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A Lemma on Almost Regular Graphs and an Alternative Proof for Bounds on $\gamma_t(P_k \square P_m)$

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ABSTRACT

Gravier *et al.* established bounds on the size of a minimal totally dominant subset for graphs $P_k \square P_m$. This paper offers an alternative calculation, based on the following lemma: Let $k, r \in \mathbb{N}$ so $k \geq 3$ and $r \geq 2$. Let H be an r-regular finite graph, and put $G = P_k \square H$. 1) If a perfect totally dominant subset exists for G, then it is minimal; 2) If $r > 2$ and a perfect totally dominant subset exists for G, then every minimal totally dominant subset of G must be perfect. Perfect dominant subsets exist for $P_k \square C_n$ when k and n satisfy specific modular conditions. Bounds for $\gamma_t(P_k \square P_m)$, for all k, m follow easily from this lemma. Note: The analogue to this result, in which we replace "totally dominant" by simply "dominant", is also true.

Keywords: Domination; Total Domination; Matrix; Linear Algebra

1. Introduction

Let $G = (V(G), E(G))$ be a graph. In this paper, each edge of a graph must have two different endpoints; also, two vertices may be linked by at most one edge. A subset *Z* of vertices is said to *totally dominate G* if every vertex of *G* has a neighbor in *Z*. We say *Z perfectly totally dominates* if every vertex has exactly one neighbor in *Z*. Next, suppose that *G* is finite. In this case, we say a totally dominant subset *Z* is *minimal* if $|Z|$ is the smallest size possible among all dominant subsets. This minimal size is denoted by $\gamma_t(G)$.

For $r \in \mathbb{N}$, we say that a graph *G* is *r*-regular if every vertex is the endpoint of exactly *r* edges. Suppose *G* is regular. A subset *Z* which perfectly totally dominates is clearly minimal. If a perfect dominant set does not exist, we can search for minimality among dominant subsets *Z* by counting "overlaps". That is, for each $v \in V(G)$, let $ol_t(v, G, Z)$ be the number of neighbors of v which lie in *Z*, minus 1. If Z_1 and Z_2 are two totally dominant subsets, then $|Z_1| < |Z_2|$ happens if and only if the sum of Z_1-overlaps is strictly less than the sum of Z_2-overlaps.

These elementary links between minimality, perfection and overlaps may fail if *G* is not regular. For arbitrary graphs, all sorts of behavior is possible. For graph theorists, a challenge is to specific assertions that apply to a broad family of graphs.

The following conventions will be used here.

(1a) For $k \in \mathbb{N}$, $k \geq 2$, let P_k, the k-*path* be the graph whose vertices are the numbers $1, 2, \cdots, k$, and whose edges are links from *i* to $i+1$ for each $1 \leq i < k$. There is an infinite member of this family: Interpret $\mathbb{Z}$ as a graph in which edges consist of links from i to $i+1$ for all i.

(1b) Let $k > 2$. The graph consisting of P_k plus an edge between 1 and k called the k-*cycle*. It is denoted by C_k.

(1c) For G and H graphs, the product graph $G \square H$ is defined as follows. The set of vertices $V(G \square H)$ is $V(G) \times V(H)$. Two vertices (x_1, y_1) and (x_2, y_2) are linked by an edge if and only if

- either $x_1 = x_2$ and $y_1 y_2$ is an edge of H, or
- $x_1 x_2$ is an edge of G and $y_1 = y_2$.

For example, for $k, n \in \mathbb{N}$, $P_k \square P_n$ is the familiar $k \times n$ grid map. A product of a list of paths and circuits by $\square$ is called a grid graph.

A product of n copies of $\mathbb{Z}$ corresponds to the set $\mathbb{Z}^n$ with the "Manhatten metric" notion of the edge: two tuples $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are linked if and only if there is an index i such that $|x_i - y_i| = 1$ and

$x_j = y_j$ for all $j \neq i$.

Tiling is the route that Gravier [1] takes in computing γ_t for grid graphs. The program begins with the work by herself, Molland and Payan [2] on the tiling question. The solution generates perfectly dominant subsets on $\mathbb{Z}^n$. Now, finite grid graphs can be interpreted as rectangular subsets, or (for products with C_n factors) as such subsets with some "opposed" sides identified. Domination becomes a problem of refining the patterns at the edges.

Our current work exploits the abundance of perfect dominations on graphs $G = P_k \Box C_n$. A calculation with matrices leads to a lower bound on $\gamma_t(G)$ that can only be attained by a perfectly totally dominant subset. Once we classify which indices k,n admit perfect dominations, an elementary trick provides upper and lower bounds for all graphs $P_k \Box C_n$. The bounds here do not improve on the earlier work, but are almost as narrow.

Suppose *H* is a finite *r*-regular graph for some natural number *r*, and put $G = P_k \Box H$ for $k \geq 3$. Then the majority of vertices of *G* have a degree $r+2$. The vertices of the degree $r+1$ form two connected subgraphs. A crude bound for a minimal totally dominant subset of *G* is $k|H|/(r+2)$. However, this bound is too low by a positive number times $|H|$.

