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Longest Hamiltonian in N$_{odd}$-Gon

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ABSTRACT

We single out the polygonal paths of $n_{odd}-1$ order that solve each of the $\left\lfloor \frac{n_{odd}}{2} \right\rfloor$ different longest non-cyclic Euclidean Hamiltonian path problems in $\mathcal{N}\left(K_{n=n_{odd}}\left(\mathrm{e}^{\mathrm{i}\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ networks by an arithmetic algorithm. As by product, the procedure determines the winding index of cyclic Hamiltonian polygonals on the vertices of a regular polygon.

Keywords: Hamiltonian Path; Extremal Problems; Euclidean Geometric Problem; Farthest Neighbor Tours; Traveling Salesman Problem; Geometry of Odd Regular Polygons

1. Introduction

Our aim implies to determine the overall lengths of every Longest Euclidean Hamiltonian Path Problems and the composition and the orderings of the directed segments that accomplish these longest Hamiltonian travels. The identification regardless of planar rotation and orientation is done with the proposed algorithm [1-3].

This paper apart from the Introduction, Conclusion and References contains §2 *Algorithm and Hamiltonian Paths in N$_{odd}$-Gons* and §3 *Maximum Hamiltonian Path Problems in N$_{odd}$-Gons*. §2 formulates specific Max. Hamiltonian Problems and postulates the algorithm for their resolutions. §3 devoted to the solution of the $\left\lfloor \frac{n_{odd}}{2} \right\rfloor$ different Max. Traveling Salesman Path Problems in N$_{odd}$-Gons [4,5].

2. Algorithm and Hamiltonian Paths in N$_{odd}$-Gons

This work is focused in the resolution of the $\left\lfloor \frac{n_{odd}}{2} \right\rfloor$ different Maximum Traveling Salesman Path Problems of order $n_{odd}-1$ with inicial point at $V_0=(-1,0)$ and final point at V_k for $1\le k\le \left\lfloor \frac{n}{2} \right\rfloor$ (see **Figure 1**) in the $\mathcal{N}\left(K_{n=n_{odd}}\left(\mathrm{e}^{\mathrm{i}\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ networks. These structures are built by the complete graph $K_{n_{odd}}$ on the odd regular polygon vertices, *i.e.* $\mathrm{e}^{\mathrm{i}\pi}\sqrt[n_{odd}]{1}$, and weighted with the Euclidean distances d_{ij} between nodes [6].

2.1. Intrinsic Geometry and Arithmetic

Let $V_0,\cdots,V_{n-1}$ be the points of the $\mathrm{e}^{\pi\mathrm{i}}\sqrt[n]{1}$ set and let them be clockwise enumerated by the integers modulo n, $\mathbb{Z}_n$, from the vertex $V_0=(-1,0)$. For each k in $0\le k\le \left\lfloor \frac{n}{2} \right\rfloor$ and each $j\in\mathbb{Z}_n$, let $L^-_{k,j}$ denote the segment that joins V_j with V_{j+k}, while $L^+_{k,j}$ denotes the one that joins V_j to $V_{j+(n-k)}=V_{j-k}$. From now onwards, L^-_k and L^+_k denote to $L^-_{k,0}$ and $L^+_{k,0}$, respectively. Let $l_{\max}$ be the diameter, it joins the vertex V_j with its opposite $V_{j+\frac{n}{2}}$, only if n is even. $L^-_{\left\lfloor \frac{n}{2} \right\rfloor}$ and $L^+_{\left\lfloor \frac{n}{2} \right\rfloor}$ respectively designate the quasi-diameters if n is odd (see **Figure 1**), [7].

If P_n symbolizes a regular n-gon inscribed in the unitary circle and with vertices in $V_0,\cdots,V_{n-1}$, P_n can be considered as the polygonal of sides $L^-_{1,0},L^-_{1,1},\cdots,L^-_{1,n-1}$ [8]. From the vectorial interpretation of the $L^{\pm}_{k,j}$ segments, $L^-_{k,j}$ can be interpreted as the resultant of the polygonal of k sides of P_n, that joins clockwise V_j to V_{j+k}, while $L^+_{k,j}$ is the resultant of the polygonal of $n-k$ sides that joins clockwise V_j to $V_{j-k}\equiv V_{j+n-k}$.

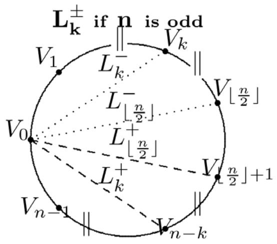

Figure 1. L_k segments in N_{odd}-Gons.

The segments $L_{k,j}^-$ and $L_{k,j}^+$ are the respective chords (or resultants) of the polygonals $sn+k$ and $rn+n-k$ consecutive sides of P_n, whichever are the integers s and r. Therefore, it is natural to associate $L_{k,j}^-$ with the integer $e\left(L_{k,j}^-\right)=k$, and likewise $L_{k,j}^+$ with the integer $e\left(L_{k,j}^+\right)=n-k\equiv -k\left(\bmod n\right)$.

Definition 2.1.1 For any integer n, L is a L_k segment if for any k, $0\le k\le \left\lfloor \frac{n}{2}\right\rfloor$, and for any $j\in\mathbb{Z}_n$, L is equal to $L_{k,j}^-$ or $L=L_{k,j}^+$.

Definition 2.1.2 If L is an L_k segment, the integer associated to L'', noted as $e(L)$, is given by:

$$e(L)=\begin{cases} k & \text{if } L=L_{k,j}^- \\ n-k\equiv -k & \text{if } L=L_{k,j}^+ \end{cases}$$

Definition 2.1.3 If $S=\{L_1,\cdots,L_j\}$ is a sequence of L_k segments, the integer associated to the path evolved by S'', is given by $e(S)=\sum_{i=1}^{j}e(L_i)\left(\bmod n\right)$.

