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Dominating Sets and Domination Polynomials of Square of Paths

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ABSTRACT

Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in V-S is adjacent to at least one vertex in S. Let P_n^2 be the square of the Path P_n and let $D(P_n^2, i)$ denote the family of all dominating sets of P_n^2 with cardinality i. Let $d(P_n^2, i) = |D(P_n^2, i)|$. In this paper, we obtain a recursive formula for $d(P_n^2, i)$. Using this recursive formula, we construct the polynomial, $D(P_n^2, x) = \sum_{i=\lceil \frac{n}{5} \rceil}^{n} d(P_n^2, i) x^i$, which we call domination polynomial of P_n^2 and obtain some properties of this polynomial.

Keywords: Domination Set; Domination Number; Domination Polynomials

1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = U_{n \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of G, if $N[S] = V$, or equivalently, every vertex in V-S is adjacent to at least one vertex in S. The domination number of a graph G is defined as the minimum size of a dominating set of vertices in G and it is denoted as $\gamma(G)$. A simple path is a path in which all its internal vertices have degree two, and the end vertices have degree one and is denoted by P_n.

Definition 1.1.: Let $D(G, i)$ be the family of dominating sets of a graph G with Cardinality i and let

$$d(G, i) = |D(G, i)|.$$

Then the domination polynomial $D(G, x)$ of G is defined as

$$D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i) x^i.$$

where $\gamma(G)$ is the domination number of G.

Result 1.2. [1]: If a graph G consists of m components

$G_1, \cdots, G_m$, then

$$D(G, x) = D(G_1, x) \cdots D(G_m, x).$$

Result 1.3. [1]: Let G_1 and G_2 be graphs of order n_1 and n_2 respectively. Then

$$D(G_1 \vee G_2, x) = \left((1+x)^{n_1} - 1 \right) \left((1+x)^{n_2} - 1 \right) + D(G_1, x) + D(G_2, x).$$

Result 1.4. [1]: Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if for all $S \subseteq V |N(S)| \ge |S|$.

Result 1.5. [1]: Let G be a graph of order n. Then for every $0 \le i \le \dfrac{n}{2}$, we have $d(G, i) \le d(G, i+1)$.

Result 1.6. [1]: For any graph G of order n, $\gamma(G \circ K_1) = n$.

Result 1.7. [1]: For any graph G of order n and $n \le m \le 2n$, we have

$$d(G \circ K_1, m) = \binom{n}{m-n} 2^{2n-m}.$$

Hence $D(G \circ K_1, x) = x^n (x+2)^n$.

Definition 1.8: The Square of a graph with the same set of vertices as G and an edge between two vertices if

and only if there is a path of length at most 2 between them. The second power of a graph is also called its Square.

Let P_n^2 be the square of the path P_n (2nd power) with n vertices. Let $D\left(P_n^2, i\right)$ be the family of dominating sets of the graph P_n^2 with cardinality i and let

$$d\left(P_n^2, i\right) = \left| D\left(P_n^2, i\right)\right|.$$

We call the polynomial

$$D\left(P_n^2, x\right) = \sum_{i=\left\lceil \frac{n}{5}\right\rceil}^{n} d\left(P_n^2, i\right) x^i$$

the domination polynomial of the graph P_n^2 [2].

In the next section, we construct the families of the dominating sets of the square of paths by recursive method.

As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also, we denote the set $\{1, 2, \cdots, n\}$ by $[n]$, throughout this paper.

2. Main Result

Let $D\left(P_n^2, i\right)$ be the family of dominating sets of P_n^2 with cardinality i. We investigate the dominating sets of P_n^2. We need the following lemma to prove our main results in this section.

Lemma 2.1. [3]: $\gamma\left(P_n^2\right) = \left\lceil \dfrac{n}{5}\right\rceil$ By Lemma 2.1 and the definition of domination number, one has the following Lemma:

Lemma 2.2. $D\left(P_n^2, i\right) = \Phi$ if and only if $i \succ n$ or $i \prec \left\lceil \dfrac{n}{5}\right\rceil$. A simple path is a path in which all its internal vertices have degree two, and the end vertices have degree one. The following Lemma follows from observation [2].

Lemma 2.3. [2]: If a graph G contains a simple path of length $5k − 1$, then every dominating set of G must contain at least k vertices of the path.

In order to find a dominating set of P_n^2, with cardinality i, we only need to consider

$$D\left(P_{n-1}^2, i-1\right), \quad D\left(P_{n-2}^2, i-1\right),$$
$$D\left(P_{n-3}^2, i-1\right), \quad D\left(P_{n-4}^2, i-1\right)$$

and $D\left(P_{n-5}^2, i-1\right)$. The families of these dominating sets can be empty or otherwise. Thus, we have eight combinations, whether these five families are empty or not. Two of these combinations are not **possible** (Lemma 2.5(i), & (ii)). Also, the combination that

$$D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-2}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right) = \Phi$$

$$D\left(P_{n-4}^2, i-1\right) = D\left(P_{n-5}^2, i-1\right) = \Phi$$

does not need to be considered because it implies that $D\left(P_n^2, i\right) = \Phi$ (See Lemma 2.5 (iii)). Thus we only need to consider five combinations or cases. We consider those cases in Theorem 2.7.

