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Edge Colorings of Planar Graphs without 6-Cycles with Two Chords*

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ABSTRACT

It is proved here that if a planar graph has maximum degree at least 6 and any 6-cycle contains at most one chord, then it is of class 1.

Keywords: Edge Coloring; Planar Graph; Cycle; Class 1

1. Introduction

All graphs considered here are finite and simple. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. If $v \in V(G)$, then its neighbor set $N_G(v)$ (or simply $N(v)$) is the set of the vertices in G adjacent to v and the degree $d(v)$ of v is $|N_G(v)|$. We denote the maximum degree of G by $\Delta(G)$. For $V' \subseteq V(G)$, we denote $N(V') = \bigcup_{u \in V'} N(u)$. A k^-, k^+-vertex is a vertex of degree k, at least k. A k (or k^+)-vertex adjacent to a vertex x is called a k(or k^+)-neighbor of x. Let $d_k(x)$, $d_{k+}(x)$ denote the number of k-neighbors, k^+-neighbors of x. A k-cycle is a cycle of length k. Two cycles sharing a common edge are said to be adjacent. Given a cycle C of length k in G, an edge $xy \in E(G) \setminus E(C)$ is called a chord of C if $x, y \in V(C)$. Such a cycle C is also called a chordal-k-cycle.

A graph is *k-edge-colorable*, if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The *edge chromatic number* of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. In 1964, Vizing showed that for every simple graph G, $\Delta(G) \le \chi'(G) \le \Delta(G)+1$. A graph G is said to be of *class* 1 if $\chi'(G) = \Delta(G)$, and of *class* 2 if $\chi'(G) = \Delta(G)+1$. A graph G is *critical* if it is connected and of class 2 and $\chi'(G-e) < \chi'(G)$ for any edge e of G. A critical graph with maximum degree Δ is called a Δ-critical graph. It is clear that every critical graph is 2-connected.

For planar graphs, more is known. As noted by Vizing [1], if C_4, K_4, the octahedron, and the icosahedron have one edge subdivided each, class 2 planar graphs are produced for $\Delta \in \{2,3,4,5\}$. He proved that every planar graph with $\Delta \ge 8$ is of class 1 (There are more general results, see [2] and [3]) and then conjectured that every planar graph with maximum degree 6 or 7 is of class 1. The case $\Delta = 7$ for the conjecture has been verified by Zhang [4] and, independently, by Sanders and Zhao [5]. The case $\Delta = 6$ remains open, but some partial results are obtained. Theorem 16.3 in [1] stated that a planar graph with the maximum degree Δ and the girth g is of class 1 if $\Delta \ge 3$ and $g \ge 8$, or $\Delta \ge 4$ and $g \ge 5$, or $\Delta \ge 5$ and $g \ge 4$. Lam, Liu, Shiu and Wu [6] proved that a planar graph G is of class 1 if $\Delta \ge 6$ and no two 3-cycles of G sharing a common vertex. Zhou [7] obtained that every planar graph with $\Delta \ge 6$ and without 4 or 5-cycles is of class 1. Bu and Wang [8] proved that every planar graph with $\Delta \ge 6$ and without 6-cycles is of class 1. Ni [9] extended the result that every planar graph with $\Delta \ge 6$ and without chordal 6-cycles is of class 1. In the note, we improve the above result by proving that every planar graph with $\Delta \ge 6$ and without 6-cycles with two chords is of class 1.

2. The Main Result and Its Proof

To prove our result, we will introduce some known lemmas.

Lemma 1. (Vizing's Adjacency Lemma [1]). *Let G be a Δ-critical graph, and let u and v be adjacent vertices of G with $d(v) = k$.*

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1) *If* $k<\Delta$, *then u is adjacent to at least* $\Delta-k+1$ *vertices of degree* Δ;

2) *If* $k=\Delta$, *then u is adjacent to at least two vertices of degree* Δ.

From the Vizing's Adjacency Lemma, it is easy to get the following corollary.

Corollary 2. *Let G be a* Δ*-critical graph. Then*

1) *Every vertex is adjacent to at most one* 2*-vertex and at least two* Δ*-vertices*;

2) *The sum of the degree of any two adjacent vertices is at least* $\Delta+2$;

3) *If* $uv\in E(G)$ *and* $d(u)+d(v)=\Delta+2$, *then every vertex of* $N(\{u,v\})\setminus\{u,v\}$ *is a* Δ*-vertex.*

Lemma 3 [4]. *Let G be a* Δ*-critical graph*, $uv\in E(G)$ *and* $d(u)+d(v)=\Delta+2$. *Then*

1) *every vertex of* $N(N(\{u,v\}))\setminus\{u,v\}$ *is of degree at least* $\Delta-1$;

2) *if* $d(u),d(v)<\Delta$, *then every vertex of* $N(N(\{u,v\}))\setminus\{u,v\}$ *is a* Δ*-vertex.*

Lemma 4 [5]. *No* Δ*-critical graph has distinct vertices x, y, z such that x is adjacent to y and z,* $d(z)<2\Delta-d(x)-d(y)+2$ *and xz is in at least* $d(x)+d(y)-\Delta-2$ *triangles not containing y.*