It should be taken into account the following facts:

- The consecutive collocation of two L_k segments from any vertex V_i determines the vertex that corresponds to collocate, from V_i and in clockwise, as many sides of P_n as correspond to the sum of the integers associated to each one of the two L_k segments. In other words, the resultant of a polygonal built by two L_k segments, is other L_k segment and its associated integer is the sum (modulo n) of the integers associated to the components of the polygonal.
- The L_k segment is $L_{e(L),j}^-$ by considering any fixed value of j, when $0\le e(L)\le \left\lfloor \frac{n}{2}\right\rfloor$. Otherwise, if $\left\lfloor \frac{n}{2}\right\rfloor < e(L)\le n$, is $L_{n-e(L),j}^+=L_{-e(L),j}^+$.

The concept of the associated integer $e(L_k)$ and its addition modulo n, deploy the following geometric correlate over the set of vertices $\{V_0=(-1,0),\cdots,V_{n-1}\}$: *For each i, $0\le i\le n-1$, the geometric place that corresponds to the vertex V_i coincides with the place that corresponds to V_{i+sn}, for each integer s. Since the segments $L_{k,0}^-$ and $L_{k,0}^+$ respectively connect the vertices V_0 to V_k and V_0 to $V_{-k}\equiv V_{n-k}$, it is clear that for any integer k between 0 and $\left\lfloor \frac{n}{2}\right\rfloor$, the vertices V_k and V_{-k} are symmetric with respect to the horizontal axis*. Given a sequence of L_k segments, henceforward the polygonal that they determine is in a one-to-one relationship with the sum of each one of these directed segments that belong to the sequence.

Since $e\left(L_{k,i}^{\pm}\right)=e\left(L_{k,j}^{\pm}\right)$, whichever i and j are, without loss of generality in the sequences of L_k segments, the second subindices of these directed segments are rooted out.

2.2. Resuming the Algorithm

Lemma 2.6 and Theorem 2.7 in [1] detail the proofs of the following algorithmic statements.

Theorem 2.2.1 The pathway determined by a sequence S of L_k segments starts and ends at the same vertex V_j if and only if $e(S)\equiv 0$.

Theorem 2.2.2 A sequence S of n L_k segments determines a Euclidean Hamiltonian cycle C_H^n of order n if and only if any proper subsequence of order $m<n$ has associated integer neither n nor a multiple of n and $e(S)\equiv 0$.

Corollary 2.2.1 A sequence $\tilde{S}$ of L_k segments of order m, $3\le m\le n$, building a Euclidean closed polygonal in $\mathcal{N}\left(K_n\left(e^{i\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ networks, passing once through certain or all $e^{i\pi}\sqrt[n]{1}$ vertices, has $e(\tilde{S})\equiv 0$. Since, $e(\tilde{S})$ is a multiple of n exists z less than or equal to $\left\lfloor \frac{n}{2}\right\rfloor$ which counts the times that $\tilde{S}$ cw. winds around the geometric centre of P_n. We named this specific integer as the "*winding index*".

2.3. Applications of the Algorithm: Winding Index in Special Cyclic Paths in N_{odd}-Gons

Let C_{Q-H}^m symbolize a cyclic polygonal in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ network, which does not repeat vertices, with the exception of the first and the last one, and which passes through certain m nodes, $m<n_{odd}$. Specially, $C_H^{n_{odd}}$ stands for Euclidean Hamiltonian cycles in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ network.

Exampe 2.1. Let $n_{odd}L_k^-$, $1\le k\le \left\lfloor \frac{n_{odd}}{2}\right\rfloor$. If k does not divide n_{odd} they are $C_H^{n_{odd}}$ s of winding index k [9]. $C_H^{n_{odd}}$: $n_{odd}L_{\left\lfloor \frac{n_{odd}}{2}\right\rfloor}^-$ is the Max TSP [10].

Exampe 2.2. Let

$$L_{l+1}^{-}, \underbrace{L_{\lfloor \frac{n}{2} \rfloor}^{+}}_{2}, L_{\lfloor \frac{n}{2} \rfloor -1}^{-}, \underbrace{L_{\lfloor \frac{n}{2} \rfloor}^{-}}_{2l}, L_{\lfloor \frac{n}{2} \rfloor -1}^{-}, \underbrace{L_{\lfloor \frac{n}{2} \rfloor}^{+}}_{2}, \quad 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

The angular cw. avance is proportional to:

$$l+1+2\left(\left\lfloor \frac{n_{odd}}{2} \right\rfloor + 1\right) + \left\lfloor \frac{n_{odd}}{2} \right\rfloor - 1 + 2l \left\lfloor \frac{n_{odd}}{2} \right\rfloor + \left\lfloor \frac{n_{odd}}{2} \right\rfloor - 1 + 2\left(\left\lfloor \frac{n_{odd}}{2} \right\rfloor + 1\right) = (l+3)\, n_{odd},$$

then $l+3$ is the winding index. Algorithmic computations render that these cycles are $C_H^{n_{odd}=7+2l}$ and $C_{Q\text{-}H}^{7+2l}$ for networks built on $n_{odd} > 7+2l$. For $L_{l+1}^{-}, \underbrace{L_{\lfloor \frac{n}{2} \rfloor}^{-}}_{2l}, L_{\lfloor \frac{n}{2} \rfloor}^{-}, L_{\lfloor \frac{n}{2} \rfloor -1}^{-}, \underbrace{L_{\lfloor \frac{n}{2} \rfloor}^{+}}_{2}$ the algorithm prompts $l+2$ as winding index and singled out them as $C_H^{n_{odd}=5+2l}$ and $C_{Q\text{-}H}^{5+2l}$ if $n_{odd} > 5+2l$. In $L_{l+1}^{-}, L_{\lfloor \frac{n}{2} \rfloor -1}^{-}, \underbrace{L_{\lfloor \frac{n}{2} \rfloor}^{-}}_{2l-1},$ $1 \le l \le \left\lfloor \frac{n}{2} \right\rfloor - 1$, the algorithm characterizes these cycles as $C_{Q\text{-}H}^{2l+1}$ in $\mathcal{N}\left(K_{n=n_{odd} \ge 5}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$ with winding index l.