Lemma 2.4.: If $Y \in D\left(P_{n-6}^2, i-1\right)$, and there exists $x \in [n]$ such that $Y \cup \{x\} \in D\left(P_n^2, i\right)$, then

$$Y \in D\left(P_{n-5}^2, i-1\right).$$

Proof: Suppose that $Y \notin D\left(P_{n-5}^2, i-1\right)$. Since

$$Y \in D\left(P_{n-6}^2, i-1\right),$$

Y contains at least one vertex labeled $n − 6$, $n − 7$ or $n − 8$.

If $n - 6 \in Y$, then $Y \in D\left(P_{n-5}^2, i-1\right)$, a contradiction. Hence $n − 7$ or $n − 8 \in Y$, but then in this case, $Y \cup \{x\} \notin D\left(P_n^2, i\right)$, for any $x \in [n]$, also a contradiction.

Lemma 2.5.:

1) If

$$D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right) = \Phi$$

then

$$D\left(P_{n-2}^2, i-1\right) = \Phi$$

2) If

$$D\left(P_{n-1}^2, i-1\right) \neq \Phi, \quad D\left(P_{n-3}^2, i-1\right) \neq \Phi$$

then

$$D\left(P_{n-2}^2, i-1\right) \neq \Phi,$$

3) If

$$D\left(P_{n-1}^2, i-1\right) = \Phi, \quad D\left(P_{n-2}^2, i-1\right) = \Phi,$$

$$D\left(P_{n-3}^2, i-1\right) = \Phi, \quad D\left(P_{n-4}^2, i-1\right) = \Phi,$$

$$D\left(P_{n-5}^2, i-1\right) = \Phi, \text{ then } D\left(P_n^2, i\right) = \Phi$$

Proof:

1) Since $D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right) = \Phi$, by Lemma 2.2, $i-1 \succ n-1$ (or) $i-1 \prec \left\lceil \dfrac{n-3}{5}\right\rceil$. In either case, we have $D\left(P_{n-2}^2, i-1\right) = \Phi$.

2) Suppose that $D\left(P_{n-2}^2, i-1\right) = \Phi$, so by Lemma 2.2, we have $i-1 \succ n-2$ or $i-1 \prec \left\lceil \dfrac{n-2}{5}\right\rceil$ If $i-1 \succ n-2$, then $i-1 \succ n-3$. Therefore $D\left(P_{n-3}^2, i-1\right) = \Phi$, a contradiction.

3) Suppose that $D\left(P_n^2, i\right) \neq \Phi$, Let $Y \in D\left(P_n^2, i\right)$. Thus, at least one vertex labeled n or $n − 1$ or $n − 2$ is in Y. If $n \in Y$, then by Lemma (2.3), at least one vertex labeled $n − 1$, $n − 2$, $n − 3$, $n − 4$ or $n − 5$ is in Y. If $n − 1 \in Y$ or

$n - 2 \in Y$, then $Y - \{n\} \in D\left(P_{n-1}^2, i-1\right)$, a contradiction.

If $n - 3 \in Y$, then $Y - \{n\} \in D\left(P_{n-2}^2, i-1\right)$, a contradiction. Now, suppose that $n - 1 \in Y$. Then, by Lemma 2.3, at least one vertex labeled $n - 2$, $n - 3$, $n - 4$, $n - 5$ or $n - 6$ is in Y. If $n - 2 \in Y$ or $n - 3 \in Y$, then $Y - \{n-1\} \in D\left(P_{n-2}^2, i-1\right)$ a contradiction. If $n - 4 \in Y$, then $Y - \{n-1\} \in D\left(P_{n-3}^2, i-1\right)$, a contradiction.

If $n - 5 \in Y$, then $Y - \{n-1\} \in D\left(P_{n-4}^2, i-1\right)$, a contradiction

If $n - 6 \in Y$, then $Y - \{n-1\} \in D\left(P_{n-5}^2, i-1\right)$, a contradiction. Therefore, $D\left(P_n^2, i\right) = \Phi$.

Lemma 2.6.: If $D\left(P_n^2, i\right) \neq \Phi$, then

1) $D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-2}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right)$
$= D\left(P_{n-4}^2, i-1\right) = \Phi$ and $D\left(P_{n-5}^2, i-1\right) \neq \Phi$ if and only if $n = 5k$ and $i = k$ for some $k \in N$;

2) $D\left(P_{n-2}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right) = D\left(P_{n-4}^2, i-1\right)$
$= D\left(P_{n-5}^2, i-1\right) = \Phi$ then $D\left(P_{n-1}^2, i-1\right) \neq \Phi$ if and only if $i = n$;

3) $D\left(P_{n-1}^2, i-1\right) = \Phi$, $D\left(P_{n-2}^2, i-1\right) \neq \Phi$,
$D\left(P_{n-3}^2, i-1\right) \neq \Phi$, $D\left(P_{n-4}^2, i-1\right) \neq \Phi$,
$D\left(P_{n-5}^2, i-1\right) \neq \Phi$, if and only if $n = 5k + 2$ and
$i = \left\lceil \dfrac{5k+2}{5} \right\rceil$ for some $k \in N$;

4) $D\left(P_{n-1}^2, i-1\right) \neq \Phi$, $D\left(P_{n-2}^2, i-1\right) \neq \Phi$,
$D\left(P_{n-3}^2, i-1\right) \neq \Phi$, $D\left(P_{n-4}^2, i-1\right) \neq \Phi$, and
$D\left(P_{n-5}^2, i-1\right) = \Phi$ if and only if $i = n - 3$.

