 applies to yield a surjective graph derangement  $f$ . A cycle  $C$  admits a dyadic graph derangement iff it is not finite of odd length; since  $G$  is bipartite, no cycle in  $f$  is finite of odd length. Thus decomposing  $f$  cycle by cycle yields a dyadic graph derangement.

Assume Theorem 13, and let  $G = (V, E)$  be a locally finite graph satisfying the Hall Condition: for all finite independent subsets  $J \subset V$ ,  $\#NJ \geq \#J$ . Let  $G_2 = (V_1, V_2, E_2)$  be the **bipartite double** of  $G$ : we put  $V_1 = V_2 = V$ . For  $x \in V$ , let  $x_1$  (resp.  $x_2$ ) denote the copy of  $x$  in  $V_1$  (resp.  $V_2$ ). For every  $e = \{x, y\} \in E$ ,

we give ourselves edges  $\{x_1, y_2\}, \{y_1, x_2\} \in E_2$ . Then  $G_2$  is locally finite bipartitioned, and it is easy to see that the Hall Condition in  $G$  implies the Cockroach Condition in  $G_2$ . By Theorem 13,  $G_2$  admits a dyadic graph derangement  $f_2$ . From  $f_2$  we construct a graph derangement  $f$  of  $G$ : for  $x \in V$ , let  $x_1$  be the corresponding element of  $V_1$ ; let  $y_2 = f_2(x_1)$ , and let  $y$  be the element of  $V$  corresponding to  $y_2$ . Then we put  $f(x) = y$ . It is immediate to see that  $f$  is a graph derangement of  $G$ . It need not be surjective, but no problem if it isn't: apply Proposition 2.

Remark 3.11: Our deduction of Theorem 9 from Theorem 13 is inspired by an unpublished manuscript of L. Levine [15].

Remark 3.12: Recall that Theorem 13 implies the Cantor-Bernstein Theorem. The above equivalence thus has the following curious consequence: Cantor-Bernstein follows almost immediately from the transversal form of Halls' Theorem.

### 3.5. Dyadic Existence Theorems

One may ask if there is also an Existence Theorem for dyadic derangements in non-bipartite graphs. The answer is yes, a quite celebrated theorem of Tutte. A later result of C. Berge gives information on dyadic graph permutations.

Let  $G = (V, E)$  be a graph. For a subset  $X \subset V$ , we denote by  $G \setminus X$  the induced subgraph with vertex set  $V \setminus X$ .

**Theorem 14 (Third Existence Theorem)** For a finite graph  $G = (V, E)$ , TFAE:

- 1)  $G$  has a dyadic graph derangement.
- 2) For every subset  $X \subset V$ , the number of connected components of  $G \setminus X$  with an odd number of vertices is at most  $\#X$ .

*Proof.* See [16].

Remark 3.13: Theorem 14 can be generalized to locally finite graphs: see [17].

A **maximum matching** of a finite graph  $G$  is a matching  $\mathcal{M}$  such that  $\#\mathcal{M}$  is maximized among all matchings of  $G$ . Thus if  $G$  admits a perfect matching, a matching  $\mathcal{M}$  is a maximum matching iff it is perfect, whereas in general the size of a maximum matching measures the deviation from a perfect matching in  $G$ .

For a finite graph  $H$ , let  $\text{odd}(H)$  be the number of connected components of  $H$  with an odd number of vertices.

**Theorem 15 (Berge's Theorem [18])** Let  $G$  be any finite graph. The size of a maximum matching in  $G$  is

$$B_G = \frac{1}{2} \left( \min_{X \subset V} \#X - \text{odd}(G \setminus X) + \#V \right).$$

We immediately deduce the following result on dyadic

graph permutations.

**Corollary 16** Let  $G$  be a finite graph. Then the least number of fixed points in a dyadic graph permutation of  $G$  is  $\#V - 2B_G$ .

### 3.6. Matchless Graph Derangements

Let  $G = (V, E)$  be a finite graph with  $\#V = n$ . Then a graph permutation of cycle type  $(n)$  is called a **Hamiltonian cycle** (or Hamiltonian circuit).

From the perspective of graph derangements it is clear that Hamiltonian cycles lie at the other extreme from dyadic derangements and permutations. Here much less is known than in the dyadic case: there is no known Hamiltonian analogue of Tutte's Theorem on perfect matchings, and in place of Berge's Theorem we have the following open question.

**Question 2** Given a finite graph  $G$ , determine the largest integer  $n$  such that  $G$  admits a graph permutation of type  $(n, 1, \dots, 1)$ . Equivalently, determine the maximum order of a vertex subgraph of  $G$  admitting a Hamiltonian cycle.

