Open Journal of Discrete Mathematics, 2013, 3, 130-136
http://dx.doi.org/10.4236/ojdm.2013.33024 Published Online July 2013 (http://www.scirp.org/journal/ojdm)

The Number of Canalyzing Functions over Any Finite Set*

Yuan Li[1#], David Murrugarra[2], John O. Adeyeye[1], Reinhard Laubenbacher[3]
[1]Department of Mathematics, Winston-Salem State University, Winston-Salem, USA
[2]School of Mathematics, Georgia Tech, Atlanta, USA
[3]Virginia Bioinformatics Institute, Virginia Tech, Blacksburg, USA
Email: #liyu@wssu.edu

Received March 1, 2013; revised May 20, 2013; accepted June 16, 2013

ABSTRACT

In this paper, we extend the definition of Boolean canalyzing functions to the canalyzing functions of multi-state case. Namely, $f:Q^n \to Q$, where $Q=\{a_1,a_2,\cdots,a_q\}$. We obtain its cardinality and the cardinalities of its various subsets (They may not be disjoint). When $q=2$, we obtain a combinatorial identity by equating our result to the formula in [1]. For a better understanding to the magnitude, we obtain the asymptotes for all the cardinalities as either $n\to\infty$ or $q\to\infty$.

Keywords: Canalyzing Function; Inclusion and Exclusion Principle

1. Introduction

The idea of canalization was initiated from Waddington, C. H. [2]. When comparing the class of canalyzing functions to other classes of functions with respect to their evolutionary plausibility as emergent control rules in genetic regulatory systems, it is informative to know the number of canalyzing functions with a given number of input variables [1]. However, the Boolean network modeling paradigm is rather restrictive, with its limit to two possible functional levels, ON and OFF, for genes, proteins, etc. Many discrete models of biological networks therefore allow variables to take on multiple states. Common used discrete multi-state model types are so-called logical models [3], Petri nets [4], and agent- based models [5].

In this paper, we generalize the concept of Boolean canalyzing rules to the multi-state case. By generalizing the results in [1], we provide formulas for the cardinalities of various subsets of canalyzing functions. We also obtain the asymptotes of these cardinalities as either $n\to\infty$ or $q\to\infty$. We obtain a combinatorial identity by equating our result to the formula in [1].

2. Preliminaries

In this section we introduce the definition of a *canalyzing function*.

Let $[n]=\{1,2,\cdots,n\}, Q=\{a_1,a_2,\cdots,a_q\}$ and $f:Q^n\to Q$.

A function is canalyzing if there is a variable x_i and an element $a\in Q$ so that the value of the function is fixed once variable x_i is fixed at a. More precisely, we have the following definitions.

Definition 2.1

1) The function $f(x_1,x_2,\cdots,x_n)$ is $\langle i:a:b\rangle$ canalyzing if $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

2) The function $f(x_1,x_2,\cdots,x_n)$ is $\langle i:a:-\rangle$ canalyzing if there exists $b\in Q$ such that $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

3) The function $f(x_1,x_2,\cdots,x_n)$ is $\langle -:a:b\rangle$ canalyzing if there exists $i\in[n]$ such that $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

4) The function $f(x_1,x_2,\cdots,x_n)$ is $\langle i:-:b\rangle$ canalyzing if there exists $a\in Q$ such that $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

5) The function $f(x_1,x_2,\cdots,x_n)$ is $\langle -:a:-\rangle$ canalyzing if there exist $i\in[n], b\in Q$ such that $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

6) The function $f(x_1,x_2,\cdots,x_n)$ is $\langle i:-:-\rangle$

---
*Supported by an award from the USA DoD # W911NF-11-10166.
#Corresponding author.

canalyzing if there exist a, $b \in Q$ such that $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

7) The function $f(x_1,x_2,\cdots,x_n)$ is $\langle -:-:b\rangle$ canalyzing if there exist $i\in[n], a\in Q$ such that $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

8) The function is $f(x_1,x_2,\cdots,x_n)$ is $\langle -:-:-\rangle$ canalyzing if there exist $i\in[n], a,b\in Q$ such that $f(x_1,\cdots,x_{i-1},a,x_{i+1},\cdots,x_n)=b$, for all $x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n$.

By abuse of notation, we also use $\langle i:-:b\rangle$ to stand for the set of all the $\langle i:-:b\rangle$ canalyzing functions, $\langle i:a:b\rangle$ will stand for the set of all the $\langle i:a:b\rangle$ canalyzing functions and etc. We use ϕ to stand for the empty set.

By the definitions, we immediately have the following propositions.

Proposition 2.2 *If* $b_1\neq b_2$, *then* $\langle i:a:b_1\rangle\cap\langle i:a:b_2\rangle=\phi$.

Proposition 2.3 *If* $b_1\neq b_2$ *and* $i_1\neq i_2$, *then* $\langle i_1:-:b_1\rangle\cap\langle i_2:-:b_2\rangle=\phi$.