5) $D\left(P_{n-1}^2, i-1\right) \neq \Phi$, $D\left(P_{n-2}^2, i-1\right) \neq \Phi$,
$D\left(P_{n-3}^2, i-1\right) \neq \Phi$, $D\left(P_{n-4}^2, i-1\right) \neq \Phi$, and
$D\left(P_{n-5}^2, i-1\right) \neq \Phi$ if and only if

$$\left\lceil \frac{n-1}{5} \right\rceil + 1 \leq i \leq n - 4.$$

Proof:
1) ($\Rightarrow$)
Since

$$D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-2}^2, i-1\right)$$
$$= D\left(P_{n-3}^2, i-1\right) = D\left(P_{n-4}^2, i-1\right) = \Phi,$$

by Lemma 2.2,

$$i-1 \succ n-1 \quad \text{or} \quad i-1 \prec \left\lceil \frac{n-2}{5} \right\rceil$$

If $i-1 \succ n-1$, then $i \succ n$ and by Lemma 2.2,

$D\left(P_n^2, i\right) = \Phi$, a contradiction.

So $i \prec \left\lceil \dfrac{n-2}{5} \right\rceil + 1$, and since $D\left(P_n^2, i\right) \neq \Phi$, we have $\left\lceil \dfrac{n}{5} \right\rceil \leq i \leq \left\lceil \dfrac{n-2}{5} \right\rceil + 1$, which implies that $n = 5k$ and $i = k$ for some $k \in N$.

($\Leftarrow$) If $n = 5k$ and $i = k$ for some $k \in N$, then by Lemma 2.2,

$$D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-2}^2, i-1\right)$$
$$= D\left(P_{n-3}^2, i-1\right) = D\left(P_{n-4}^2, i-1\right) = \Phi,$$

and $D\left(P_{n-5}^2, i-1\right) \neq \Phi$.

2) ($\Rightarrow$) Since

$$D\left(P_{n-2}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right)$$
$$= D\left(P_{n-4}^2, i-1\right) = D\left(P_{n-5}^2, i-1\right) = \Phi,$$

by Lemma 2.2,

$$i-1 \succ n-2 \quad \text{or} \quad i-1 \prec \left\lceil \frac{n-3}{5} \right\rceil.$$

If

$$i-1 \prec \left\lceil \frac{n-3}{5} \right\rceil, \text{ then } i-1 \prec \left\lceil \frac{n-1}{5} \right\rceil$$

and hence $D\left(P_{n-2}^2, i-1\right) = \Phi$, a contradiction.
So $i-1 \succ n-2$. Also, $D\left(P_{n-1}^2, i-1\right) \neq \Phi$.

Therefore $i \succ n-1$. Therefore $i \geq n$. But $i \leq n$. Hence $i = n$.

($\Leftarrow$) If $i = n$, then by lemma 2.2,

$$D\left(P_{n-2}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right)$$
$$= D\left(P_{n-4}^2, i-1\right) = D\left(P_{n-5}^2, i-1\right) = \Phi$$

then $D\left(P_{n-1}^2, i-1\right) \neq \Phi$.

3) ($\Rightarrow$) Since $D\left(P_{n-1}^2, i-1\right) = \Phi$, by Lemma 2.2,
$i-1 \succ n-1$ or $i-1 \prec \left\lceil \dfrac{n-1}{5} \right\rceil$.

If $i-1 \succ n-1$, then $i-1 \succ n-2$, by Lemma 2.2,

$$D\left(P_{n-2}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right)$$
$$= D\left(P_{n-4}^2, i-1\right) = D\left(P_{n-5}^2, i-1\right) = \Phi,$$

a contradiction.

Therefore $i \prec \left\lceil \dfrac{n-1}{5} \right\rceil + 1$.

But $i-1 \geq \left\lceil \dfrac{n-2}{5} \right\rceil$, because $D\left(P_{n-2}^2, i-1\right) \neq \Phi$.

Therefore $i \geq \left\lceil \dfrac{n-2}{5} \right\rceil + 1$.

Hence, $\left\lceil\dfrac{n-2}{5}\right\rceil+1\le i\prec\left\lceil\dfrac{n-1}{5}\right\rceil+1$.

This holds only if $n=5k+2$ and

$$i=k+1=\left\lceil\frac{5k+2}{5}\right\rceil ,$$

for some $k\in N$.