Exampe 2.3. Table 1 deploys cycles living in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$ $\forall\, n_{odd} \ge n_{odd}^{\min}$.

Exampe 2.4. Table 2 shows Euclidean Hamiltonian cycles in special $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$ networks.

3. Maximum Hamiltonian Path Problems in N_{odd}-Gons

In $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$ network for $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$, we study the trajectories built by a single L_k^{-} segment, $\left(\left\lfloor \frac{n}{2} \right\rfloor - k\right)$ directed $L_{\lfloor \frac{n}{2} \rfloor -1}^{\pm}$ segments, and $\left(\left\lfloor \frac{n}{2} \right\rfloor + k\right)$ directed segments $L_{\lfloor \frac{n}{2} \rfloor}^{\pm}$, that is (1).

Table 1. C_{Q-H}^{m} **in** $N\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$.

$N\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$; $1 \le k \le \lfloor n/2 \rfloor - 1$	
$C_{Q-H}^{m}: \quad \forall\, n_{odd} \ge n_{odd}^{\min}$	Winding Index
$C_{Q-H}^{4}: \quad L_{\lfloor n/2 \rfloor -1}^{-}, L_{\lfloor n/2 \rfloor}^{-}, L_{\lfloor n/2 \rfloor -1}^{+}, L_{\lfloor n/2 \rfloor}^{+}$; $n_{odd} \ge 5$	2
$C_{Q-H}^{4}: \quad L_{\lfloor n/2 \rfloor -1}^{-}, 3L_{\lfloor n/2 \rfloor}^{+}$; $n_{odd} \ge 5$	2
$C_{Q-H}^{6}: \quad L_{\lfloor n/2 \rfloor -1}^{+}, \underbrace{L_{\lfloor n/2 \rfloor}^{+}}_{2}, L_{\lfloor n/2 \rfloor -1}^{-}, \underbrace{L_{\lfloor n/2 \rfloor}^{-}}_{2}$; $n_{odd} \ge 7$	3
$C_{Q-H}^{6}: \quad \underbrace{L_{\lfloor n/2 \rfloor}^{-}, L_{\lfloor n/2 \rfloor -1}^{+}}_{2}, L_{\lfloor n/2 \rfloor}^{-}, L_{\lfloor n/2 \rfloor -1}^{-}$; $n_{odd} \ge 7$	3
$C_{Q-H}^{2k+1}: \quad L_{k}^{-}, 2k\, L_{\lfloor n/2 \rfloor}^{-}$; $n_{odd} \ge 5$	k
$C_{Q-H}^{2k+1}: \quad L_{k}^{-}, \underbrace{L_{\lfloor n/2 \rfloor}^{+}, L_{\lfloor n/2 \rfloor -1}^{-}}_{k}$; $n_{odd} \ge 5$	k

Table 2. $C_H^{n_{odd}}$ **in** $N\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$.

$C_H^{n_{odd}}$ in $N\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$; Winding Index $\lfloor n/2 \rfloor$
$C_H^{n_{odd}=4t+1}$
$L_1^{-}, \underbrace{L_{\lfloor n/2 \rfloor}^{-}, L_{\lfloor n/2 \rfloor -1}^{-}}_{(n-1)/4}, \underbrace{L_{\lfloor n/2 \rfloor}^{+}}_{2}, \underbrace{L_{\lfloor n/2 \rfloor -1}^{+}, L_{\lfloor n/2 \rfloor}^{+}}_{(n-1)/4-1}$
$C_H^{n_{odd}=4t+3}$
$L_1^{-}, \underbrace{L_{\lfloor n/2 \rfloor}^{-}, L_{\lfloor n/2 \rfloor -1}^{-}}_{(n-3)/4}, \underbrace{L_{\lfloor n/2 \rfloor}^{-}}_{2} \underbrace{L_{\lfloor n/2 \rfloor -1}^{+}, L_{\lfloor n/2 \rfloor}^{+}}_{(n-3)/4}$
$C_H^{n_{odd}=4t+3}$
$L_1^{-}, \underbrace{L_{\lfloor n/2 \rfloor}^{+}}_{2}, \underbrace{L_{\lfloor n/2 \rfloor -1}^{-}}_{2}, \underbrace{L_{\lfloor n/2 \rfloor}^{-}, L_{\lfloor n/2 \rfloor -1}^{-}}_{(n-3)/4-1}, \underbrace{L_{\lfloor n/2 \rfloor}^{+}}_{2}, \underbrace{L_{\lfloor n/2 \rfloor -1}^{+}, L_{\lfloor n/2 \rfloor}^{+}}_{(n-3)/4-1}$

3.1. Lengths of Relevant Pathways

Our present concern is to study the Euclidean lengths and the composition of the directed segments that build the trajectories given by (1).

$$\hat{L}_k^{-} + pL_{\lfloor \frac{n}{2} \rfloor -1}^{\pm} + (n-p-1)\, L_{\lfloor \frac{n}{2} \rfloor}^{\pm}, \quad p = \left\lfloor \frac{n}{2} \right\rfloor - k. \qquad (1)$$