We find a subtler minimal bound using matrices. The computation also shows that

(2a) A perfect totally dominant subset is minimal, and assumes the bound;

(2b) A minimal subset cannot have fewer members than a perfect subset; and

(2c) Unless $r = 2$ and n is odd, if a perfect totally dominant subset exists, then every minimal subset is perfect.

The conclusions follow from a formula which, for *Z* a totally dominant subset, determines $|Z|$ is a sum over $v \in V(G)$ of $ol_t(v,Z,G)\cdot\omega_j$, where each ω_j is a non-zero weight associated to row j of v.

Remark. A variation on total domination is (simple) domination. A subset dominates (non-totally) if each vertex *v* either has a neighbor in *Z* *or* belongs to *Z*. A dominant subset *Z* is perfect (non-totally) if for each vertex v, either

(3a) $v \in Z$ and *v* has no neighbors in *Z*, or

(3b) $v \notin Z$ and *v* has exactly one neighbor in *Z*.

Our theory implies that, in this context, if a perfect dominant subset exists, it is minimal and every minimal dominant subset is perfect.

1.1. Sample Perfect Behavior

A proof of minimality has two parts: first, exhibit a subset; then prove no smaller totally dominant subset can exist. The examples here are drawn from Gravier [1].

Assume *n* is even. In this case, $P_k \Box C_n$ is bipartite. Identify C_n with $\mathbb{Z}/n\mathbb{Z}$ in the standard way. We can "color" the vertices: we say (i,j) (where j is read $\bmod(n)$) is *black* if $i+j$ is even and *white* if $i+j$ is odd. Then every edge links a black vertex with a white one. If *Z* dominates $P_k \Box C_n$, then the set of black members of *Z* dominates all white vertices, and the white vertices of *Z* dominate all the black. Consequently, a minimal dominant subset is a disjoint union of two minimal "color" dominant subsets; each a subset of one color vertices that dominates all vertices of the other color. Furthermore, the "shift by 1" automorphism of $P_k \Box C_k$ identifies the sets of different colored vertices.

Figure 1 shows a pattern of vertices of one color. Provided that k is odd, this pattern will totally dominate all vertices of the opposite color.

If k is even, this pattern does not quite work. Instead, as illustrated in **Figure 2** for $k = 8$, one can build a pattern by taking triangular wedges of the first pattern, and pairing them with a skew reflection. The latter pattern can be repeated throughout $P_k \Box C_n$ provided that $2(k+1)$ divides n.

The contribution of this paper is an alternate construction of a lower bound. The bound is met for these perfect subsets. Next, using these subsets, one can establish a general upper bound for $P_k \Box P_m$ for all m.

1.2. A Tie with Perfection

Gravier [1] proves that the set Z consisting of the middle row of $P_3 \Box P_n$, for any n, is a minimal totally dominant subset. Obviously, this choice of minimal

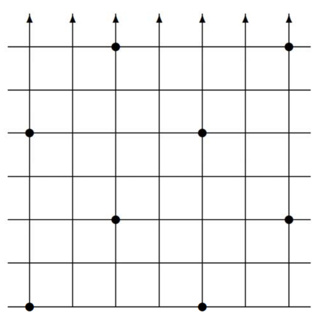

Figure 1. One color dominance, *k* odd.

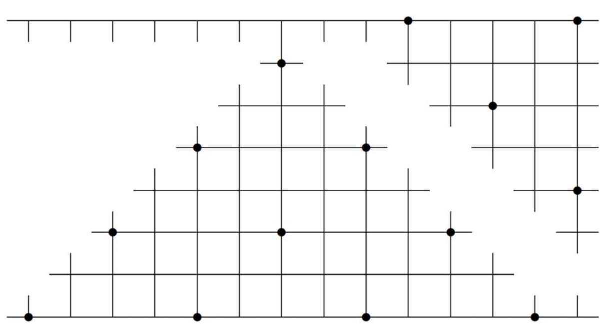

Figure 2. One color dominance, *k* = 8.

subset produces many overlaps. By rotating 3×3 blocks, we can produce other minimal dominant sets with fewer overlaps, as in **Figure 3**. Furthermore, if n is a multiple of 4, there is a variation which is a perfect total domination of $P_3\Box C_n$, as in **Figure 4**. The flexibility in the number of vertices which are dominated by more than one member of Z reflects the presence of vertices of two degrees, namely 3 and 4.

In this example, the size of a minimal, imperfect totally dominant subset "ties" the size of a perfect totally dominant set. Can a minimal subset be smaller than a perfect one? We prove that a tie is rare, and that beating is impossible.

1.3. Weights

We have two sets of theorems based on series.

Definition 1 *Let r be a real number. Let $\Xi[r]$ be the set of infinite sequences of real numbers $\{a_i\}_{i=0}^{\infty}$ such that*

$$\forall i>1,\ a_i = ra_{i-1} - a_{i-2}.$$

Clearly, $\Xi[r]$ is a real vector space, and the function $\{a_i\}_{i=1}^{\infty} \mapsto (a_0, a_1)$ is a linear isomorphism from it onto $\mathbb{R}^2$.