By the definitions, we have

$$\langle -:-:-\rangle=\bigcup_{b\in Q}\langle -:-:b\rangle=\bigcup_{a\in Q}\langle -:a:-\rangle=\bigcup_{i\in[n]}\langle i:-:-\rangle,$$

$$\langle -:-:b\rangle=\bigcup_{i\in[n]}\langle i:-:b\rangle=\bigcup_{a\in Q}\langle -:a:b\rangle,$$

$$\langle -:a:-\rangle=\bigcup_{b\in Q}\langle -:a:b\rangle=\bigcup_{i\in[n]}\langle i:a:-\rangle,$$

$$\langle i:-:-\rangle=\bigcup_{a\in Q}\langle i:a:-\rangle=\bigcup_{b\in Q}\langle i:-:b\rangle,$$

$$\langle i:-:b\rangle=\bigcup_{a\in Q}\langle i:a:b\rangle,$$

$$\langle i:a:-\rangle=\bigcup_{b\in Q}\langle i:a:b\rangle,$$

$$\langle -:a:b\rangle=\bigcup_{i\in[n]}\langle i:a:b\rangle.$$

For any set S, we use $|S|$ to stand for its cardinality. We use $C(n,k)=\frac{n!}{k!(n-k)!}$ to stand for the binomial coefficients. As usual, $C(n,k)$ should be explained as zero once $k>n$.

Obviously, for the above notations, the cardinality are same for different values of i,a and b. In other words, we have $|\langle i_1:a_1:b_1\rangle|=|\langle i_2:a_2:b_2\rangle|$, $|\langle i:-:b\rangle|=|\langle j:-:c\rangle|$, $|\langle -:a:b\rangle|=|\langle -:c:d\rangle|$ and etc.

3. Enumeration

Theorem 3.1 *Given* $i\in[n], a,b\in Q$, *the number of* $\langle i:a:b\rangle$ *canalyzing functions is* $q^{q^n-q^{n-1}}$. *In other words, we have* $|\langle i:a:b\rangle|=q^{q^n-q^{n-1}}$.

Proof: A function in the set $\langle i:a:b\rangle$ is uniquely determined by its value on inputs $(x_1,\cdots,x_n)$ with $x_i\neq a$. There are $(q-1)q^{n-1}=q^n-q^{n-1}$ such inputs, and the function can take q different values. Thus $|\langle i:a:b\rangle|=q^{q^n-q^{n-1}}$. □

Because $\langle i:a:-\rangle=\bigcup_{b\in Q}\langle i:a:b\rangle$, by Proposition 2.2, we get

Theorem 3.2 *The number of all the* $\langle i:a:-\rangle$ *canalyzing function is*

$$q\left(q^{q^n-q^{n-1}}\right)=q^{q^n-q^{n-1}+1}.$$

Lemma 3.3 *We have* $\left|\bigcap_{j=1}^{k}\langle i:a_j:b\rangle\right|=q^{q^n-kq^{n-1}}$ *for any* $\{a_1,a_2,\cdots,a_k\}\subset Q$.

Proof: A function in the set $\bigcap_{j=1}^{k}\langle i:a_j:b\rangle$ is uniquely determined by it values on inputs $(x_1,\cdots,x_n)$ with $x_i\notin\{a_1,\cdots,a_k\}$. There are $(q-k)q^{n-1}=q^n-kq^{n-1}$ such inputs. □

Theorem 3.4 *Given* $i\in[n]$ *and* $b\in Q$, *the number of* $\langle i:-:b\rangle$ *canalyzing functions is* $q^{q^n}-\left(q^{q^{n-1}}-1\right)^q$. *In other words, we have* $|\langle i:-:b\rangle|=q^{q^n}-\left(q^{q^{n-1}}-1\right)^q$.

Proof: By Inclusion and Exclusion Principle, we have

$$\begin{aligned}|\langle i:-:b\rangle|&=\left|\bigcup_{a\in Q}\langle i:a:b\rangle\right|\\&=\bigcup_{a\in Q}|\langle i:a:b\rangle|-\sum_{\{a_1,a_2\}\subset Q}|\langle i:a_1:b\rangle\cap\langle i:a_2:b\rangle|\\&\quad+\cdots+(-1)^{k-1}\sum_{\{a_1,a_2,\cdots,a_k\}\subset Q}\left|\bigcap_{j=1}^{k}\langle i:a_j:b\rangle\right|+\cdots+(-1)^{q-1}\\&=C(q,1)q^{q^n-q^{n-1}}-C(q,2)q^{q^n-2q^{n-1}}\\&\quad+\cdots+(-1)^{k-1}C(q,k)q^{q^n-kq^{n-1}}+\cdots+(-1)^{q-1}\\&=\sum_{k=1}^{q}(-1)^{k-1}C(q,k)q^{q^n-kq^{n-1}}\\&=q^{q^n}\sum_{k=1}^{q}\left(-C(q,k)\left(-q^{-q^{n-1}}\right)^k\right)\\&=q^{q^n}\left(1-\left(1-q^{-q^{n-1}}\right)^q\right)=q^{q^n}-\left(q^{q^{n-1}}-1\right)^q.\end{aligned}$$ □

Similar to Lemma 3.3, we have

Lemma 3.5 *If* $\{i_1,i_2,\cdots,i_k\}\subset[n]$, *then* $\left|\bigcap_{j=1}^{k}\langle i_j:a:b\rangle\right|=q^{(q-1)^k q^{n-k}}$.

Based on this lemma, we can get the following result.