Since for $n \in IN$ the lengths L_k of the segments $L_k^{\pm}$, $1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$ verify the following relationships:

$$\mathfrak{L}\left(L_{\lfloor \frac{n}{2} \rfloor - i}\right) - \mathfrak{L}\left(L_{\lfloor \frac{n}{2} \rfloor -(i+1)}\right) < \mathfrak{L}\left(L_{\lfloor \frac{n}{2} \rfloor -(i+1)}\right) - \mathfrak{L}\left(L_{\lfloor \frac{n}{2} \rfloor -(i+2)}\right),$$

$$0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 3.$$

Therefore, the overall traveled Euclidean lengths of the pathways (1) are given by:

$$\mathfrak{L}\left(\hat{L}_k^{-}\right) + p\mathfrak{L}\left(L_{\lfloor \frac{n}{2} \rfloor -1}^{\pm}\right) + (n-p-1)\,\mathfrak{L}\left(L_{\lfloor \frac{n}{2} \rfloor}^{\pm}\right), \quad p = \left\lfloor \frac{n}{2} \right\rfloor - k. \qquad (2)$$

Therein, precisely we focusing on the Euclidean Hamiltonian cycles, $C_H^{n_{odd}}$ s, which accomplish the lengths (2) in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right), d_{n\times n}^{E}\right)$ network.

Next Theorem establishes the composition of the directed segments that give birth to the sequences with overall traveled lengths (2).

Theorem 3.1.1 The overall traveled lengths (2) in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{\pi i}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ are accomplished for any sequence built by a single L_k^-, $\left(\left\lfloor\frac{n}{2}\right\rfloor-k-\alpha\right)$ $L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}$, α $L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}$, β $L^+_{\left\lfloor\frac{n}{2}\right\rfloor}$ and $\left(\left\lfloor\frac{n}{2}\right\rfloor+k-\beta\right)$ $L^-_{\left\lfloor\frac{n}{2}\right\rfloor}$ directed segments if $p=\left\lfloor\frac{n}{2}\right\rfloor-k$ and $1\le k\le\left\lfloor\frac{n}{2}\right\rfloor$ if the conditions in (3) are satisfied.

$$\begin{cases}3\alpha+\beta=2p\\ 0\vee\left\lceil p+\frac{1}{3}-\frac{n}{3}\right\rceil\le\alpha\le\frac{2}{3}p\end{cases}\quad(3)$$

Proof

$$\frac{2\pi}{n}k+\frac{2\pi}{n}\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left[\left(\left\lfloor\frac{n}{2}\right\rfloor-k\right)-\alpha\right]+\frac{2\pi}{n}\left(\left\lfloor\frac{n}{2}\right\rfloor+2\right)\alpha$$

$$+\frac{2\pi}{n}\left\lfloor\frac{n}{2}\right\rfloor\left[\left(\left\lfloor\frac{n}{2}\right\rfloor+k\right)-\beta\right]+\frac{2\pi}{n}\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)\beta=2m\pi$$

$$n\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)+2k+1+3\alpha+\beta=mn$$

From the constraints $0\le\alpha\le\left\lfloor\frac{n}{2}\right\rfloor-k$ and $0\le\beta\le\left\lfloor\frac{n}{2}\right\rfloor+k$ follows $2k+1+3\alpha+\beta=\left[m-\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right]n\le 2n-1$, m should be $m=\left\lfloor\frac{n}{2}\right\rfloor$ and hence $2k+1+3\alpha+\beta=n\Rightarrow 3\alpha+\beta=2p$. Therefore, the admissible couples (α,β) for the lengths (2) should verified (3). □

Backward recurrence over the traveled length in steepest descent steps from the max $n_{odd}L\left(L^-_{\left\lfloor\frac{n}{2}\right\rfloor}\right)$ to $\hat{L}_1^-$ constraint and the lack of Hamiltonian cycles for

$$\hat{L}^-_{\left\lfloor\frac{n}{2}\right\rfloor-p}+\underbrace{(p-1)L^{\pm}_{\left\lfloor\frac{n}{2}\right\rfloor-1}+(n-p)L^{\pm}_{\left\lfloor\frac{n}{2}\right\rfloor}}_{\nexists P_H^{n-1}},\quad 1\le p\le\left\lfloor\frac{n}{2}\right\rfloor-2$$

state that (4) is the Euclidean Hamiltonian Maximum Path length when $\hat{L}^-_{\left\lfloor\frac{n}{2}\right\rfloor-p}=\hat{L}_k^-$ is rooted out.

$$\underbrace{pL\left(L^{\pm}_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)+\left(n_{odd}-p-1\right)L\left(L^{\pm}_{\left\lfloor\frac{n}{2}\right\rfloor}\right)}_{\mathrm{Max}L\left(P_H^{n_{odd}-1}\right)},\quad 1\le p\le\left\lfloor\frac{n}{2}\right\rfloor-1.\quad(4)$$

3.2. Specific Directed Segments for the Max. Traveling Salesman Path Problems in N$_{odd}$-Gons

We confirm in Theorem (3.3.1), Theorem (3.3.2) and Theorem (3.3.3) the existence of Euclidean Hamiltonian cycles that attain the overall Euclidean lengths given by the sequences (1) and the assignments (3).

1) For p_{odd} if $\alpha=\left\lfloor\frac{p_{odd}}{2}\right\rfloor$ and $\beta=\left\lfloor\frac{p_{odd}}{2}\right\rfloor+2$ in (3) exists $C_H^{n_{odd}}$ s with overall traveled length (2). See Theorem (3.3.1) at pg. 4.