For r real, let $i \mapsto \lambda(r,i)$ be the unique member of $\Xi[r]$ such that $\lambda(r,0)=0$ and $\lambda(r,1)=1$. Observe that $\lambda(r,2)=r$.

In the opening section, we defined the overlap function $ol_t(v,G,Z)$ for totally dominant subsets *Z* of a graph *G*. In addition, for *G* a graph and *Z* a dominant (but possibly not totally) subset, and $v\in V(G)$, let $ol(v,G,Z)$ be $ol_t(v,G,Z)$ if $v\notin Z$ and $ol_t(v,G,Z)+1$ if $v\notin Z$. For $k>3$, $G=P_k\Box H$ for some graph H and $v\in V(G)$, define $row(v)$ the *row of* v to be the first coordinate of v.

Lemma 2 *Let $r,k\in\mathbb{N}$ such that $r\geq 2$ and $k\geq 3$.*

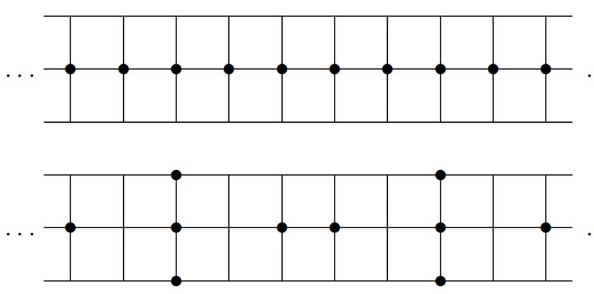

Figure 3. Two ways to totally dominate $P_3\Box P_n$.

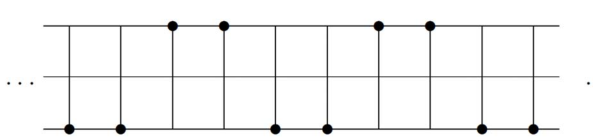

Figure 4. Perfect domination in $P_3\Box C_{4m}$.

For each integer $1\leq j\leq k$, put

$$\omega_j = \lambda(r,k+1) + (-1)^{j+1}\lambda(r,k+1-j) + (-1)^{k+j}\lambda(r,j).$$

For each $1\leq j\leq k$,

(4a) $\omega_j \geq 0$, *and*

(4b) $\omega_j = 0$ *if and only if* $r=2$, k *is odd and* j *is even.*

We refer to $\omega_1,\cdots,\omega_k$ as the weight system for parameters r,k.

Definition 3 *Let $r,k\in\mathbb{N}$ such that $r\geq 2$ and $k\geq 3$. Let $\omega_1,\cdots,\omega_k$ be the weight system for r,k. Also, let $v_1,\cdots,v_k$ be the weight system for parameters $r+1,k$. Define*

$$\mu(r,k) = \frac{(rk+2k+2)\lambda(r,k+1) + 2\lambda(r,k) + (-1)^{k+1}2}{(r+2)^2\lambda(r,k+1)}.$$

Suppose H is an r-regular graph, and put $n=|H|$ and $G=P_k\Box H$. Define two functions on $Z\subseteq V(G)$:

$$\mathrm{score}_t(Z) = \sum_{v\in V(G)} ol_t(v,Z,G)\cdot\omega_{\mathrm{row}(v)}$$

$$\mathrm{score}(Z) = \sum_{v\in V(G)} ol_t(v,Z,G)\cdot v_{\mathrm{row}(v)}$$

Theorem 4 *Assume the hypothesis and construction of Lemma* 2 *and Definition* 3. *Let H be a finite graph, and put $n=|H|$ and $G=P_k\Box H$.*

(A) If $Z\subseteq V(G)$ *is totally dominant, then*

$$|Z| = n\mu(r,k) + \frac{\mathrm{score}_t(Z)}{(r+2)\lambda(r,k+1)}.$$

(B) If $Z\subseteq V(G)$ *is dominant, then*

$$|Z| = n\mu(r+1,k) + \frac{\mathrm{score}(Z)}{(r+3)\lambda(r+1,k+1)}.$$

A trivial consequence of this theorem and the preceding lemma is:

Corollary 5 *Assume the hypothesis of Theorem* 4.

(A) *Suppose $r\geq 3$. If Z_1,Z_2 are totally dominant subsets of G, then*

$$|Z_1|<|Z_2| \Leftrightarrow \mathrm{score}_t(Z_1) < \mathrm{score}_t(Z_2).$$

(B) *If Z_1,Z_2 are dominant subsets of G, then*

$$|Z_1|<|Z_2| \Leftrightarrow \mathrm{score}(Z_1) < \mathrm{score}(Z_2).$$

2. Modeled with Matrices

Our results are based on a simple linear algebra model. For convenience,

(5) For $k\in\mathbb{N}$, let $\mathrm{Ind}(k)=\{1,\cdots,k\}$.