To be convenient, for a plane graph *G*, let $F(G)$ be the face set of *G*. A face of a graph is said to be *incident* with all edges and vertices in its boundary. Two faces sharing an edge *e* are said to be *adjacent* at *e*. A degree of a face *f*, denoted by $d_G(f)$ is the number of edges incident with *f* where each cut edge is counted twice. A k^-, k^+-face is a face of degree *k*, at least *k*. A *k*-face of *G* is denoted by $[v_1,v_2,\cdots,v_k]$ if it is incident with $v_1,v_2,\cdots,v_k$ along its boundary. A 3-face $[x,y,z]$ of *G* is called an (i,j,k)-face if $d(x)=i\le d(y)=j\le d(z)=k$. For a vertex $x\in V(G)$, we denote by $f_k(v)$ the number of *k*-faces incident with *v*.

Lemma 5 [4,5]. *If G is a planar graph with* $\Delta(G)\ge 7$, *then G is of class* 1.

Lemma 6 [8]. *If G is a graph of class* 2, *then G contains a k-critical subgraph for each k satisfying* $2\le k\le\Delta(G)$.

Theorem 7. *Let G be a planar graph with* $\Delta\ge 6$. *If any* 6*-cycle contains at most one chord, then G is of class* 1.

Proof. Suppose that *G* is a counterexample to our theorem with the minimum number of edges and suppose that *G* is embedded in the plane. Then *G* is a 6-critical graph by Lemmas 5 and 6, and it is 2-connected. By Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$, we have

$$\sum_{v\in V(G)}(d(v)-4)+\sum_{f\in F(G)}(d(f)-4)=-8<0$$

We define *ch* to be the initial charge. Let $ch(x)=d(x)-4$ for each $x\in V\cup F$. So $\sum_{x\in V\cup F}ch(x)<0$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x\in V\cup F$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum. If we can show that $ch'(x)\ge 0$ for each $x\in V\cup F$, then we get an obvious contradiction $0\le\sum_{x\in V\cup F}ch'(x)=\sum_{x\in V\cup F}ch(x)<0$. which completes our proof.

The discharging rules are defined as follows.

R1: Every 5^+-face *f* sends $\frac{d(f)-4}{d(f)}$ to each incident vertex.

R2: Every 2-vertex receives 1 from each adjacent vertex.

R3: Every 3-vertex receives $\frac{1}{3}$ from each adjacent vertex.

R4: Let *f* be a 3-face [*x*,*y*,*z*] with $d(x)\le d(y)\le d(z)$. If $2\le d(x)\le 4$ and $\min\{d(y),d(z)\}\ge 5$, then *f* receives $\frac{1}{2}$ from *y*, $\frac{1}{2}$ from *z*; If $d(x)=d(y)=4$ and $d(z)=6$ then *z* sends 1 to *f*; If $\min\{d(x),d(y),d(z)\}\ge 5$, then *x*, *y*, *z* sends $\frac{1}{3}$ to *f*, respectively.

R5: If a 5-vertex *v* is adjacent to a 6-vertex *x* and incident with a (3,5,6)-face [*u*,*v*,*w*] such that $ux\notin E(G)$ and $w\ne x$, then *x* sends $\frac{1}{5}$ to *v*.

Now, let's began to check $ch'(x)\ge 0$ for all $x\in V\cup F$. Let $f\in F(G)$. Then $d(f)\ge 3$. If $d(f)\ge 5$, then $ch'(f)\ge ch(f)-(d(f)-4)=0$ by R1. If $d(f)=4$, then $ch'(f)=ch(f)=0$. If $d(f)=3$, then $ch'(f)\ge ch(f)+\max\left\{2\times\frac{1}{2},1,3\times\frac{1}{3}\right\}=0$ by R4. Let $w\in V(G)$. Then $d(w)\ge 2$. If $d(w)=2$, then $ch'(w)=ch(w)+2\times\frac{1}{2}=0$ by R2. If $d(w)=3$, then *w* is adjacent to three 5^+-vertices by Corollary 2, and it follows that $ch'(w)=ch(w)+3\times\frac{1}{3}=0$ by R3. If $d(w)=4$, then $ch'(w)=ch(w)=0$.

Since any 6-cycle of G contains at most one chord, we have the following claim.