2) For p_{even}

a) $\alpha=\frac{p_{even}}{2}=\beta$ in (3) exists $C_H^{n_{odd}}$ s with whole traveled length (2). See Theorem (3.3.2) at pg. 5,

b) $\alpha=\frac{p_{even}}{2}-1$ and $\beta=\alpha+4$ in (3) exists $C_H^{n_{odd}}$ s with whole traveled length (2). See Theorem (3.3.3) at pg. 5.

3.3. Orderings of the Directed Segments for the Max. Traveling Salesman Paths in N$_{odd}$-Gons

$P_H^{n_{odd}-1}$ symbolizes any Euclidean Hamiltonian path that resolves the Max Traveling Salesman Path Problems with initial vertex $V_0=(-1,0)$ and final vertex V_k, that is whichever be the bridge, $\hat{L}_k^-=\hat{L}^-_{\left\lfloor\frac{n}{2}\right\rfloor-p}$ for $1\le k\le\left\lfloor\frac{n}{2}\right\rfloor$, between the starting and ending points.

Observation 3.1 Proofs of Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.3.3 result from direct application of Theorem 2.2.2 of the proposed algorithm.

Theorem 3.3.1 Let $p_{odd}=\left\lfloor\frac{n}{2}\right\rfloor-k$ an odd integer for $k\in\left\{1,\cdots,\left\lfloor\frac{n}{2}\right\rfloor-1\right\}$. The pathways (5) and (6) build $P_H^{n_{odd}-1}$s in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{\pi i}\sqrt[n]{1}\right),\left(d_{i,j}\right)_{n\times n}\right)$ networks if $n_{odd}\ge 2p_{odd}+3$.

$$\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor}+L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}}_{\left\lfloor\frac{p_{odd}}{2}\right\rfloor}+\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor}}_{2}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{\left\lfloor\frac{p_{odd}}{2}\right\rfloor+1}+\tilde{m}\,L^-_{\left\lfloor\frac{n}{2}\right\rfloor}\quad(5)$$

$$\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor}+L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}}_{\left\lfloor\frac{p_{odd}}{2}\right\rfloor-i}+\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor}}_{2}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{\left\lfloor\frac{p_{odd}}{2}\right\rfloor-i}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}+\tilde{m}\,L^-_{\left\lfloor\frac{n}{2}\right\rfloor}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}$$

$$+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}}_{i-1}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{2}+\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}+L^+_{\left\lfloor\frac{n}{2}\right\rfloor}}_{i},\quad 1\le i\le\left\lfloor\frac{p_{odd}}{2}\right\rfloor.\quad(6)$$

for $\tilde{m}=(n-1)-(2p_{odd}+2)$. □

Let $\overrightarrow{F.R.}$ and $\overleftarrow{B.R.}$ denote, respectively, the forward and backward readings of any sequence of $L_k^{\pm}$ segments.

Corollary 3.3.1 In $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ networks if $n_{odd}\geq 2p_{odd}+3$, forward and backward readings of the sequences (5) and (6) are $P_H^{n_{odd}-1}$. Consequently, $\overrightarrow{F.R.}$ and $\overleftarrow{B.R.}$ of the sequence (5) and (6) account for 2 plus to $2\left\lfloor\frac{p_{odd}}{2}\right\rfloor$ distinct sequences, respectively. Furthermore, $\overrightarrow{F.R.}$ and $\overleftarrow{B.R.}$ of the pathway (5) and paths (6) build $(p_{odd}+1)$ $C_H^{n_{odd}}$ s if the directed segment $\hat{L}^-_{\left\lfloor\frac{n}{2}\right\rfloor-p_{odd}}$ is initially appended to these sequences. □

Theorem 3.3.2 Let $p_{even}=\left\lfloor\frac{n}{2}\right\rfloor-k$ an even integer for $k\in\left\{1,\cdots,\left\lfloor\frac{n}{2}\right\rfloor-1\right\}$. The pathways (7) and (8) build $P_H^{n_{odd}-1}$ s in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{\pi i}\sqrt[n]{1}\right),\left(d_{i,j}\right)_{n\times n}\right)$ networks if $n_{odd}\geq 2p_{even}+3$, with $\beta=\alpha=\frac{p_{even}}{2}$ is the number of $L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}$ and $L^+_{\left\lfloor\frac{n}{2}\right\rfloor}$, respectively.

$$\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor}+L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}}_{\frac{p_{even}}{2}}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{2}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{\frac{p_{even}}{2}}+\tilde{m}\,L^-_{\left\lfloor\frac{n}{2}\right\rfloor} \tag{7}$$

$$\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor}+L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}}_{\frac{p_{even}}{2}-1-i}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{2}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{\frac{p_{even}}{2}-2-i}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}+\tilde{m}\,L^-_{\left\lfloor\frac{n}{2}\right\rfloor}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}$$

$$+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor}+L^-_{\left\lfloor\frac{n}{2}\right\rfloor-1}}_{i}+\underbrace{L^-_{\left\lfloor\frac{n}{2}\right\rfloor}}_{2}+\underbrace{L^+_{\left\lfloor\frac{n}{2}\right\rfloor-1}+L^+_{\left\lfloor\frac{n}{2}\right\rfloor}}_{i+1},\quad 0\leq i\leq\frac{p_{even}}{2}-2, \tag{8}$$

for $\tilde{m}=(n-1)-(2p_{even}+2)$ □

Corollary 3.3.2 In $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ networks if $n_{odd}\geq 2p_{even}+3$, forward and backward readings of the sequences (7) and (8) are $P_H^{n_{odd}-1}$. Particularly, the enumeration of the distinct $P_H^{n_{odd}-1}$ s given birth from the forward and backward readings of the sequences (8) depend on the $\frac{p_{even}}{2}$ evenness. Specifically,