($\Leftarrow$) If $n=5k+2$ and $i=\left\lceil\dfrac{5k+2}{5}\right\rceil$ for some $k\in N$, then by Lemma 2.2,

$$D\left(P_{n-1}^{2},i-1\right)=\Phi , \quad D\left(P_{n-2}^{2},i-1\right)\ne\Phi ,$$

$$D\left(P_{n-3}^{2},i-1\right)\ne\Phi , \quad D\left(P_{n-4}^{2},i-1\right)\ne\Phi$$

and

$$D\left(P_{n-5}^{2},i-1\right)\ne\Phi .$$

4) ($\Rightarrow$) Since $D\left(P_{n-5}^{2},i-1\right)=\Phi$, by Lemma 2.2,

$$i-1\succ n-5 \quad\text{or}\quad i-1\prec\left\lceil\frac{n-5}{5}\right\rceil .$$

Since $D\left(P_{n-2}^{2},i-1\right)\ne\Phi$, by Lemma 2.2,

$$\left\lceil\frac{n-2}{5}\right\rceil+1\le i\le n-1 .$$

Therefore $i-1\prec\left\lceil\dfrac{n-5}{5}\right\rceil$ is not possible. Therefore $i-1\succ n-5$. Therefore $i\succ n-4$.

Therefore $i\ge n-3$.
Since $D\left(P_{n-3}^{2},i-1\right)=\Phi$, $i-1\le n-4$.
Therefore $i\le n-3$. Hence $i=n-3$.
($\Leftarrow$) if $i=n-3$ then
By Lemma 2.2,

$$D\left(P_{n-1}^{2},i-1\right)\ne\Phi , \quad D\left(P_{n-2}^{2},i-1\right)\ne\Phi ,$$

$$D\left(P_{n-3}^{2},i-1\right)=\Phi , \quad D\left(P_{n-4}^{2},i-1\right)\ne\Phi$$

and

$$D\left(P_{n-5}^{2},i-1\right)=\Phi .$$

5) ($\Rightarrow$) Since $D\left(P_{n-1}^{2},i-1\right)\ne\Phi$, $D\left(P_{n-2}^{2},i-1\right)\ne\Phi$ $D\left(P_{n-3}^{2},i-1\right)\ne\Phi$, and $D\left(P_{n-5}^{2},i-1\right)=\Phi$, then by applying Lemma (2.2),

$$\left\lceil\frac{n-1}{5}\right\rceil\le i-1\le n-1 , \quad \left\lceil\frac{n-2}{5}\right\rceil\le i-1\le n-2 ,$$

$$\left\lceil\frac{n-3}{5}\right\rceil\le i-1\le n-3 , \quad \left\lceil\frac{n-3}{5}\right\rceil\le i-1\le n-4$$

and

$$\left\lceil\frac{n-3}{5}\right\rceil\le i-1\le n-5 .$$

So

$$\left\lceil\frac{n-1}{5}\right\rceil\le i-1\le n-5$$

and hence $\left\lceil\dfrac{n-1}{5}\right\rceil+1\le i\le n-4$.

($\Leftarrow$) If $\left\lceil\dfrac{n-1}{5}\right\rceil+1\le i\le n-4$, then the result follows from Lemma 2.2.

Theorem 2.7.: For every $n\ge6$ and $i\ge\left\lceil\dfrac{n}{5}\right\rceil$,

1) If

$$D\left(P_{n-1}^{2},i-1\right)=D\left(P_{n-2}^{2},i-1\right)$$
$$=D\left(P_{n-3}^{2},i-1\right)=D\left(P_{n-4}^{2},i-1\right)=\Phi$$

and

$$D\left(P_{n-5}^{2},i-1\right)\ne\Phi$$

then

$$D\left(P_{n}^{2},i\right)=\left\{\left\{3,8,\cdots,n-7,n-2\right\}\right\} .$$

2) If

$$D\left(P_{n-2}^{2},i-1\right)=D\left(P_{n-3}^{2},i-1\right)$$
$$=D\left(P_{n-4}^{2},i-1\right)=D\left(P_{n-5}^{2},i-1\right)=\Phi$$

and

$$D\left(P_{n-1}^{2},i-1\right)\ne\Phi$$

then

$$D\left(P_{n}^{2},i\right)=\left\{[n]\right\} .$$

3) If

$$D\left(P_{n-1}^{2},i-1\right)=\Phi , \quad D\left(P_{n-2}^{2},i-1\right)\ne\Phi ,$$

$$D\left(P_{n-3}^{2},i-1\right)\ne\Phi , \quad D\left(P_{n-4}^{2},i-1\right)\ne\Phi$$

and

$$D\left(P_{n-5}^{2},i-1\right)\ne\Phi$$

then

$$\begin{aligned}
&D\left(P_{n}^{2},i\right)\\
&=\left\{\left\{3,8,\cdots,n-4,n-2\right\},X_{1}\bigcup\{n\},X_{2}\bigcup\{n-1\},\right.\\
&\quad X_{3}\bigcup\{n-2\}\big/X_{1}\in D\left(P_{n-3}^{2},i-1\right),\\
&\quad \left. X_{2}\in D\left(P_{n-4}^{2},i-1\right),X_{3}\in D\left(P_{n-5}^{2},i-1\right)\right\}
\end{aligned} \qquad (2.1)$$