Claim 1. *Let f, f', f'' be three faces incident with w such that f' is adjacent to f and f''. If f and f'' are* 3*-faces, then f' must be a* 5^+*-face.*

Suppose that $d(w)=5$. We have $ch(w)=1$, $f_3(w)\le 3$, $\min\{d(u)\,|\,u\in N(w)\}\ge 3$, $d_3(w)\le 1$ and $d_6(w)\ge 2$. Let $w_0,w_1,\cdots,w_4$ be neighbors of *w* and $f_0,f_1,\cdots,f_4$ be faces incident with *w* such that f_i is incident with w_i and w_{i+1}, for all $i\in\{0,1,\cdots,4\}$, where $w_5=w_0$. If all neighbors of *w* are 5^+-vertices, then $ch'(w)\ge ch(w)+3\times\frac{1}{3}=0$ by R4. Suppose that $\min\{d(w_i):0\le i\le 4\}=4$. If $f_3(w)\le 2$, then

$ch'(w) \ge 1 - 2\times\frac{1}{2} = 0$ by R4; Otherwise, without loss of generality, assume that f_4, f_0, f_2 are 3-faces. Then f_1 and f_3 are 5^+-faces by Claim 1. By Lemma 4, $d_4(v) = 1$. So w sends at most $\left(2\times\frac{1}{2}+\frac{1}{3}\right)$ to its adjacent 3-faces. At the same time, w receives at least $2\times\frac{1}{5}$ from f_1 and f_3 by R1, and it follows that $ch'(w) \ge 1 + \frac{2}{5} - \left(2\times\frac{1}{2}+\frac{1}{3}\right) > 0$. Suppose that $d_3(w) = 1$, without loss of generality, assume that $d(w_1) = 3$. Then $d_6(w) = 4$ by Lemma 1. If $f_3(w) \le 1$, or $f_3(w) = 2$ and ww_1 is not incident with a 3-face, then $ch'(w) \ge 1 - \frac{1}{3} - \max\left\{\frac{1}{2}, 2\times\frac{1}{3}\right\} = 0$ by R3 and R4; Otherwise, $f_3(w) \ge 2$ and then w is incident with a 5^+-face. If $f_3(w) = 2$, then $ch'(w) \ge 1 + \frac{1}{5} - \frac{1}{3} - \left(\frac{1}{2}+\frac{1}{3}\right) > 0$; Otherwise, w is incident with two 5^+-faces. If ww_1 is not incident with a 3-face, then $ch'(w) \ge 1 + 2\times\frac{1}{5} - \frac{1}{3} - 3\times\frac{1}{3} > 0$ by R3 and R4; Otherwise, w receives at least $2\times\frac{1}{5}$ from its neighbors by R5, and it follows that $ch'(w) \ge 1 + 4\times\frac{1}{5} - \frac{1}{3} - \left(2\times\frac{1}{2}+\frac{1}{3}\right) > 0$.

In the following we check the case that $d(w) = 6$. Thus we have $ch(w) = 2$, $f_3(w) \le 4$, $d_2(w) \le 1$ and $d_6(w) \ge 2$ by Lemma 1.

Case 1. w sends positive charge to some adjacent 5-vertex v (ref. R5).

Suppose that v is incident with a (3,5,6)-face $[u,v,x]$ such that $wu \notin E(G)$ and $w \ne x$ (see R5). Then x may sends $\frac{1}{5}$ to v by R5. At the same time, w is adjacent to five 6-vertices by Lemma 3, that is, $d_6(w) = 5$. Since $f_3(w) \le 4$, $ch'(w) = 2 - \frac{1}{5} - 4\times\frac{1}{3} > 0$.

Case 2. w sends no charge to its adjacent 5^+-vertices.

Let $k = \min\{d(u) \mid u \in N(w)\}$. If $k \ge 5$, then $ch'(w) \ge 2 - 4\times\frac{1}{3} > 0$. Suppose that $k = 4$. Then $d_6(w) \ge 3$ by Lemma 1 and w may be incident with a (4,4,6)-face. If $f_3(w) \le 3$, then $ch'(w) \ge 2 - 1 - \frac{1}{2} - \frac{1}{3} > 0$; Otherwise, $f_{5^+}(w) \ge 2$ and it follows that $ch'(w) \ge 2 + \frac{1}{5} - \max\left\{1 + \frac{1}{2} + 2\times\frac{1}{3}, 4\times\frac{1}{2}\right\} > 0$.

Suppose that $k = 3$. Then $d_6(w) \ge 6 - 3 + 1 = 4$ by Lemma 1. If $d_{5^+}(w) \ge 5$, then $ch'(w) \ge 2 - \frac{1}{3} - 2\times\left(\frac{1}{2}+\frac{1}{3}\right) = 0$; Otherwise, w is incident with two 4-vertices u,v, then u and v are incident with at most one 3-face by Lemma 4 since $d(u)+d(v)+d(w) \le 3+4+6 < 14$. So $f_3(w) \le 3$ and it follows that $ch'(w) \ge 2 - 2\times\frac{1}{2} - 2\times\frac{1}{3} > 0$ by R3 and R4.

Suppose that $k = 2$, that is, w is adjacent to a 2-vertex v. Then $d_6(w) = 5$ by Lemma 1. If $f_3(w) = 4$, then $f_{5^+}(w) = 2$ and it follows that $ch'(w) \ge 2 + \frac{2}{5} - 1 - \frac{1}{2} - 2\times\frac{1}{3} > 0$; Otherwise, $ch'(w) \ge 2 - 1 - \max\left\{\frac{1}{2} + 2\times\frac{1}{3} - \frac{1}{5}, 3\times\frac{1}{3}\right\} = 0$.

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