Theorem 3.6 *We have*

$$|\langle -:a:b\rangle|=\sum_{1\le k\le n}(-1)^{k-1}C(n,k)q^{(q-1)^k q^{n-k}}.$$

Proof: By Inclusion and Exclusion Principle, we have

$$\left|\langle -:a:b\rangle\right| = \left|\bigcup_{i\in[n]}\langle i:a:b\rangle\right|$$
$$= \sum_{1\le i\le n}\left|\langle i:a:b\rangle\right| - \sum_{1\le i<j\le n}\left|\langle i:a:b\rangle\cap\langle j:a:b\rangle\right|$$
$$+\cdots+(-1)^{k-1}$$
$$\cdot\sum_{1\le i_1<\cdots<i_k\le n}\left|\bigcap_{j=1}^{k}\langle i_j:a:b\rangle\right|+\cdots+(-1)^{n-1}\left|\bigcap_{j=1}^{n}\langle j:a:b\rangle\right|$$
$$= C(n,1)q^{(q-1)q^{n-1}} - C(n,2)q^{(q-1)^2q^{n-2}}$$
$$+\cdots+(-1)^{k-1}C(n,k)q^{(q-1)^kq^{n-k}}+\cdots+(-1)^{n-1}q^{(q-1)^n}$$
$$= \sum_{1\le k\le n}(-1)^{k-1}C(n,k)q^{(q-1)^kq^{n-k}}. \qquad \square$$

From the above theorem, we can get the following result.

Theorem 3.7 *We have*

$$\left|\langle -:a:-\rangle\right| = q\sum_{1\le k\le n}(-1)^{k-1}C(n,k)q^{(q-1)^kq^{n-k}}$$

Proof: Because $\langle -:a:-\rangle = \bigcup_{b\in Q}\langle -:a:b\rangle$, by Theorem 3.6, we just need to show $\langle -:a:b_1\rangle\cap\langle -:a:b_2\rangle=\phi$ if $b_1\ne b_2$. Suppose $f\in\langle -:a:b_1\rangle\cap\langle -:a:b_2\rangle$, then there exist i_1 and $i_2\in[n]$ such that $f\in\langle i_1:a:b_1\rangle\cap\langle i_2:a:b_2\rangle$ since $\langle -:a:b\rangle=\bigcup_{i\in[n]}\langle i:a:b\rangle$. If $i_1=i_2$, we get a contradiction by Proposition 2.2. If $i_1\ne i_2$, we get a contradiction by Proposition 2.3 since $\langle i_1:a:b_1\rangle\subset\langle i_1:-:b_1\rangle$ and $\langle i_2:a:b_2\rangle\subset\langle i_2:-:b_2\rangle$. □

Now, we are going to find the formula for the number of all the canalyzing functions with given canalyzed value b. In other words, the formula of $\left|\langle -:-:b\rangle\right|$.

Let $S_b=\{\langle i:a:b\rangle \mid i\in[n],a\in Q\}$ for any $b\in Q$. By Inclusion and Exclusion Principle, we have

$$\left|\langle -:-:b\rangle\right| = \left|\bigcup_{i\in[n]}\bigcup_{a\in Q}\langle i:a:b\rangle\right| = \sum_{k=1}^{nq}(-1)^{k-1}N_k.$$

where

$$N_k = \sum_{s\subset S_b,|s|=k}\left|\bigcap_{T\in s}T\right|.$$

In order to evaluate N_k, we write all the members in S_b as the following $n\times q$ matrix.

$$A=\begin{pmatrix}\langle 1:a_1:b\rangle & \langle 1:a_2:b\rangle & \cdots & \langle 1:a_q:b\rangle\\ \langle 2:a_1:b\rangle & \langle 2:a_2:b\rangle & \cdots & \langle 2:a_q:b\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle n:a_1:b\rangle & \langle n:a_2:b\rangle & \cdots & \langle n:a_q:b\rangle\end{pmatrix}$$

For any $s\subset S_b$ with $|s|=k$, we will choose k elements from the above matrix to form s.

Suppose k_1 of its elements are from the first row (there are $C(q,k_1)$ ways to do so). Let these k_1 elements be $\langle 1:a_{11}:b\rangle,\langle 1:a_{12}:b\rangle,\cdots,\langle 1:a_{1k_1}:b\rangle$.

Suppose k_2 of its elements are from the second row (there are $C(q,k_2)$ ways to do so). Let these k_2 elements be $\langle 2:a_{21}:b\rangle,\langle 2:a_{22}:b\rangle,\cdots,\langle 2:a_{2k_2}:b\rangle$.

$\cdots$

Suppose k_n of its elements are from the last row (there are $C(q,k_n)$ ways to do so). Let these k_n elements be $\langle n:a_{n1}:b\rangle,\langle n:a_{n2}:b\rangle,\cdots,\langle n:a_{nk_n}:b\rangle$.

$$k_1+k_2+\cdots+k_n=k, 0\le k_i\le q, i=1,2,\cdots,n.$$

Similar to Lemma 3.3, we have

Lemma 3.8 *Let* s *be the subset of* S_b *as mentioned above, then* $\left|\bigcap_{T\in s}T\right| = q^{(q-k_1)(q-k_2)\cdots(q-k_n)}$.