A general graph derangement is close to being a partition of the vertex set into Hamiltonian cycles, *except* that if  $x$  and  $y$  are adjacent vertices, then sending  $x$  to  $y$  and  $y$  to  $x$  meets the requirements of a graph derangement but does not give a Hamiltonian subcycle because the edge from  $x$  to  $y$  is being used twice. Let us say a graph derangement is **matchless** if each cycle has length at least 3. The above Existence Theorems lead us naturally to the following question.

**Question 3** Is there an Existence Theorem for matchless graph derangements?

As noted above, there is no known Existence Theorem for Hamiltonian cycles. Since a Hamiltonian cycle is a graph derangement of a highly restricted kind, one might hope that Question 3 is somewhat more accessible.

## 4. Checkerboards Revisited

### 4.1. Universal and Even Universal Graphs

Let  $G$  be a graph on the vertex set  $[n] = \{1, \dots, n\}$  such that  $\text{Der}G \neq \emptyset$ . As in §2.2, it is natural to inquire about the possible cycle types of graph derangements (and also graph permutations) of  $G$ . We say  $G$  is **universal** if for every partition  $\mathfrak{p}$  of  $n$  there is a graph derangement of  $G$  with cycle type  $\mathfrak{p}$ . For instance, the complete graph  $K_n$  is (rather tautologously) universal.

If  $n \geq 5$  and  $G$  is bipartite, by Corollary 4  $G$  is *not* universal, because the only possible cycle types are even. Thus for graphs known to be bipartite the more interesting condition is that every possible even partition of  $[n]$  occurs as the cycle type of a graph derangement of  $G$ : we call such graphs **even universal**.

**4.2. On the Even Universality of Checkerboard Graphs**

**Proposition 17** For all  $n \in \mathbb{Z}^+$ , the checkerboard graph  $R_{2,n}$  is even universal.

*Proof.* This is an easy inductive argument which we leave to the reader.

**Proposition 18** Let  $n \geq 4$  be an even number. Then:

1) If  $a_1, \dots, a_k$  are even numbers greater than 2 such that  $4 + \sum_{i=1}^k a_i = 3n$ , then there is no graph derangement of  $R_{3,n}$  of cycle type  $(a_1, \dots, a_k, 4)$ .

2) It follows that  $R_{3,n}$  is not even universal.

*Proof.* 1) A 4-cycle on any checkerboard graph must be a  $(2 \times 2)$ -square. By symmetry, we may assume that the  $(2 \times 2)$ -square is placed so as to occupy portions of the top two rows of  $R_{m,n}$ . The two vertices immediately underneath the square cannot be part of any Hamiltonian cycle in the complement of the square, so any graph derangement containing a 4-cycle must also contain a 2-cycle directly underneath the 4-cycle. Thus the cycle type  $(a_1, \dots, a_k, 4)$  is excluded.

2) Since  $n \geq 4$  is even,  $3n$  is even and at least 12, so there are even partitions of  $3n$  of the above form:  $(4, 4, \dots, 4)$  if  $n$  is divisible by 4;  $(6, 4, 4, \dots, 4)$  otherwise.  $\square$

**Proposition 19** If  $m$  is odd and  $n$  is divisible by 4, then  $R_{m,n}$  admits no graph derangement of cycle type  $(4, 4, \dots, 4)$  and is therefore not even universal.

*Proof.* Left to the reader.  $\square$

Example 4.1  $(G_{3,4})$ : There are  $p\left(\frac{12}{2}\right) = 11$  even

partitions of 12. By Proposition 19, we cannot realize the cycle types  $(8, 4)$ ,  $(4, 4, 4)$  by graph derangements. We can realize the other nine:

- $(12), (10, 2), (8, 2, 2), (6, 6), (6, 4, 2), (6, 2, 2, 2),$
- $(4, 4, 2, 2), (4, 2, 2, 2, 2), (2, 2, 2, 2, 2, 2).$

Example 4.2  $(G_{4,4})$ : Of the  $p\left(\frac{16}{2}\right) = 22$  even partitions of 16, we can realize 20:

- $(16), (14, 2), (12, 4), (12, 2, 2), (10, 4, 2), (10, 2, 2, 2),$
- $(8, 8), (8, 6, 2), (8, 4, 4), (8, 4, 2, 2),$
- $(8, 2, 2, 2), (6, 6, 2, 2), (6, 4, 4, 2), (6, 4, 2, 2, 2),$
- $(6, 2, 2, 2, 2, 2), (4, 4, 4, 4), (4, 4, 4, 2, 2),$
- $(4, 4, 2, 2, 2, 2), (4, 2, 2, 2, 2, 2, 2), (2, 2, 2, 2, 2, 2, 2, 2).$

We cannot realize:

- $(10, 6), (6, 6, 4).$

Indeed, the only order 6 cycle of a checkerboard graph is the  $2 \times 3$  checkerboard subgraph. Removing any 6-

cycle leaves one of the corner vertices pendant and hence not part of any cycle of order greater than 2.