4) If $D\left(P_{n-3}^{2},i-1\right)=\Phi$, $D\left(P_{n-2}^{2},i-1\right)\ne\Phi$, $D\left(P_{n-1}^{2},i-1\right)\ne\Phi$,

then

$$D\left(P_n^2, i\right) = \left\{[n] - \{x\}, \ x \in [n]\right\}.$$

5) If

$$D\left(P_{n-1}^2, i-1\right) \neq \Phi, \quad D\left(P_{n-2}^2, i-1\right) \neq \Phi,$$

$$D\left(P_{n-3}^2, i-1\right) \neq \Phi, \quad D\left(P_{n-4}^2, i-1\right) \neq \Phi, \quad D\left(P_{n-5}^2, i-1\right) \neq \Phi$$

then

$$D\left(P_n^2, i\right) = \left\{X_1 \cup \{n\}, X_2 \cup \{n-1\}, X_3 \cup \{n-2\} \big/ X_1 \in D\left(P_{n-1}^2, i-1\right), X_2 \in D\left(P_{n-2}^2, i-1\right), X_3 \in D\left(P_{n-3}^2, i-1\right),\right.$$
$$\left.\{\{n-2\} \cup (X_4 - X_3)\}, \{\{n\} \cup (X_4 \cap X_3)\}, \{\{n-1\} \cup (X_5 - X_4), \big/ X_4 \in D\left(P_{n-4}^2, i-1\right), X_5 \in D\left(P_{n-5}^2, i-1\right)\}\right\} \tag{2.2}$$

Proof:

1) Since

$$D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-2}^2, i-1\right)$$
$$= D\left(P_{n-3}^2, i-1\right) = D\left(P_{n-4}^2, i-1\right) = \Phi$$

and $D\left(P_{n-5}^2, i-1\right) \neq \Phi$ by Lemma 2.6 i) $n = 5k$, and $i = k$ for some $k \in N$.

Clearly the set $\{3, 8, \cdots, n-7, n-2\}$ has $\frac{n}{5}$ elements.

By the definition of P_n^2, 3 has joining with 1, 2, 4, 5, and 8 has joining with 6, 7, 9, 10. Therefore 3 and 8 covers all the vertices up to 10 for $n = 10$. Proceeding like this, we obtain that $\{3, 8, \cdots, n-7, n-2\}$ covers all Vertices up to n. The other sets with cardinality $\frac{n}{5}$ are $\{2, 7, 12, \cdots, n-3\}$, $\{4, 9, \cdots, n-1\}$ etc. In the first set, $n-3$ does not cover the vertex n. The second set does not cover the vertex 1 and so on.

Therefore $\{3, 8, \cdots, n-7, n-2\}$ is the only dominating set of cardinality $\frac{n}{5} = k = i$.

2) We have

$$D\left(P_{n-2}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right)$$
$$= D\left(P_{n-4}^2, i-1\right) = D\left(P_{n-5}^2, i-1\right) = \Phi$$

and

$$D\left(P_{n-1}^2, i-1\right) \neq \Phi.$$

By Lemma 2.6 (ii), we have $i = n$. So,

$$D\left(P_n^2, i\right) = \left\{[n]\right\}.$$

3) We have

$$D\left(P_{n-1}^2, i-1\right) = \Phi, \quad D\left(P_{n-2}^2, i-1\right) \neq \Phi,$$

$$D\left(P_{n-3}^2, i-1\right) \neq \Phi, \quad D\left(P_{n-4}^2, i-1\right) \neq \Phi$$

and

$$D\left(P_{n-5}^2, i-1\right) \neq \Phi.$$

By Lemma 2.6 (iii), $n = 5k + 2$ and

$$i = \left\lceil \frac{5k+2}{5} \right\rceil = k+1$$

for some $k \in N$.

Since

$$X = \{3, 8, \cdots, 5k-2\} \in D\left(P_{5k}^2, k\right),$$

$$X \cup \{5k+2\} \in D\left(P_{5k+2}^2, k+1\right).$$

Also, if $X \in D\left(P_{5k+1}^2, k\right)$ then

$$X \cup \{5k+1\} \in D\left(P_{5k+2}^2, k+1\right)$$

Therefore, we have

$$\left\{\{3, 8 \cdots 5k-2, 5k+1\}, X_1 \cup \{n\}, X_2 \cup \{n-1\}, X_3 \cup \{n-2\} \big/ X_1 \in D\left(P_{n-3}^2, i-1\right),\right.$$
$$\left. X_2 \in D\left(P_{n-4}^2, i-1\right), X_3 \in D\left(P_{n-5}^2, i-1\right)\right\} \subseteq D\left(P_{5k+2}^2, k+1\right) \tag{2.3}$$

Now let $Y \in D\left(P_{5k+2}^2, k+1\right)$. Then $5k + 2$, or $5k + 1$ is in Y. If $5k + 2 \in Y$, then by Lemma 3, at least one vertex labeled $5k + 1$, $5k$ or $5k - 1$ is in Y. If $5k + 1$ or $5k$ is in Y, then

$$Y - \{5k+2\} \in D\left(P_{5k+1}^2, k\right),$$

a contradiction; because $D\left(P_{5k+1}^2, k\right) = \Phi$.