Hence,

$$N_k = \sum_{k_1+\cdots+k_n=k}C(q,k_1)\cdots C(q,k_n)q^{(q-k_1)\cdots(q-k_n)}.$$

We get

Theorem 3.9 *For any* $b\in Q$, *we have*

$$\left|\langle -:-:b\rangle\right|$$
$$=\sum_{k=1}^{nq}(-1)^{k-1}\left(\sum_{k_1+k_2+\cdots+k_n=k}\left(\prod_{j=1}^{n}C(q,k_j)\right)q^{\prod_{j=1}^{n}(q-k_j)}\right).$$

In order to evaluate $\langle i:-:-\rangle$, we need two more lemmas. Their proofs are similar to that of Lemma 3.3 and we omit them.

Lemma 3.10 *If* $\{a_1,a_2,\cdots,a_k\}\subset Q$ *and* $\{b_1,b_2,\cdots,b_k\}\subset Q$, *then*

$$\left|\bigcap_{j=1}^{k}\langle i:a_j:b_j\rangle\right| = q^{(q-k)q^{n-1}}.$$

Lemma 3.11 *If* $a_{11},\cdots,a_{1k_1};a_{21},\cdots,a_{2k_2};\cdots;a_{r1},\cdots,a_{rk_r}$ *are* $k_1+\cdots+k_r$ *distinct elements of* Q, $\{b_1,\cdots,b_r\}\subset Q$. *Then,*

$$\left|\left(\bigcap_{j=1}^{k_1}\langle i:a_{1j}:b_1\rangle\right)\cap\left(\bigcap_{j=1}^{k_2}\langle i:a_{2j}:b_2\rangle\right)\cdots\cap\left(\bigcap_{j=1}^{k_r}\langle i:a_{rj}:b_r\rangle\right)\right|$$
$$= q^{(q-k_1-k_2-\cdots-k_r)q^{n-1}}$$

Now, we are ready to find the cardinality of $\langle i:-:-\rangle$.

Theorem 3.12 *We have*

$$\left|\langle i:-:-\rangle\right|$$
$$= q!\sum_{k=1}^{q}\frac{(-1)^{k-1}q^{(q-k)q^{n-1}}}{(q-k)!}\left(\sum_{k_1+\cdots+k_q=k,0\le k_i\le q}\frac{1}{k_1!k_2!\cdots k_q!}\right).$$

Proof: First, we have $\langle i:-:-\rangle=\bigcup_{a\in Q}\bigcup_{b\in Q}\langle i:a:b\rangle$.

Let $S_i = \{\langle i:a:b\rangle \mid a,b\in Q\}$, we get $\left|\langle i:-:-\rangle\right| = \sum_{k=1}^{q^2}(-1)^{k-1}N_k$. Where

$$N_k = \sum_{s\subset S_i,|s|=k}\left|\bigcap_{T\in s}T\right|.$$

In order to evaluate N_k, we write all the elements in S_i as the following $q\times q$ matrix.

$$B=\begin{pmatrix}\langle i:a_1:b_1\rangle & \langle i:a_2:b_1\rangle & \cdots & \langle i:a_q:b_1\rangle\\ \langle i:a_1:b_2\rangle & \langle i:a_2:b_2\rangle & \cdots & \langle i:a_q:b_2\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle i:a_1:b_q\rangle & \langle i:a_2:b_q\rangle & \cdots & \langle i:a_q:b_q\rangle\end{pmatrix}$$

For any $s\subset S_i$ with $|s|=k$, we will choose k elements from the above matrix to form s.

Suppose k_1 of it elements are from the first row (There are $C(q,k_1)$ ways to do so). Let these k_1 elements be $\langle i:a_{11}:b_1\rangle,\langle i:a_{12}:b_1\rangle,\cdots,\langle i:a_{1k_1}:b_1\rangle$.

Suppose k_2 of its elements are from the second row, we must choose these elements from different columns, otherwise the intersection will be ϕ by Proposition 2.2 (There are $C(q-k_1,k_2)$ ways to do so). Let these k_2 elements be $\langle i:a_{21}:b_2\rangle,\langle i:a_{22}:b_2\rangle,\cdots,\langle i:a_{2k_2}:b_2\rangle$

$\cdots$

Suppose k_q of its elements are from the last row (There are $C(q-k_1-k_2-\cdots-k_{q-1},k_q)$ ways to do so). Let these k_q elements be

$\langle i:a_{q1}:b_q\rangle,\langle i:a_{q2}:b_q\rangle,\cdots,\langle i:a_{qk_q}:b_q\rangle$.

where $k_1+k_2+\cdots+k_q=k, 0\le k_i\le q, i=1,2,\cdots,q$. We have

$$\begin{aligned}N_k &= \sum_{s\subset S_i,|s|=k}\left|\bigcap_{T\in s}T\right|\\ &= \sum_{k_1+\cdots+k_q=k,0\le k_i\le q}C(q,k_1)C(q-k_1,k_2)\\ &\qquad\cdots C(q-k_1-\cdots-k_{q-1},k_q)I_{k_1k_2\cdots k_q}\end{aligned}$$

where

$$I_{k_1k_2\cdots k_q}=$$
$$\left|\left(\bigcap_{j=1}^{k_1}\langle i:a_{1j}:b_1\rangle\right)\cap\left(\bigcap_{j=1}^{k_2}\langle i:a_{2j}:b_2\rangle\right)\cdots\cap\left(\bigcap_{j=1}^{k_q}\langle i:a_{qj}:b_q\rangle\right)\right|$$

By Lemma 3.11, we know

$$I_{k_1k_2\cdots k_q}=q^{(q-k_1-k_2-\cdots-k_q)q^{n-1}}=q^{(q-k)q^{n-1}}.$$

This number is zero if $k>q$.