1) If $\frac{p_{even}}{2}$ is odd, since $\left(\frac{p_{even}}{2}\right)_{odd}-1-i\neq i+1$ every sequence in (8) is not a palindrome [1]. Moreover, the $\left(\frac{p_{even}}{2}\right)_{odd}-1$ sequences defined in (8) are in couples $\overrightarrow{F.R.}$ and $\overleftarrow{B.R.}$ Specifically, the $\overrightarrow{F.R.}$ path $P_H^{n_{odd}-1}$ determined by $i=0$ coincides to $\overleftarrow{B.R.}$ path $P_H^{n_{odd}-1}$ determined by $i=\frac{p_{even}}{2}-2$, $i=1$ $\overrightarrow{F.R.}$ path coincides with $\overleftarrow{B.R.}$ of the sequence defined by $i=\frac{p_{even}}{2}-3$ and so on. That is the $\overrightarrow{F.R.}$ paths defined by (8) with $i\in\left\{0,\cdots,\frac{\frac{p_{even}}{2}-3}{2}\right\}$ coincide with the $\overleftarrow{B.R.}$ paths determined by (8) with $i\in\left\{\frac{p_{even}}{2}-2,\cdots,\frac{\frac{p_{even}}{2}-1}{2}\right\}$.

Therefore, exists $\left(\frac{p_{even}}{2}\right)_{odd}-1$ distinct $P_H^{n_{odd}-1}$ s which correspond with each one of the $\overrightarrow{F.R.}$ determined by (8). Since $\overrightarrow{F.R.}$ of (7) is different to its $\overleftarrow{B.R.}$, both $P_H^{n_{odd}-1}$ s should be added to the final enumeration. In conclusion, the distinct $P_H^{n_{odd}-1}$ s are $\left(\frac{p_{even}}{2}\right)_{odd}+1$.

2) If $\frac{p_{even}}{2}$ is even, since $\left(\frac{p_{even}}{2}\right)_{even}-1-i=i+1$, then $i=\frac{\left(\frac{p_{even}}{2}\right)_{even}-2}{2}=\frac{p_{even}}{4}-1$ this index in (8) builds a $P_H^{n_{odd}-1}$ which is a palindrome [1]. In addition, $\overrightarrow{F.R.}$ paths defined by (8) with $i\in\left\{0,\cdots,\frac{p_{even}}{4}-2\right\}$ coincide with the $\overleftarrow{B.R.}$ paths determined by (8) with $i\in\left\{\frac{p_{even}}{2}-2,\cdots,\frac{p_{even}}{4}\right\}$. Therefore, exists $\left(\frac{p_{even}}{2}\right)_{even}-2$ distinct $P_H^{n_{odd}-1}$ s which correspond with each one of $\overrightarrow{F.R.}$ paths determined by (8). Since $\overrightarrow{F.R.}$ of (7) is different to its $\overleftarrow{B.R.}$, both $P_H^{n_{odd}-1}$ s should be added to the final enumeration. In conclusion, the distinct $P_H^{n_{odd}-1}$ s are $\left(\frac{p_{even}}{2}\right)_{even}+1$. □

Theorem 3.3.3 Let $p_{even}=\left\lfloor\frac{n}{2}\right\rfloor-k$ an even integer for $k \in \left\{1,\cdots,\left\lfloor \frac{n}{2} \right\rfloor - 1\right\}$. The pathways (9) build $P_H^{n_{odd}-1}$s in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{\pi i}\sqrt[n]{1}\right),\left(d_{i,j}\right)_{n\times n}\right)$ networks if $n_{odd} \geq 2p_{even}+3$, meanwhile $\alpha = \frac{p_{even}}{2} - 1 \geq 0$, is the number of $L^+_{\left\lfloor \frac{n}{2} \right\rfloor -1}$ and $\beta = \alpha + 4 = \frac{p_{even}}{2} + 3$ the amount of $L^+_{\left\lfloor \frac{n}{2} \right\rfloor}$.

$$\underbrace{L^+_{\left\lfloor \frac{n}{2} \right\rfloor} + L^+_{\left\lfloor \frac{n}{2} \right\rfloor -1}}_{\frac{p_{even}}{2}-1-i} + \underbrace{L^+_{\left\lfloor \frac{n}{2} \right\rfloor}}_{2} + \underbrace{L^-_{\left\lfloor \frac{n}{2} \right\rfloor -1} + L^-_{\left\lfloor \frac{n}{2} \right\rfloor}}_{\frac{p_{even}}{2}-1-i} + L^-_{\left\lfloor \frac{n}{2} \right\rfloor -1} + \tilde{m}L^-_{\left\lfloor \frac{n}{2} \right\rfloor} + L^-_{\left\lfloor \frac{n}{2} \right\rfloor -1}$$
$$+ \underbrace{L^-_{\left\lfloor \frac{n}{2} \right\rfloor} + L^-_{\left\lfloor \frac{n}{2} \right\rfloor -1}}_{i} + \underbrace{L^+_{\left\lfloor \frac{n}{2} \right\rfloor}}_{2} + \underbrace{L^+_{\left\lfloor \frac{n}{2} \right\rfloor -1} + L^+_{\left\lfloor \frac{n}{2} \right\rfloor}}_{i},$$
$$0 \leq i \leq \frac{p_{even}}{2} - 1. \tag{9}$$

for $\tilde{m} = (n-1) - (2p_{even}+2)$. □

Corollary 3.3.3 In $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ networks if $n_{odd} \geq 2p_{even}+3$, forward and backward readings of the sequences (9) are $P_H^{n_{odd}-1}$s. Particularly, the enumeration of the distinct $P_H^{n_{odd}-1}$s given birth from the forward and backward readings of the sequences (9) depend on the $\frac{p_{even}}{2} - 1$ evenness. Specifically,