Notation 6. Let $k\in\mathbb{N}$. We identify the real vector space $\mathbb{R}^k$ with length *k* column vectors. We use transpose notation to write these horizontally:

$$(z_1,\cdots,z_k)^{\mathrm{T}} \quad \text{for} \quad \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix}.$$

For each $1 \le i \le k$, let π_i be the projection function from each vector $(z_1,\cdots,z_k)$ to its *i*-coordinate z_i. Also define a linear functional $\mathbb{R}^k \to \mathbb{R}$

$$sum(z) = \sum_{i=1}^{k} \pi_i(z).$$

We denote the zero vector by $\hat{0}$.

In what follows, let $k, r \in \mathbb{N}$, and let H be a finite, r-regular graph. Put $G = P_k \Box H$.

For $Z \subseteq V(G)$, define the *row count vector z* for *Z* to be $(z_1,\cdots,z_k)^{\mathrm{T}}$ in which z_i is the number of members of *Z* in the i-th row. Obviously, $sum(z) = |Z|$.

Now suppose $Z \subseteq V(G)$ totally dominates, and let $z = (z_1,\cdots,z_k)$ be its row count vector. Let $1 \le i \le k$. The sum of $ol_t(v, Z, G)$ over all *v* in the *i*-th row, plus $|H|$, equals

$$\begin{aligned} &rz_1 + z_2, && \text{for } i = 1, \\ &z_i + rz_{i+1} + z_{i+1}, && \text{for } 1 \le i \le k-1, \text{and} \\ &z_{k-1} + rz_k, && \text{for } i = k. \end{aligned} \tag{6}$$

In particular,

(7a) If *Z* totally dominates, then each of these expressions must be $\ge |H|$, and

(7b) If *Z* perfectly totally dominates, then each of these expressions must equal $|H|$.

If we replace *totally domination* with simple *domination*, the analogous assertions hold after the r terms in (6) are changed to $r+1$.

These remarks motivate our next definition.

Definition 7 *Let* r *be a real number and let* k *be a natural number* >1. *Define* $L[r,k]$ *to be the* $k \times k$ *matrix such that*

$$\forall i, j \in \mathrm{Ind}(k), L[r,k]_{i,j} = \begin{cases} r & \text{if } i = j, \\ 1 & \text{if } i - j \text{ is 1 or } -1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $L[r,k]$ is symmetric.

Also, for these parameters, define $M[r,k]$ *to be the* $k \times k$ *matrix such that*

$$\forall i, j \in \mathrm{Ind}(k),$$

$$M[r,k]_{i,j} = \begin{cases} \lambda(-r,i)\lambda(-r,k+1-j) & \text{if } i \le j, \\ \lambda(-r,j)\lambda(-r,k+1-i) & \text{if } j \le i. \end{cases}$$

Note that the case $i = j$ *is covered in both parts of this conditional definition.*

As we shall see, the matrix $M[r,k]$ is essentially $L[r,k]^{-1}$.

3. Relevant Sequences

There is a discrete analogy to *convexity* for functions of a single real variable. We recall some basics.

Definition 8 *Let* $\{a_i\}_{i=0}^{\infty}$ *be a sequence of real numbers, starting at index* 0. *We say that the sequence is convex if*

$$\forall i \in \mathbb{N},\ a_{i+1} - a_i \ge a_i - a_{i-1}.$$

We say the sequence is strictly convex if $a_{i+1} - a_i > a_i - a_{i-1}$ *for each* i.

Lemma 9 *Let* $\{a_i\}_{i=0}^{\infty}$ *be a convex sequence. For* $u, v \in \mathbb{N}$,

$$a_{u+v} \ge a_u + a_v - a_0.$$

Moreover, $a_{u+v} = a_u + a_v - a_0$ *if and only if there is a number* t *such that*

$$\forall i \in \mathrm{Ind}(u+v),\ a_i = t + a_{i-1}.$$

Proof. We may interchange *u* and *v* without loss of generality. Hence, assume $u \ge v$. For each $i \in \mathbb{N}$, put $b_i = a_i - a_{i-1}$. Then $\{b_i\}_{i=1}^{\infty}$ is a weakly increasing sequence. Then

$$\begin{aligned} &a_{u+v} - a_0 - (a_u - a_0) - (a_v - a_0) \\ &= \left(\sum_{i=1}^{u+v} b_i\right) - \left(\sum_{i=1}^{u} b_i\right) - \left(\sum_{i=1}^{v} b_i\right) \\ &\Leftrightarrow a_{u+v} - a_u - a_v + a_0 = \left(\sum_{i=1}^{v} b_{u+i}\right) - \left(\sum_{i=1}^{v} b_i\right) \\ &\Leftrightarrow a_{u+v} - a_u - a_v + a_0 = \left(\sum_{j=1}^{v} b_{u+v+1-j}\right) - \left(\sum_{i=1}^{v} b_i\right) \\ &\Leftrightarrow a_{u+v} - a_u - a_v + a_0 = \left(\sum_{i=1}^{v} \left(b_{u+v+1-i} - b_i\right)\right) \end{aligned} \tag{8}$$

Observe that

$$u + v + 1 - i \ge i \Leftrightarrow (u - v) + 2(v - i) + 1 \ge 0.$$

For each index *i* in the last sum, the term has the format $b_p - b_q$ where $p > q$. Therefore

$$a_{u+v} - a_u - a_v + a_0 \ge 0.$$

Now suppose $a_{u+v} - a_u - a_v + a_0 = 0$. Then every term in the final sum of (8) must be 0. When $i = 1$, we get $b_{u+v} - b_1 = 0$. Since b_i is an increasing sequence, it follows that $b_i = b_1$ for every index $i \le u + v$. □

We focus on the sequences $\lambda(r,i)$ of Definition 1. The first remark is that the sign can be separated from the magnitude.