A straightforward computing shows that

$$C(q,k_1)C(q-k_1,k_2)\cdots C(q-k_1-\cdots-k_{q-1},k_q)=\frac{q!}{k_1!k_2!\cdots k_q!(q-k)!}.$$

Hence, we get

$$\begin{aligned}&\left|\langle i:-:-\rangle\right|\\ &=\sum_{k=1}^{q^2}(-1)^{k-1}N_k=\sum_{k=1}^{q}(-1)^{k-1}N_k\\ &=\sum_{k=1}^{q}(-1)^{k-1}\left(\sum_{k_1+\cdots+k_q=k,0\le k_i\le q}\frac{q!}{k_1!k_2!\cdots k_q!(q-k)!}q^{(q-k)q^{n-1}}\right)\\ &=q!\sum_{k=1}^{q}\frac{(-1)^{k-1}q^{(q-k)q^{n-1}}}{(q-k)!}\sum_{k_1+\cdots+k_q=k,0\le k_i\le q}\frac{1}{k_1!k_2!\cdots k_q!}.\end{aligned}$$

□

Now we begin to evaluate $\left|\langle -:-:-\rangle\right|$.

Theorem 3.13 *We have*

$$\left|\langle -:-:-\rangle\right|=\sum_{k=1}^{q}(-1)^{k-1}U_k+\sum_{k=1}^{nq}(-1)^{k-1}V_k.$$

where

$$\begin{aligned}U_k &= n\sum_{t_1+t_2+\cdots+t_q=k,0\le t_i\le q}\frac{q!}{t_1!t_2!\cdots t_q!(q-k)!}q^{(q-k)q^{n-1}}\\ &=\frac{nq!}{(q-k)!}q^{(q-k)q^{n-1}}\sum_{t_1+t_2+\cdots+t_q=k,0\le t_i\le q}\frac{1}{t_1!t_2!\cdots t_q!}\end{aligned}$$

and

$$V_k=q\sum_{k_1+\cdots+k_n=k,0\le k_i\le k-1,0\le k_i\le q}\left(\prod_{j=1}^{n}C(q,k_j)\right)q^{\prod_{j=1}^{n}(q-k_j)}$$

Proof: Let

$$S=\{\langle i:a:b\rangle \mid a,b\in Q, i\in[n]\}.$$

We have

$$\langle -:-:-\rangle=\bigcup_{i\in[n]}\bigcup_{a\in Q}\bigcup_{b\in Q}\langle i:a:b\rangle$$

and $\left|\langle -:-:-\rangle\right|=\sum_{k=1}^{nq^2}(-1)^{k-1}N_k$, where

$$N_k=\sum_{s\subset S,|s|=k}\left|\bigcap_{T\in s}T\right|.$$

We write all the nq^2 elements of *S* as the following n matrices.

$$M_1=\begin{pmatrix}\langle 1:a_1:b_1\rangle & \langle 1:a_1:b_2\rangle & \cdots & \langle 1:a_1:b_q\rangle\\ \langle 1:a_2:b_1\rangle & \langle 1:a_2:b_2\rangle & \cdots & \langle 1:a_2:b_q\rangle\\ \vdots & \vdots & \iddots & \vdots\\ \langle 1:a_q:b_1\rangle & \langle 1:a_q:b_2\rangle & \cdots & \langle 1:a_q:b_q\rangle\end{pmatrix}$$

$$M_2=\begin{pmatrix}\langle 2:a_1:b_1\rangle & \langle 2:a_1:b_2\rangle & \cdots & \langle 2:a_1:b_q\rangle\\ \langle 2:a_2:b_1\rangle & \langle 2:a_2:b_2\rangle & \cdots & \langle 2:a_2:b_q\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle 2:a_q:b_1\rangle & \langle 2:a_q:b_2\rangle & \cdots & \langle 2:a_q:b_q\rangle\end{pmatrix}$$

$$\vdots$$

$$M_n=\begin{pmatrix}\langle n:a_1:b_1\rangle & \langle n:a_1:b_2\rangle & \cdots & \langle n:a_1:b_q\rangle\\ \langle n:a_2:b_1\rangle & \langle n:a_2:b_2\rangle & \cdots & \langle n:a_2:b_q\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle n:a_q:b_1\rangle & \langle n:a_q:b_2\rangle & \cdots & \langle n:a_q:b_q\rangle\end{pmatrix}$$