1) If $\frac{p_{even}}{2}$ is odd, *i.e.* $\left(\frac{p_{even}}{2}\right)_{odd} - 1$ is even, then $\left(\frac{p_{even}}{2}\right)_{odd} - 1 - i = i$, therefore the sequence in (9) build by this index $i = \frac{\left(\frac{p_{even}}{2}\right)_{odd} - 1}{2}$ is a palindrome [1]. Moreover, $\left(\frac{p_{even}}{2}\right)_{odd} - 1$ sequences defined in (9) are in couples $\overrightarrow{F.R.}$ and $\overleftarrow{B.R.}$ with the exception of that given birth by the index $i = \frac{\left(\frac{p_{even}}{2}\right)_{odd} - 1}{2}$ which its $\overrightarrow{F.R.}$ and $\overrightarrow{B.R.}$ is exactly the same pathway at all. Specifically, the $\overrightarrow{F.R.}$ path $P_H^{n_{odd}-1}$ determined by $i = 0$ coincides to $\overrightarrow{B.R.}$ path $P_H^{n_{odd}-1}$ determined by $i = \left(\frac{p_{even}}{2}\right)_{odd} - 1$, $i = 1$ $\overrightarrow{F.R.}$ path coincides with $\overrightarrow{B.R.}$ of the sequence defined by $i = \left(\frac{p_{even}}{2}\right)_{odd} - 2$ and so on, until the index $i = \frac{\left(\frac{p_{even}}{2}\right)_{odd} - 1}{2}$ at which $\overrightarrow{F.R.}$ and $\overrightarrow{B.R.}$ beget only one path. That is the $\overrightarrow{F.R.}$ paths defined by (9) with the downgraded indexes
$$i \in \left\{0,\cdots,\frac{\left(\frac{p_{even}}{2}\right)_{odd} - 1}{2} - 1\right\}$$
coincide with the $\overrightarrow{B.R.}$ paths determined by (9) with
$$i \in \left\{\left(\frac{p_{even}}{2}\right)_{odd} - 1,\cdots,\frac{\left(\frac{p_{even}}{2}\right)_{odd} - 1}{2} + 1\right\}.$$

In conclusion, exists $\left(\frac{p_{even}}{2}\right)_{odd}$ distinct $P_H^{n_{odd}-1}$s which correspond with each one of the $\overrightarrow{F.R.}$ path determined by (9).

2) If $\left(\frac{p_{even}}{2}\right)_{even}$ is even, *i.e.* $\frac{p_{even}}{2} - 1$ is odd, since $\left(\frac{p_{even}}{2}\right)_{even} - 1 - i \neq i$, then sequences (9) build $P_H^{n_{odd}-1}$s none of them are palindrome [1]. In addition, $\overrightarrow{F.R.}$ paths of the indexes $\left\{0,\cdots,\frac{\frac{p_{even}}{2} - 2}{2}\right\}$ coincides with $\overrightarrow{B.R.}$ paths of the downgraded indexes $\left\{\frac{p_{even}}{2} - 1,\cdots,\frac{\frac{p_{even}}{2} - 2}{2} + 1\right\}$, respectively. In conclusion, exists $\left(\frac{p_{even}}{2}\right)_{even}$ distinct $P_H^{n_{odd}-1}$s vis-à-vis with each one of the $\overrightarrow{F.R.}$ path determined by (9). □

Observation 3.2 Corollary 3.3.1, Corollary 3.3.2 and Corollary 3.3.3 result from Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.3.3, respectively.

In conclusion, the $P_H^{n_{odd}-1}$s which resolve the Max. Euclidean Hamiltonian Path Problems with the $\hat{L}_{\left\lfloor \frac{n}{2} \right\rfloor - p}$ as the bridge between the endings of the Hamiltonian paths are evolved by the sequences (5) and (6) if p_{odd}. Otherwise by the orderings (7)-(9). Moreover, with the exception of the palindromes their backward readings also resolve these specific Max. Traveling Salesman

Problems.

3.4. Bicoupled N_{odd}-Gons TSP Conjeture

We choose the geometric paths that start up at $C=(-1,0)$ of the quasi-spherical mirror of unitary radius, touch n times-including the last touching-anywhere on the hollowed mirror, and end up at $B=(\cos\beta,\sin\beta)$, with $-\pi\le\beta\le 0$. In this geometry each n array of angles $(\alpha_1,\cdots,\alpha_{n-1},\beta)$, see **Figure 2**, denoted (α_i,β), determines a path with $n+1$ vertices-including the initial and arrival points- and n linear branches, [8,11,12]. This path may have two or more coincident vertices and linear branches shrunk to a point. For each $\beta\in[-\pi,0]$ the $n-1$ angles $\alpha_i\in IR$ are selected (see **Figure 2**) as independent variables of the overall traveled length function of the paths $F_n(\alpha_i,\beta)$.

The length of the geometric path determined by (α_i,β), is given by (10)

$$F_n(\alpha_i,\beta)=\sum_{i=0}^{n}\sqrt{2-2\cos(\alpha_i-\alpha_{i-1})}\quad \alpha_0=\pi,\alpha_n=\beta. \tag{10}$$

When $\beta=-\pi$, $(B\equiv C)$, for any polygonal cyclic trajectory, there is an n-array $(\alpha_1,\cdots,\alpha_{n-1},-\pi)$ which characterizes them. In particular, amongst these pathways are those that have as vertices the $e^{\pi i}\sqrt[m]{1}$ points, with $m\le n$. See [10] Theorem 2.1.1. and Appendix A, from page 78 to 80 [8]. Let $H^{2n_{odd}}(\alpha_i,r)$

$$H^{2n_{odd}}(\alpha_i,r)=\sum_{i=1}^{n_{odd}}2\sqrt{1+r^2-2r\cos(\alpha_i-\alpha_{i-1})}, \tag{11}$$

be a generalized length of (10), where α_i are the analogous angular parameters with the restraints $\alpha_0=\pi$ and $\alpha_{n_{odd}}=-\pi$, and r in $(0,1)$ is the structural parameter for the locations of the coupled vertices of the inner n_{odd}-polygon, $\mathcal{N}\left(K_{N=2n_{odd}}\left(e^{i\pi}\sqrt[n_{odd}]{1},re^{i\pi}\sqrt[n_{odd}]{1}\right),d^E_{N\times N}\right)$ networks [3].