Lemma 10 *Let r be a real number. Then*

$$\forall i \in \mathbb{N},\ (-1)^{i+1} \lambda(r,i) = \lambda(-r,i).$$

Proof. Trivial. □

Many of the positive sequences $\lambda(r,i)$ are convex.

Lemma 11 *Each member of* $\Xi[2]$ *is a linear sequence.*

Proof. Trivial. □

Lemma 12 *Let* $r>2$, *and let* $\{b_i\}\in\Xi[r]$ *such that* $b_1\ge b_0\ge 0$. *If* $b_1>0$, *then* $\{b_i\}$ *is increasing and strictly convex. Furthermore,* $b_i=b_{i-1}$ *can occur only if* $i=1$.

Proof. For $i\ge 2$, we can rewrite the relation $b_i=rb_{i-1}-b_{i-2}$ as

(9a) $b_i=(r-2)b_{i-1}+b_{i-1}+(b_{i-1}-b_{i-2})$, and

(9b) $(b_i-b_{i-1})=(r-2)b_{i-1}+(b_{i-1}-b_{i-2})$.

Use the two identities to induct on the double hypothesis that both

$$b_i>b_{i-1}>0 \text{ and } (b_i-b_{i-1})>(b_{i-1}-b_{i-2})>0.$$ □

Corollary 13 *Let* $r\in\mathbb{R}$ *and* $k\in\mathbb{N}$ *such that* $|r|\ge 2$. *Then* $\lambda(r,k)\ne 0$.

Proof. This is an easy consequence of this lemma and Lemma 10. □

The next two propositions play roles in our analysis.

Lemma 14 *Let* r *be a real number other than* 2. *For* $k\ge 1$,

$$\sum_{i=1}^{k}\lambda(r,i)=\frac{\lambda(r,k+1)-\lambda(r,k)-1}{r-2}. \tag{10}$$

Proof. In what follows, a sum from any integer m to $m-1$ is defined to be 0. For this proof, we abbreviate $\lambda(k)$ for $\lambda(r,k)$.

For each $k\in\mathbb{N}\cup\{0\}$, define

$$s_k=\sum_{i=0}^{k}\lambda(r,i).$$

Then for $k\ge 2$,

$$\begin{aligned}s_k&=\lambda(0)+\lambda(1)+\sum_{i=2}^{k}\left[r\lambda(i-1)-\lambda(i-2)\right]\\&=1+r\cdot\left(\sum_{j=1}^{k-1}\lambda(j)\right)-\sum_{j=0}^{k-2}\lambda(j)\\&=1+r\cdot s_{k-1}-s_{k-2}.\end{aligned}$$

Define a new sequence by $t_i=s_i+\dfrac{1}{r-2}$. Replace $s_i=t_i-\dfrac{1}{r-2}$ into the previous relation to get

$$\forall k\ge 2,\ t_k=r\cdot t_{k-1}-t_k.$$

Hence, $\{t_i\}$ belongs to $\Xi[r]$.

Now

$$t_0=s_0+\frac{1}{r-2}=\frac{1}{r-2}$$
$$t_1=s_1+\frac{1}{r-2}=\frac{r-1}{r-2}.$$

In the vector space $\mathbb{R}^2$,

$$\left(\frac{1}{r-2},\frac{r-1}{r-2}\right)=\frac{1}{r-2}(1,r)-\frac{1}{r-2}(0,1).$$

The sequences t_i and

$$i\mapsto\frac{1}{r-2}\lambda(i+1)-\frac{1}{r-2}\lambda(i)$$

both belong to $\Xi[r]$, and agree on the first two indices. Hence, they are the same sequence. This gives the equality of (10). □

Lemma 15 *Let* r *be a real number, and let* $j,k\in\mathbb{N}$ *such that* $k\ge j$. *Then*

$$\begin{aligned}\lambda(r,k+1)=&\lambda(r,j)\lambda(r,k+2-j)\\&-\lambda(r,j-1)\lambda(r,k+1-j).\end{aligned}$$

Proof. We write $\lambda(i)$ for $\lambda(r,i)$ in this argument. If $k=j$, then $\lambda(k+2-j)=\lambda(2)=r$, $\lambda(k+1-j)=\lambda(1)=1$, and the result follows from the recursive definition.