We combine all the above M_i to form a $nq\times q$ matrix M whose first q rows are M_1, the second q rows are $M_2,\cdots$, the last q rows are M_n. In other words, we have

$$M=\begin{pmatrix}\langle 1:a_1:b_1\rangle & \langle 1:a_1:b_2\rangle & \cdots & \langle 1:a_1:b_q\rangle\\ \langle 1:a_2:b_1\rangle & \langle 1:a_2:b_2\rangle & \cdots & \langle 1:a_2:b_q\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle 1:a_q:b_1\rangle & \langle 1:a_q:b_2\rangle & \cdots & \langle 1:a_q:b_q\rangle\\ \langle 2:a_1:b_1\rangle & \langle 2:a_1:b_2\rangle & \cdots & \langle 2:a_1:b_q\rangle\\ \langle 2:a_2:b_1\rangle & \langle 2:a_2:b_2\rangle & \cdots & \langle 2:a_2:b_q\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle 2:a_q:b_1\rangle & \langle 2:a_q:b_2\rangle & \cdots & \langle 2:a_q:b_q\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle n:a_1:b_1\rangle & \langle n:a_1:b_2\rangle & \cdots & \langle n:a_1:b_q\rangle\\ \langle n:a_2:b_1\rangle & \langle n:a_2:b_2\rangle & \cdots & \langle n:a_2:b_q\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle n:a_q:b_1\rangle & \langle n:a_q:b_2\rangle & \cdots & \langle n:a_q:b_q\rangle\end{pmatrix}$$

We are going to choose k elements from M to form the intersection. In order to get a possible non empty intersection, we know all these k elements must come from either the same M_i (for some fixed i) or all of them from the same column of M by Proposition 2.3.

Each M_i is in fact the transpose of B and each column of M is all the elements of A (As sets, they are equal). Hence, a typical intersection is either the one in Theorem 3.9 or the one in Theorem 3.12. But these two cases are not disjoint.

Suppose we choose k_i elements from

$$M_i, i=1,2,\cdots,n, k_1+k_2+\cdots+k_n=k,\\ 0\le k_i\le k, i=1,2,\cdots,n.$$

If there exist i such that $k_i=k$, then $k_j=0,\forall j\ne i$. This implies the intersection looks like the one in Lemma 3.11 and $k\le q$.

If $0\le k_i\le k-1,\forall i\in[n]$, then the intersection looks like the one in Lemma 3.8 and $k\le nq$.

The above two cases are disjoint now. By Lemma 3.11 and Lemma 3.8, we get

$$N_k=\sum_{s\subset S,|s|=k}\left|\bigcap_{T\in s}T\right|=\sum_{k_1+\cdots+k_n=k,0\le k_i\le k}\\ =\sum_{\exists i,k_i=k}+\sum_{k_i\le k-1,i=1,\cdots,n}=U_k+V_k$$

where (Note: there are n matrices $M_1,M_2,\cdots,M_n$ and q columns of M)

$$U_k=n\sum_{t_1+t_2+\cdots+t_q=k,0\le t_i\le q}\frac{q!}{t_1!t_2!\cdots t_q!(q-k)!}q^{(q-k)q^{n-1}}\\ =\frac{nq!}{(q-k)!}q^{(q-k)q^{n-1}}\sum_{t_1+t_2+\cdots+t_q=k,0\le t_i\le q}\frac{1}{t_1!t_2!\cdots t_q!}$$

$$V_k=q\sum_{k_1+\cdots+k_n=k,0\le k_i\le k-1,0\le k_i\le q}\left(\prod_{j=1}^{n}C(q,k_j)\right)q^{\prod_{j=1}^{n}(q-k_j)}$$

Hence,

$$\left|\langle -:-:-\rangle\right|=\sum_{k=1}^{nq^2}(-1)^{k-1}N_k\\ =\sum_{k=1}^{nq^2}(-1)^{k-1}(U_k+V_k)=\sum_{k=1}^{q}(-1)^{k-1}U_k+\sum_{k=1}^{nq}(-1)^{k-1}V_k,$$

□

In the following, we will reduce the formula

$$\left|\langle -:-:-\rangle\right|$$

when $q=2$ and compare it with the one in [1]. We have

$$\left|\langle -:-:-\rangle\right|=\sum_{k=1}^{2}(-1)^{k-1}U_k+\sum_{k=1}^{2n}(-1)^{k-1}V_k.$$

where

$$U_k=n\sum_{t_1+t_2=k,0\le t_i\le 2}\frac{2!}{t_1!t_2!(2-k)!}2^{(2-k)2^{n-1}}$$

$$V_k=2\sum_{k_1+\cdots+k_n=k,0\le k_i\le k-1,0\le k_i\le 2}\left(\prod_{j=1}^{n}C(2,k_j)\right)2^{\prod_{j=1}^{n}(2-k_j)}$$

A simple calculation shows that

$$U_1=4n2^{2^{n-1}}=C(n,1)2^2 2^{2^{n-1}}$$

and

$$U_2 = 4n.$$

$V_1 = 0$ since the condition of the sum is not satisfied.

$$V_2 = 2\sum_{k_1+\cdots+k_n=2,0\le k_i\le 1}\left(\prod_{j=1}^{n} C(2,k_j)\right)2^{\prod_{j=1}^{n}(2-k_j)}$$
$$= C(n,2)2^3 2^{2^{n-2}}$$

Note, $C(n,2)$ is the number of solutions of the equation $k_1+\cdots+k_n = 2, 0\le k_i \le 1$.