Herein, see **Figure 3**, we pose the following conjeture: Are Max. TSPs in bilayer $\mathcal{N}\left(K_{N=2n_{odd}}\left(e^{i\pi}\sqrt[n_{odd}]{1},re^{i\pi}\sqrt[n_{odd}]{1}\right),d^E_{N\times N}\right)$ networks baited for the regular shapes of the Max. TSP in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n_{odd}]{1}\right),d^E_{n\times n}\right)$ networks?

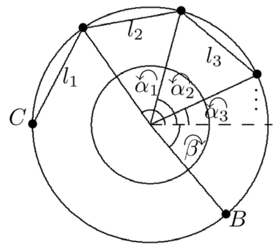

Figure 2. Measure of α_i angular parameter.

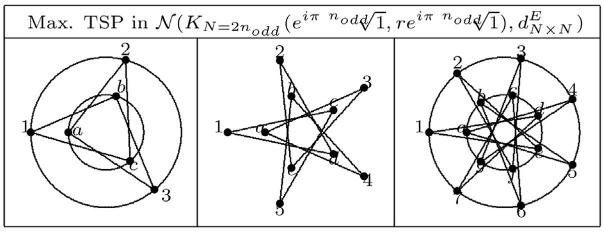

Figure 3. Max TSP in coupled N_{odd}-Gons.

4. Conclusion

This paper is an offspring of a series of previous works about Hamiltonian maximum overall traveled lengths in $\mathcal{N}\left(K_n\left(e^{\pi i}\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$ networks. Herein are singled out all the Euclidean Hamiltonian pathways that resolve the $\left\lfloor \frac{n}{2}\right\rfloor$ different maximum traveled Hamiltonian paths of order $n_{odd}-1$ in $\mathcal{N}\left(K_{n=n_{odd}}\left(e^{i\pi}\sqrt[n_{odd}]{1}\right),d^E_{n\times n}\right)$ networks. As a by-product the proposed algorithm allow us to determine the winding index of specific cyclic polygonals. The approach is a step forward on the intrinsic geometry and inherent arithmetic of the vertex locus of the N_{odd}-Gons.

REFERENCES

[1] B. I. Niel, "Every Longest Hamiltonian Path in Even N-Gons," *Discrete Mathematics, Algorithms and Applications*, Vol. 4, No. 4, 2012, p. 16. doi:10.1142/S1793830912500577

[2] B. I. Niel, "Viajes Sobre N_{odd}-Gons," EAE, 2012.

[3] B. I. Niel, W. A. Reartes and N. B. Brignole, "Every Longest Hamiltonian Path in Odd N_{odd}-Gons," *SIAM Conference on Discrete Mathematics*, Austin, 14-17 June 2010, p. 42.

[4] D. Applegate, R. Bixby, V. Chavatal and W. Cook, "Traveling Salesman Problem: A Computational Study," Princeton University Press, Princeton, 2006.

[5] A. Barvinok, E. K. Gimadi and A. I. Serdyukov, "The Maximum Traveling Salesman Problem," In: G. Gutin and A. P. Punnen, Eds., *The Traveling Salesman Problem and Its Variations*, Kluwer Academic Publishers. Dordrecht, 2002.

[6] F. Buckly and F. Harary, "Distance in Graphs," Addison-Wesley Publishing Co., Boston, 1990.

[7] S. P. Fekete, H. Meijer, A. Rohe and W. Tietze, "Solving a 'Hard' Problem to Approximate an 'Easy' One: Heuristics for Maximum Matchings and Maximum Traveling Salesman Problems," *Journal of Experimental Algorithms*, Vol. 7, 2002, 11 Pages.

[8] A. Kirillov, "On Regular Polygons, Euler's Function, and Fermat Numbers," In: S. Tabachnikov, Ed., *Kvant Selecta*: *Algebra and Analysis*, Amer Mathematical Society, Providence, 1999, pp. 87-98.

[9] H. S. M. Coxeter, "Introduction to Geometry," John Wiley & Sons, Inc., Hoboken, 1963.

[10] B. I. Niel, "Geometry of the Euclidean Hamiltonian Suboptimal and Optimal Paths in the $N\left(K_n\left(\sqrt[n]{1}\right),\left(d_{ij}\right)_{n\times n}\right)$'s Networks," *Proceedings of the VIII Dr. Antonio A. R. Monteiro*, *Congress of Mathematics*, 26-28 May 2005, Bahía Blanca, pp. 67-84. http://inmabb.criba.edu.ar/cm/actas/pdf

[11] W. R. Hamilton, "On a General Method of Expressing the Paths of Light, and of the Planets, by the Coefficients of a Characteristic Function," Vol. I, Dublin University Review and Quarterly Magazine, Dublin, 1833, pp. 795-826.

[12] B. I. Niel, "Hamilton's Real Find on Geometric Optics in a Hamiltonian Play," *Proceedings of Modelling and Simulation*, *MS*'2004, Lyon, 5-7 July 2004, pp. 9.9-9.13