The remaining cases follow from a proof is by induction on j. The inductive hypothesis is

$$\forall k>j,$$
$$\lambda(k+1)=\lambda(j)\lambda(k+2-j)-\lambda(j-1)\lambda(k+1-j).$$

For $j=1$, this follows from the fact that $\lambda(1)=1$ and $\lambda(0)=0$.

Assume $j\in\mathbb{N}$ for which the inductive hypothesis is true. Let $k\in\mathbb{N}$ so $k>j+1$. Then

$$\begin{aligned}&\lambda(j+1)\lambda(k+2-(j+1))-\lambda(j)\lambda(k+1-(j+1))\\&=\left[r\lambda(j)-\lambda(j-1)\right]\lambda(k+1-j)-\lambda(j)\lambda(k-j)\\&=r\lambda(j)\lambda(k+1-j)-\lambda(j)\lambda(k-j)\\&\quad-\lambda(j-1)\lambda(k+1-j)\\&=\lambda(j)\left[r\lambda(k+1-j)-\lambda(k-j)\right]-\lambda(j-1)\lambda(k+1-j)\\&=\lambda(j)\lambda(k+2-j)-\lambda(j-1)\lambda(k+1-j)\\&=\lambda(k+1).\end{aligned}$$

□

4. The Inverse Matrices

We can now prove

Lemma 16 *Let* $k\in\mathbb{N}$ *and* $r\in\mathbb{R}$. *The matric product* $L[r,k]\cdot M[r,k]$ *is* $-\lambda(-r,k+1)$ *times the identity matrix.*

Proof. For this argument, let $L=L[r,k]$ and $M=M[r,k]$ and, for each $i\in\mathbb{N}\cup\{0\}$, let $\lambda(i)=\lambda(-r,i)$. Let $u,v\in\operatorname{Ind}(k)$. We prove the lemma by comparing the u,v entry of $L\cdot M$ and $-\lambda(k+1)$ times the u,v entry of the identity matrix. There are a lot of cases.

Case $u=1$. For any $v\in\operatorname{Ind}(k)$,

$$\left(L\cdot M\right)_{1,v} = 2M_{1,v} + M_{2,v}.$$

Suppose $v=1$. Recall that $\lambda(1)=1$. Therefore

$$rM_{1,1} + M_{2,1} = r\lambda(k) + \lambda(k-1) = -\lambda(k+1).$$

Suppose $v\neq 1$. Recall that $\lambda(2) = -r$. Then $v \geq 2$, and

$$rM_{1,v} + M_{2,v} = r\lambda(k+1-v) + (-r)\lambda(k+1-v) = 0.$$

Case $u=k$. For any $v\in \mathrm{Ind}(k)$,

$$\left(L\cdot M\right)_{k,v} = M_{k-1,v} + rM_{k,v}.$$

If $v=k$, this is

$$\begin{aligned}&\lambda(k-1)\lambda(1) + r\lambda(k)\lambda(1)\\ &= \lambda(k-1) + r\lambda(k) = -\lambda(k+1).\end{aligned}$$

Now suppose $v\neq k$. Then $v\leq k-1$, and

$$\begin{aligned}\left(L\cdot M\right)_{k,v} &= \lambda(v)\lambda(2) + r\lambda(v)\lambda(1)\\ &= \lambda(v)(-r+r) = 0.\end{aligned}$$

Case $1<u<k$. For any $v\in\mathrm{Ind}(k)$,

$$\left(L\cdot M\right)_{u,v} = M_{u-1,v} + rM_{r,v} + M_{r+1,v}.$$

There are three subcases here. First, suppose $v\leq u-1$. Then

$$\begin{aligned}&\left(L\cdot M\right)_{u,v}\\ &= \lambda(v)\lambda\big(k+1-(u-1)\big) + r\lambda(v)\lambda(k+1-u)\\ &\quad+\lambda(v)\lambda\big(k+1-(u+1)\big)\\ &= \lambda(v)\big(\lambda(k+2-u) + r\lambda(k+1-u) + \lambda(k-u)\big)\end{aligned}$$

The recursive definition states that $\lambda(k+2-u) = -r\lambda(k+1-u) - \lambda(k-u)$. Hence, the expression equals 0.

Next, suppose $v\geq u+1$. Then

$$\begin{aligned}&\left(L\cdot M\right)_{u,v}\\ &= \lambda(u-1)\lambda(k+1-v) + r\lambda(u)\lambda(k+1-v)\\ &\quad+\lambda(u+1)\lambda(k+1-v)\\ &= \lambda(k+1-v)\big(\lambda(u-1) + r\lambda(u) + \lambda(u+1)\big).\end{aligned}$$

Again, the recursive definition implies that this expression is 0.