When $3\le k\le 2n$,

$$V_k = 2\sum_{k_1+\cdots+k_n=k,0\le k_i\le 2}\left(\prod_{j=1}^{n} C(2,k_j)\right)2^{\prod_{j=1}^{n}(2-k_j)}$$
$$= C(n,k)2^{k+1}2^{2^{n-k}} + \sum_{1\le t\le\left\lfloor\frac{k}{2}\right\rfloor} C(n,t)C(n-t,k-2t)2^{k-2t+1}$$

Note, $C(n,t)C(n-t,k-2t)$ is the number of solutions of the equation $k_1+\cdots+k_n=2$, $0\le k_i\le 2$ with exactly t components equal to 2.
hence, when $q=2$,

$$\left|\langle -:-:-\rangle\right|$$
$$= -4n + \sum_{1\le k\le n}(-1)^{k+1}C(n,k)2^{k+1}2^{2^{n-k}}$$
$$+\sum_{3\le k\le 2n}(-1)^{k-1}\sum_{1\le t\le\left\lfloor\frac{k}{2}\right\rfloor}C(n,t)C(n-t,k-2t)2^{k-2t+1}$$

When $n=1,2,3,4$, one can obtain (without calculator) the sequence 4, 14, 120, 3514. These results are consistent with those in [1]. By [1], the cardinality of $\left|\langle -:-:-\rangle\right|$ should be

$$\left|\langle -:-:-\rangle\right|$$
$$= 2\left((-1)^n - n\right) + \sum_{1\le k\le n}(-1)^{k+1}C(n,k)2^{k+1}2^{2^{n-k}}.$$

So, we obtain the following combinatorial identity(for any positive integer n).

$$\sum_{3\le k\le 2n}(-1)^{k-1}\sum_{1\le t\le\left\lfloor\frac{k}{2}\right\rfloor}C(n,t)C(n-t,k-2t)2^{k-2t+1}$$
$$= 2\left((-1)^n + n\right)$$

The left sum should be explained as 0 if $n=1$. As usual, $C(n,k)$ is 0 if $k>n$.

From Theorem 3.1, we know $\left|\langle i:a:b\rangle\right| = q^{q^n - q^{n-1}}$, since $\langle -:-:-\rangle = \bigcup_{i\in[n]}\bigcup_{a\in\mathbb{F}}\bigcup_{b\in\mathbb{F}}\langle i:a:b\rangle$, we obtain

$$\left|\langle -:-:-\rangle\right| \le nq^2 q^{(q-1)q^{n-1}}.$$

In order to get an intuitive idea about the magnitude of all the cardinality numbers, We will find their asymptote as $n\to\infty$ or $q\to\infty$.

We have the following notation

Definition 3.14 *We call* $f(x)\overset{x}{\cong} g(x)$ *if*

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1.$$

Now, we can list all the cardinalities asymptotically.

Theorem 3.15 *If* $n\ge 4$ *and* $q\ge 2$, *then*

$$\left|\langle i:a:b\rangle\right| = q^{(q-1)q^{n-1}}; \left|\langle i:a:-\rangle\right| = qq^{(q-1)q^{n-1}};$$
$$\left|\langle i:-:b\rangle\right| \overset{n}{\cong} qq^{(q-1)q^{n-1}}, \left|\langle i:-:b\rangle\right| \overset{q}{\cong} qq^{(q-1)q^{n-1}};$$
$$\left|\langle -:a:b\rangle\right| \overset{n}{\cong} nq^{(q-1)q^{n-1}}, \left|\langle -:a:b\rangle\right| \overset{q}{\cong} nq^{(q-1)q^{n-1}};$$
$$\left|\langle -:a:-\rangle\right| \overset{n}{\cong} nqq^{(q-1)q^{n-1}}, \left|\langle -:a:-\rangle\right| \overset{q}{\cong} nqq^{(q-1)q^{n-1}};$$
$$\left|\langle -:-:b\rangle\right| \overset{n}{\cong} nqq^{(q-1)q^{n-1}}, \left|\langle -:-:b\rangle\right| \overset{q}{\cong} nqq^{(q-1)q^{n-1}};$$
$$\left|\langle i:-:-\rangle\right| \overset{n}{\cong} q^2q^{(q-1)q^{n-1}}, \left|\langle i:-:-\rangle\right| \overset{q}{\cong} q^2q^{(q-1)q^{n-1}};$$
$$\left|\langle -:-:-\rangle\right| \overset{n}{\cong} nq^2q^{(q-1)q^{n-1}}, \left|\langle -:-:-\rangle\right| \overset{q}{\cong} nq^2q^{(q-1)q^{n-1}}.$$

Proof:

The first two rows are Theorem 3.1 and Theorem 3.2.

We will give a proof of the last row, the others are similar and easier.

$$\left|\langle -:-:-\rangle\right| = \sum_{k=1}^{q}(-1)^{k-1}U_k + \sum_{k=1}^{nq}(-1)^{k-1}V_k.$$

$$U_1 = \frac{nq!}{(q-1)!}q^{(q-1)q^{n-1}}\sum_{t_1+t_2+\cdots+t_q=1,0\le t_i\le q}\frac{1}{t_1!t_2!\cdots t_q!}$$
$$= nq^2q^{(q-1)q^{n-1}}.$$

When $2\le k\le q$, we have

$$U_k = \frac{nq!}{(q-k)!}q^{(q-k)q^{n-1}}\sum_{t_1+t_2+\cdots+t_q=k,0\le t_i\le q}\frac{1}{t_1!t_2!\cdots t_q!}$$
$$\le nq!q^{(q-2)q^{n-1}}\sum_{0\le t_i\le q,i=1,2,\cdots,q}1 = nq!q^{(q-2)q^{n-1}}(q+1)^q.$$