Example 4.3  $(G_{3,6})$ : There are  $p\left(\frac{18}{2}\right) = 30$  even partitions of 18. We can realize these 23 of them:

- $(18), (16, 2), (14, 2, 2), (12, 6), (12, 4, 2),$
- $(12, 2, 2, 2), (10, 6, 2), (10, 4, 2, 2),$
- $(10, 2, 2, 2, 2), (8, 8, 2), (8, 6, 2, 2), (8, 4, 2, 2, 2),$
- $(8, 2, 2, 2, 2, 2), (6, 6, 6), (6, 6, 4, 2),$
- $(6, 6, 2, 2, 2), (6, 4, 4, 2, 2), (6, 4, 2, 2, 2, 2),$
- $(6, 2, 2, 2, 2, 2, 2), (4, 4, 4, 2, 2, 2),$
- $(4, 4, 2, 2, 2, 2, 2), (4, 2, 2, 2, 2, 2, 2, 2),$
- $(2, 2, 2, 2, 2, 2, 2, 2, 2).$

We cannot realize the following ones:

- $(14, 4), (10, 8), (10, 4, 4), (8, 6, 4),$
- $(8, 4, 4, 2), (6, 4, 4, 4), (4, 4, 4, 4, 2).$

Of these, all but  $(10, 8)$  and  $(4, 4, 4, 4, 2)$  are excluded by Proposition 18, and we leave it to the reader to check that these two “exceptional” cases cannot occur.

Example 4.5  $(G_{4,5})$ : There are  $p\left(\frac{20}{2}\right) = 42$  even partitions of 20. We can realize 39 of them. We cannot realize:

- $(8, 8, 4), (8, 4, 4, 4), (4, 4, 4, 4, 4).$

No  $(8, 8, 4)$ : to place a 4-cycle in  $G_{4,5}$  without leaving a pendant vertex, we must place it in a corner, without loss of generality the upper left corner. The only 8-cycle which can fit into the remaining lower left corner is the rectangular one, and this leaves a pendant vertex at the lower right corner.

No  $(8, 4, 4, 4)$ : Any placement of three 4-cycles in  $G_{4,5}$  gives two of them in either the top half or the bottom half, and without loss of generality the top half. Any such placement leaves a pendant vertex in the top row.

No  $(4, 4, 4, 4, 4)$ : This follows from Proposition 19. Alternately, the argument of the previous paragraph works here as well.

Example 4.6  $((G_{4,6}))$ : There are  $p\left(\frac{24}{12}\right) = 77$  even partitions of 24. They can all be realized by graph derangements:  $G_{4,6}$  is even universal.

By analyzing the above examples, we found the following additional families of excluded cycle types. The proofs are left to the reader.

**Proposition 20** Let  $n = 2k + 1$  be an odd integer



greater than 1. Then  $G_{4,n}$  admits no matchless graph derangement with  $2k-1$  or more 4-cycles.

**Proposition 21** For any non-negative integer  $k$ ,  $G_{4,6k+4}$  admits no graph derangement of the cycle type  $(6, 6, \dots, 6, 4)$ . Thus these graphs are not even universal.

**Proposition 22** For any odd integer  $m$  and  $k \in \mathbb{Z}^+$ ,  $G_{m,4k+2}$  admits no graph derangement of the cycle type  $(6, 4, \dots, 4)$ . Thus these graphs are not even universal.

In particular, for  $R_{m,n}$  to be even universal, it is necessary that  $m$  and  $n$  both are even. Proposition 21 shows that having  $m$  and  $n$  both even is not sufficient; however, we have found no examples of excluded even cycle types of  $R_{m,n}$  when  $m, n$  are both even and at least 6. Such considerations, and others, lead to the following conjecture, made by J. Laison in collaboration with the author.

**Conjecture 23** There is a positive integer  $N_0$  such that for all  $k, l \geq N_0$ , the checkerboard graph  $R_{2k,2l}$  is even universal.

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