There remains only the subcase $u=v$.

$$\begin{aligned}&\left(L\cdot M\right)_{u,u}\\ &= \lambda(u-1)\lambda(k+1-u) + r\lambda(u)\lambda(k+1-u)\\ &\quad+\lambda(u)\lambda(k-u)\\ &= \lambda(u-1)\lambda(k+1-u)\\ &\quad+\lambda(u)\big[r\lambda(k+1-u) + \lambda(k-u)\big]\\ &= \lambda(u-1)\lambda(k+1-u) - \lambda(u)\lambda(k+2-u).\end{aligned}$$

By Lemma 15, this equals $-\lambda(k+1)$. □

Lemma 17 *Let* $k\in\mathbb{N}$, $r\in\mathbb{R}$ *and* $j\in\mathrm{Ind}(k)$. *Assume* $r\neq -2$. *Then*

$$\sum_{i=1}^{k} M[r,k]_{i,j} = \sum_{i=1}^{k} M[r,k]_{j,i}$$

equals

$$\frac{\lambda(-r,k+1-j) + \lambda(-r,j) - \lambda(-r,k+1)}{r+2}.$$

Proof. Put $M = M[r,k]$ and, for each index i, $\lambda(i) = \lambda(-r,i)$. Split the sum from $i=1$ to k of $M_{i,j}$ at index j:

$$\begin{aligned}\sum_{i=1}^{k} M_{i,j} &= \sum_{i=1}^{j}\lambda(i)\lambda(k+1-j) + \sum_{i=j+1}^{k}\lambda(j)\lambda(k+1-i)\\ &= \lambda(k+1-j)\sum_{i=1}^{j}\lambda(i) + \lambda(j)\sum_{i=j+1}^{k}\lambda(k+1-i)\end{aligned}$$

In the previous line, the first sum is determined by Lemma 14. Recall the parameter is $-r$, not r

$$\begin{aligned}\lambda(k+1-j)\sum_{i=1}^{j}\lambda(i) &= \frac{\lambda(k+1-j)}{-r-2}\big(\lambda(j+1) - \lambda(j) - 1\big)\\ &= \frac{\lambda(k+1-j)}{r+2}\big(\lambda(j) - \lambda(j+1) + 1\big)\end{aligned}$$

In the second sum, change index to $p = k+1-i$. One can use the same Lemma.

$$\begin{aligned}&\lambda(j)\sum_{i=j+1}^{k}\lambda(k+1-i)\\ &= \lambda(j)\sum_{p=1}^{k-j}\lambda(p)\\ &= \frac{\lambda(j)}{r+2}\big(\lambda(k-j) - \lambda(k+1-j) + 1\big).\end{aligned}$$

Add the two terms to get

$$\begin{aligned}&\sum_{i=1}^{k} M_{i,j}\\ &= \frac{1}{r+2}\big(\lambda(k+1-j)\lambda(j) - \lambda(k+1-j)\lambda(j+1)\\ &\quad+\lambda(k+1-j) + \lambda(j)\lambda(k-j)\\ &\quad-\lambda(j)\lambda(k+1-j) + \lambda(j)\big)\\ &= \frac{1}{r+2}\big(\lambda(j)\lambda(k-j) - \lambda(k+1-j)\lambda(j+1)\big)\\ &\quad+\lambda(k+1-j) + \lambda(j)\end{aligned}$$

By Lemma 15, this is the stated formula. □

At last, we introduce weights. Define ω_j as in the statement of Lemma 2.

Corollary 18 *Let* $k\in\mathbb{N}$, $r\in\mathbb{R}$. *Assume* $r\geq 2$, *and let* $\{\omega_j\}$ *be the weight system for* r,k.

(A) *If* $r = 2$ *and* k is *odd and* j *is even*, *then* $\omega_j = 0$.

(B) *If* $r = 2$ *and either* k *is even or* j *is odd*, *then* $\omega_j > 0$.

(C) *If* $r > 2$, *then* $\omega_j > 0$.

(D) *Let* $x \in \mathbb{R}^n$. *Expand* $L[r,k] \cdot x$ *as* $(b_1, \cdots, b_n)^{\mathrm{T}}$. *Then*

$$sum(x) = \frac{1}{(r+2)\lambda(r,k+1)} \sum_{j=1}^{k} \omega_j \cdot b_j. \quad (11)$$

Proof. We start with Part (D), as that is our motivation. Given

$$x = (x_1, \cdots, x_n) \text{ and } b = L[r,k] \cdot x = (b_1, \cdots, b_k),$$

it follows that

$$x = L[r,k]^{-1} b.$$

By Lemma 16, for each $1 \le i \le k$,

$$x_i = \frac{1}{-\lambda(-r,k+1)} \sum_{j=1}^{k} M[r,k]_{i,j} b_j.$$

From Lemma 17,

$$sum(x)$$
$$= \frac{1}{-\lambda(-r,k+1)} \sum_{i,j} M[r,k] \cdot b_j$$
$$= \sum_{j=1}^{k} \frac{\lambda(-r,k+1) - \lambda(-r,k+1-j) - \lambda(-r,j)}{(r+2)\lambda(-r,k+1)} \cdot b_j.$$

Now replace each $\lambda(-r,i)$ by $(-1)^{i+1} \lambda(r,i)$. The b_j-coefficient becomes $\omega_j / ((r+2)\lambda(r,k+1))$.