Hence,

$$\frac{\left|\sum_{k=2}^{q}(-1)^{k-1}U_k\right|}{U_1} \le \frac{qnq!(q+1)^q q^{(q-2)q^{q-1}}}{nq^2q^{(q-1)q^{n-1}}} \le \frac{q^{q-1}(q+1)^q}{q^{q^{n-1}}}$$
$$\le \frac{q^{q-1}(q)^{2q}}{q^{q^{n-1}}} \le \frac{q^{3q}}{q^{q^{n-1}}}.$$

So, $\lim_{n\to\infty}\frac{\left|\sum_{k=2}^{q}(-1)^{k-1}U_k\right|}{U_1} = 0$.

$V_1 = 0$ since the condition of the sum is not satisfied.

When $2\le k\le nq$, we have

$$V_k = q \sum_{k_1+\cdots+k_n=k, 0\le k_i\le k-1, 0\le k_i\le q} \left(\prod_{j=1}^{n} C(q,k_j)\right) q^{\prod_{j=1}^{n}(q-k_j)}$$
$$\le q \sum_{0\le k_i\le q, i=1,2,\cdots,n} (q!)^n q^{(q-1)^2 q^{n-2}}$$
$$= q(q+1)^n (q!)^n q^{(q-1)^2 q^{n-2}}.$$

Note, $\prod_{j=1}^{n}(q-k_j) \le (q-1)^2 q^{n-2}$. Hence,

$$\left|\sum_{k=1}^{nq}(-1)^{k-1} V_k\right| \le nqq(q+1)^n (q!)^n q^{(q-1)^2 q^{n-2}}.$$

We obtain

$$\frac{\left|\sum_{k=1}^{nq}(-1)^{k-1} V_k\right|}{U_1} \le \frac{nqq(q+1)^n (q!)^n q^{(q-1)^2 q^{n-2}}}{nq^2 q^{(q-1)q^{n-1}}}$$
$$= \frac{(q+1)^n (q!)^n}{q^{(q-1)q^{n-2}}} \le \frac{(q+1)^n q^{qn}}{q^{(q-1)q^{n-2}}} \le \frac{q^{(2+q)n}}{q^{q^{n-2}}}.$$

Hence,

$$\lim_{n\to\infty} \frac{\left|\sum_{k=1}^{nq}(-1)^{k-1} V_k\right|}{U_1} = 0.$$

In summary, we obtain

$$\lim_{n\to\infty} \frac{|\langle -:-:-\rangle|}{nq^2 q^{(q-1)q^{n-1}}} = 1.$$

In other words,

$$|\langle -:-:-\rangle| \overset{n}{\cong} nq^2 q^{(q-1)q^{n-1}}.$$

From the above proof, it is also clear that we have

$$\lim_{q\to\infty} \frac{|\langle -:-:-\rangle|}{nq^2 q^{(q-1)q^{n-1}}} = 1.$$

In other words,

$$|\langle -:-:-\rangle| \overset{q}{\cong} nq^2 q^{(q-1)q^{n-1}}.$$

□

When $q = 2$, the first equation of the last row in the above theorem has been obtained in [1].

4. Conclusion

In this paper, we generalized the definition of Boolean canalyzing functions to the functions of multi-state case. Using Inclusion and Exclusion Principle, we get formulas for the cardinality all such functions and the cardinalities of its various subsets. When $q = 2$, we derive an interesting combinatorial identity by equating our formula to the one in [1]. Finally, for a better understanding to the magnitudes, we provide all the asymptotes of these cardinalities as either $n \to \infty$ or $q \to \infty$.

5. Acknowledgements

This work was initiated when the first and the third authors visited Virginia Bioinformatics Institute at Virginia Tech in June 2010. We thank Alan Veliz-Cuba and Franzeska Hinkelmann for many useful discussions. The first and the third authors thank Professor Reinhard Laubenbacher for his hospitality and for introducing them to Discrete Dynamical Systems. The third author was supported in part by a Research Initiation Program (RIP) award at Winston-Salem State University.

We greatly appreciate an anonymous reader. Because of his insightful comments, in this paper, the proofs for many lemmas are simplified, the results are more general (On any finite set instead of finite field).

REFERENCES

[1] W. Just, I. Shmulevich and J. Konvalina, "The Number and Probability of Canalyzing Functions," *Physica D*, Vol. 197, No. 3-4, 2004, pp. 211-221. doi:10.1016/j.physd.2004.07.002

[2] C. H. Waddington, "The Strategy of the Genes," George Allen and Unwin, London, 1957.

[3] R. Thomas and R. D'Ari, "Biological Feedback," CRC Press, Boca Raton, 1989.

[4] L. Steggles, *et al.*, "Qualitatively Modelling and Analyzing Genetic Regulatory Networks: A Petri Net Approach," *Bioinformatics*, Vol. 23, No. 3, 2007, pp. 336-343. doi:10.1093/bioinformatics/btl596

[5] M. Pogson, *et al.*, "Formal Agent-Based Modelling of Intracellular Chemical Interactions," *Biosystems*, Vol. 85, No. 1, 2006, pp. 37-45. doi:10.1016/j.biosystems.2006.02.004