Hence $5k - 1 \in Y$, $5k \notin Y$ and $5k + 1 \notin Y$.
Therefore, $Y = X \cup \{5k+2\}$ for some $X \in D\left(P_{5k}^2, k\right)$;

that is $Y = \{3, 8, \cdots, 5k-2, 5k+1\}$. Now suppose that $5k + 1 \in Y$ and $5k + 2 \notin Y$. By Lemma 2.3 at least one vertex labeled $5k$, $5k - 1$ or $5k - 2$ is in Y. If $5k \in Y$ then

$$Y - \{5k+1\} \in D\left(P_{5k}^2, k\right) = \{3, 8, \cdots, 5k-2\},$$

a contradiction because $5k \notin X$ for all $X \in D\left(P_{5k}^2, k\right)$.

Therefore $5k - 1$, or $5k - 2$ is in Y, but $5k \notin Y$.

Thus $Y = X \cup \{5k+1\}$ for some $X \in D\left(P_{5k+1}^2, k\right)$.

So

$$D\left(P_{5k+2}^2, k+1\right) \subseteq \left\{\{3, 8, \cdots, 5k-2, 5k+1\}, X_1 \cup\{n\}, X_2 \cup\{n-1\}, X_3 \cup\{n-2\}/X_1 \in D\left(P_{n-3}^2, i-1\right),\right.$$
$$\left. X_2 \in D\left(P_{n-4}^2, i-1\right), X_3 \in D\left(P_{n-5}^2, i-1\right)\right\} \subseteq D\left(P_{5k+2}^2, k+1\right) \tag{2.4}$$

From (2.3) and (2.4),

$$D\left(P_n^2, i\right) = \left\{\{3, 8, \cdots, n-4, n-2\},\right.$$
$$\left. X_1 \cup\{n\}, X_2 \cup\{n-1\}, X_3 \cup\{n-2\}/X_1 \in D\left(P_{n-3}^2, i-1\right), X_2 \in D\left(P_{n-4}^2, i-1\right), X_3 \in D\left(P_{n-5}^2, i-1\right)\right\}.$$

4) If

$$D\left(P_{n-3}^2, i-1\right) = \Phi,$$
$$D\left(P_{n-2}^2, i-1\right) \neq \Phi,$$
$$D\left(P_{n-1}^2, i-1\right) \neq \Phi,$$

by Lemma 2.6 (iv) $i = n-1$.

Therefore $D\left(P_n^2, i\right) = \left\{[n] - \{x\} \ \ x \in [n]\right\}$.

5) $D\left(P_{n-1}^2, i-1\right) \neq \Phi$, $D\left(P_{n-2}^2, i-1\right) \neq \Phi$, $D\left(P_{n-3}^2, i-1\right) \neq \Phi$, $D\left(P_{n-4}^2, i-1\right) \neq \Phi$, $D\left(P_{n-5}^2, i-1\right) \neq \Phi$.

Let $X_1 \in D\left(P_{n-2}^2, i-1\right)$, so at least one vertex labeled $n-1$, $n-2$ or $n-3$ is in X_1. If $n-1$, $n-2$ or $n-3$ $\in X_1$, then

$$X_1 \cup\{n\} \in D\left(P_n^2, i\right).$$

Let $X_2 \in D\left(P_{n-2}^2, i-1\right)$, then $n-2$ or $n-3$ or $n-4$ is in X_2.

If $n-2$, $n-3$ or $n-4 \in X_2$, then

$$X_2 \cup\{n-1\} \in D\left(P_n^2, i\right).$$

Now let $X_3 \in D\left(P_{n-3}^2, i-1\right)$ then $n-3$, $n-4$ or $n-5$ is in X_3. If $n-3$ or $n-4$ or $n-5 \in X_3$, then

$$X_3 \cup\{n-2\} \in D\left(P_n^2, i\right).$$

Now let $X_4 \in D\left(P_{n-4}^2, i-1\right)$, then $n-4$, or $n-5$ or $n-6$ is in X_4.

If $n-4 \in X_4$, then $X_4 \cup\{n\} \in D\left(P_n^2, i\right)$. If $n-5 \in X_4$, then

$$X_4 \cup\{n-1\} \in D\left(P_n^2, i\right).$$

If $n-6 \in X_4$, then $X_4 \cup\{n-2\} \in D\left(P_n^2, i\right)$. Thus we have

$$\left\{X_1 \cup\{n\}, X_2 \cup\{n-1\}, X_3 \cup\{n-2\}/X_1 \in D\left(P_{n-1}^2, i-1\right), X_2 \in D\left(P_{n-2}^2, i-1\right), X_3 \in D\left(P_{n-3}^2, i-1\right)\right\}$$
$$\cup\left\{\{\{n-2\} \cup (X_4 - X_3)\}, \{\{n\} \cup (X_4 \cap X_3)\}, \{\{n-1\} \cup (X_5 - X_4)\}/X_4 \in D\left(P_{n-4}^2, i-1\right), X_5 \in D\left(P_{n-5}^2, i-1\right)\right\} \subseteq D\left(P_n^2, i\right) \tag{2.5}$$

Now suppose that $n-1 \in Y$, $n \in Y$ then by Lemma 2.3, at least one vertex labeled $n-2$, $n-3$ or in Y. If $n-2 \in Y$ then $Y = (X_4 - X_3) \cup\{n-2\}$ for some

$$X_4 \in D\left(P_{n-4}^2, i-1\right), \ \ X_3 \in D\left(P_{n-3}^2, i-1\right) \ .$$

Now suppose that, $n-5 \in Y$ & $n-4 \in Y$, then by

Lemma 2.3 one vertex labeled $n-6$, $n-7$, in Y.