Recall Lemma 12. Then $\{\lambda(r,i)\}_i$ is a non-negative and convex sequence, and $\lambda(r,0) = 0$. Convexity implies that

$$\lambda(r,k+1) + (-1)^{j+1} \lambda(r,k+1-j) + (-1)^{k+j} \lambda(r,j)$$

is positive *unless*

(12a) $j+1$ and $k+j$ are both odd, and

(12b) $\{\lambda(r,i)\}$ is not strictly convex.

This remark establishes all our conclusions except in the case when $r = 2$, k is odd and j is even. Assume these parameters, and we know $\lambda(2,i) = i$ for all i, and (A) follows. □

This corollary proves Lemma 2.

Corollary 19 *Let* $k \in \mathbb{N}$, $r \in \mathbb{R}$. *Assume* $r \ge 2$, *and let* $\{\omega_j\}$ *be the overlap weights for* (k,r). *Let* $\hat{1}$ *be the vector in which every entry is* 1, *that is* $(1,1,\cdots,1)$. *Then*

$$sum\left(L[r,k]^{-1} \cdot \hat{1}\right) = \mu(r,k),$$

where $\mu(r,k)$ *is defined in Definition* 3.

Proof. The easiest way is to get this formula is

(13a) Start with the formulas in Lemma 17;

(13b) Sum the terms over j using Lemma 14; and

(13c) Convert all $\lambda(-r,i)$ to $(-1)^{i+1} \lambda(r,i)$. □

The observation of (6) completes the proof of all the propositions in Section 1.3.

5. When *r* = 2

The numerical calculations allow us to add some secondary comments on the examples of Sections and 1.1 and 1.2. Fix $r = 2$, and put $L = L[2,k]$. Then $\lambda(2,i) = i$ for all indices i. If Z is a perfect totally dominant subset of $P_k \Box C_n$ and z is its row-count vector, then $L \cdot z = n\hat{1}$.

If k is odd,

$$L^{-1}(n\hat{1}) = (n/2, 0, n/2, 0, \cdots, n/2).$$

Consequently, a perfect totally dominant subset cannot exist if n is odd. However, since $\omega_j = 0$ for j even, there may be totally dominant subsets whose size "ties" the estimate for a perfect subset. In the case $k = 3$, the set consisting of the middle row has row-count $(0,n,0)$. Its image under L is $(n,3n,n)$.

If k is even, the i-th coordiante of $L^{-1} \cdot (n\hat{1})$ is

$$\frac{n}{2(k+1)} \text{ times } \begin{cases} k+1-i & \text{if } i \text{ is odd}, \\ i & \text{if } i \text{ is even}. \end{cases}$$

The entries are integral if and only if $k+1$ divides *n*. Unlike the case when k is odd, $sum\left(L^{-1}(n\hat{1})\right)$ cannot be matched by the size of an imperfect dominant subset.

Near Perfect

Now

$$\mu(2,k) = \begin{cases} \dfrac{2k^2 + 4k}{8(k+1)} & \text{if } k \text{ is even}, \\ \dfrac{k+1}{4} & \text{if } k \text{ is odd}. \end{cases}$$

Proposition 20 *For* $k, n \ge 4$, $\mu(2,k) n \le \gamma_t(P_k \Box P_n) \le \mu(2,k)(n+2)$.

Proof. There is $d \in \mathbb{N}$ and $Z \subseteq P_k \Box C_d$ such that

(14a) $n+2$ divides d,

(14b) Z is a totally dominant subset of $P_k \Box C_d$,

(14c) $|Z| = \mu(2,k) d$.

Partition $P_k \Box C_m$ into subsets $Y_1, \cdots, Y_m$ where each Y_i consists of $n+2$ successive columns. For at least one index i, $|Y_i \cap Z| \le \mu(2,k)(n+2)$. Choose such an index. Identify $P_k \Box P_n$ with Y', the subgraph of columns 2 through $n+1$ of Y_i. Let $Z_1 = Z \cap Y'$. Any member of Y' which is not dominated by Z_1 is dominated by exactly one member of Z in either the 1st or

$n+2$ column; furthermore, each member of either column dominates just one member of Y'. Consequently, we can expand Z_1 to a totally dominant Z_2 for Y' of size $\le |Y_i \cap Z|$. □

6. Extended Functigraphs

Our lower bound uses only a few aspects of the graphs $P_k \square H$. Consequently, the calculation applies to a slightly larger family of graphs.

Fix $k,r,n \in \mathbb{N}$ with $n,k,r > 2$. Let $H_1,\cdots,H_k$ be a list of r-regular graphs, each with n vertices. For each $1 \le i < k$, let $h_i : V(H_i) \to V(H_{i+1})$ be a bijection. Define the *extended functigraph* on this data to be G in which

(15a) $V(G)$ is the (disjoint) union $\bigcup_{i=1}^{k} V(H_i)$, and

(15b) $E(G)$ is union of $\bigcup_{i=1}^{k} E(H_i)$ with

$$\{vh_i(v) : 1 \le i < k \wedge v \in V(H_i)\}.$$

Then the assertions of Theorem 4, and its Corollary, apply to G.

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