If $n-4 \in Y$, then $Y = (X_5 - X_4) \cup\{n-1\}$ for

$$X_4 \in D\left(P_{n-4}^2, i-1\right), \ \ X_5 \in D\left(P_{n-5}^2, i-1\right),$$

So

$$D\left(P_n^2, i\right) \subseteq \left\{X_1 \cup\{n\}, X_2 \cup\{n-1\}, X_3 \cup\{n-2\}/X_1 \in D\left(P_{n-1}^2, i-1\right), X_2 \in D\left(P_{n-2}^2, i-1\right), X_3 \in D\left(P_{n-3}^2, i-1\right)\right\}$$
$$\cup\left\{\{\{n-2\} \cup (X_4 - X_3)\}, \{\{n\} \cup (X_4 \cap X_3)\}, \{\{n-1\} \cup (X_5 - X_4)\}/X_4 \in D\left(P_{n-4}^2, i-1\right), X_5 \in D\left(P_{n-5}^2, i-1\right)\right\} \tag{2.6}$$

From (2.5) & (2.6)

$$D\left(P_n^2, i\right) = \left\{X_1 \cup\{n\}, X_2 \cup\{n-1\}, X_3 \cup\{n-2\}/X_1 \in D\left(P_{n-1}^2, i-1\right), X_2 \in D\left(P_{n-2}^2, i-1\right), X_3 \in D\left(P_{n-3}^2, i-1\right)\right\}$$
$$\cup\left\{\{\{n-2\} \cup (X_4 - X_3)\}, \{\{n\} \cup (X_4 \cap X_3)\}, \{\{n-1\} \cup (X_5 - X_4)\}/X_4 \in D\left(P_{n-4}^2, i-1\right), X_5 \in D\left(P_{n-5}^2, i-1\right)\right\}$$

3. Domination polynomial of P_n^2

Let $D\left(P_n^2, x\right) = \sum_{i=\lceil \frac{n}{5} \rceil}^{n} d\left(P_n^2, i\right) x^i$ be the domination polynomial of a path P_n^2. In this section, we derive the expression for $D\left(P_n^2, x\right)$.

Theorem 3.1.

a) If $D\left(P_n^2, i\right)$ is the family of dominating sets with cardinality i of P_n^2, then

$$d\left(P_n^2, i\right) = d\left(P_{n-1}^2, i-1\right) + d\left(P_{n-2}^2, i-1\right) + d\left(P_{n-3}^2, i-1\right) + d\left(P_{n-4}^2, i-1\right) + d\left(P_{n-5}^2, i-1\right)$$

where $d\left(P_n^2, i\right) = \left|D\left(P_n^2, i\right)\right|$.

b) For every $n \geq 6$,

$$D\left(P_n^2, x\right) = x\left[D\left(P_{n-1}^2, x\right) + D\left(P_{n-2}^2, x\right) + D\left(P_{n-3}^2, x\right) + D\left(P_{n-4}^2, x\right) + D\left(P_{n-5}^2, x\right)\right]$$

with the initial values

$$D\left(P_1^2, x\right) = x, \quad D\left(P_2^2, x\right) = x^2 + 2x,$$

$$D\left(P_3^2, x\right) = x^3 + 3x^2 + 3x,$$

$$D\left(P_4^2, x\right) = x^4 + 4x^3 + 6x^2 + 2x,$$

$$D\left(P_5^2, x\right) = x^5 + 5x^4 + 10x^3 + 8x^2 + x.$$

Proof:

a) Using (i), (ii), (iii), (iv) and (v) of Theorem 2.7, we prove (a) part.

1) Suppose (i) of Theorem 2.7 holds.

From (v), we have

$$D\left(P_n^2, i\right) = \left\{\Phi \cup \{n-1\} \cup (X_5) \ X_5 \in D\left(P_{n-5}^2, i-1\right)\right\}.$$

Since

$$D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-2}^2, i-1\right) = \Phi,$$

$$D\left(P_{n-3}^2, i-1\right) = D\left(P_{n-4}^2, i-1\right) = \Phi$$

$$\left|D\left(P_{n-1}^2, i-1\right)\right| = \left|D\left(P_{n-2}^2, i-1\right)\right| = 0$$

$$\left|D\left(P_{n-3}^2, i-1\right)\right| = \left|D\left(P_{n-4}^2, i-1\right)\right| = 0$$

Therefore $\left|D\left(P_n^2, i\right)\right| = \left|D\left(P_{n-5}^2, i-1\right)\right|$.

Therefore $d\left(P_n^2, i\right) = d\left(P_{n-5}^2, i-1\right)$.

Therefore, in this case $d\left(P_n^2, i\right) = d\left(P_{n-5}^2, i-1\right)$ holds.

2) Suppose (ii) of Theorem 2.7 holds.

From (v), we have

$$D\left(P_n^2, i\right) = \left\{\Phi \cup \{n\} \cup (X_1) \ X_1 \in D\left(P_{n-1}^2, i-1\right)\right\}.$$

Since $D\left(P_{n-1}^2, i-1\right) = D\left(P_{n-3}^2, i-1\right) = \Phi$

$$D\left(P_{n-4}^2, i-1\right) = D\left(P_{n-5}^2, i-1\right) = \Phi$$

$$\left|D\left(P_{n-2}^2, i-1\right)\right| = \left|D\left(P_{n-3}^2, i-1\right)\right| = 0$$

$$\left|D\left(P_{n-4}^2, i-1\right)\right| = \left|D\left(P_{n-5}^2, i-1\right)\right| = 0$$

Therefore $\left|D\left(P_n^2, i\right)\right| = \left|D\left(P_{n-1}^2, i-1\right)\right|$

Therefore $d\left(P_n^2, i\right) = d\left(P_{n-1}^2, i-1\right)$.

Therefore, In this case $d\left(P_n^2, i\right) = d\left(P_{n-1}^2, i-1\right)$ holds.

3) Suppose (iii) of Theorem 2.7 holds.

From (v), we have

$$D\left(P_n^2, i\right) = \left\{\Phi \cup \{n-1\} \cup (X_2), \{n-2\} \cup (X_3), \cup (X_4 - X_3) \cup \{n-2\}, (X_4 \cap X_3) \cup \{n\},\right.$$
$$\left.(X_5 - X_4) \cup \{n-1\} \big/ X_2 \in D\left(P_{n-2}^2, i-1\right), X_3 \in D\left(P_{n-3}^2, i-1\right), X_4 \in D\left(P_{n-4}^2, i-1\right) \text{ and } X_5 \in D\left(P_{n-5}^2, i-1\right)\right\}.$$

Since

$$D\left(P_{n-1}^2, i-1\right) = \Phi, \quad \left|D\left(P_{n-1}^2, i-1\right)\right| = 0.$$

Therefore

$$\left|D\left(P_n^2, i\right)\right| = \left|D\left(P_{n-2}^2, i-1\right)\right| \cup \left|D\left(P_{n-3}^2, i-1\right)\right|$$
$$\cup \left|D\left(P_{n-4}^2, i-1\right)\right| \cup \left|D\left(P_{n-5}^2, i-1\right)\right|.$$

Therefore

$$d\left(P_n^2, i\right) = d\left(P_{n-2}^2, i-1\right)$$
$$+ d\left(P_{n-3}^2, i-1\right) + d\left(P_{n-4}^2, i-1\right).$$
$$+ d\left(P_{n-5}^2, i-1\right).$$

4) Suppose (v) of Theorem 2.7 holds.

From (v), we have

$$D\left(P_n^2, i\right) = \left\{\{n\} \cup (X_1), \{n-1\} \cup (X_2), \{n-2\} \cup (X_3), \cup (X_4 - X_3) \cup \{n-2\}, (X_4 \cap X_3) \cup \{n\},\right.$$
$$(X_5 - X_4) \cup \{n-1\} \big/ X_1 \in D\left(P_{n-1}^2, i-1\right), X_2 \in D\left(P_{n-2}^2, i-1\right), X_3 \in D\left(P_{n-3}^2, i-1\right),$$
$$\left. X_4 \in D\left(P_{n-4}^2, i-1\right) \text{ and } X_5 \in D\left(P_{n-5}^2, i-1\right)\right\}.$$

Therefore

$$\left|D\left(P_n^2,i\right)\right| = \left|D\left(P_{n-1}^2,i-1\right)\right| \cup \left|D\left(P_{n-2}^2,i-1\right)\right| \cup \left|D\left(P_{n-3}^2,i-1\right)\right| \cup \left|D\left(P_{n-4}^2,i-1\right)\right| \cup \left|D\left(P_{n-5}^2,i-1\right)\right|.$$

Therefore

$$\sum d\left(P_n^2,i\right) = d\left(P_{n-1}^2,i-1\right) + d\left(P_{n-2}^2,i-1\right) + d\left(P_{n-3}^2,i-1\right) + d\left(P_{n-4}^2,i-1\right) + d\left(P_{n-5}^2,i-1\right).$$

Therefore, we have the theorem.

b)

$$d\left(P_n^2,i\right)x^i = d\left(P_{n-1}^2,i-1\right)x^i + d\left(P_{n-2}^2,i-1\right)x^i + d\left(P_{n-3}^2,i-1\right)x^i + d\left(P_{n-4}^2,i-1\right)x^i + d\left(P_{n-5}^2,i-1\right)x^i$$

$$\sum d\left(P_n^2,i\right)x^i = \sum d\left(P_{n-1}^2,i-1\right)x^i + \sum d\left(P_{n-2}^2,i-1\right)x^i + \sum d\left(P_{n-3}^2,i-1\right)x^i + \sum d\left(P_{n-4}^2,i-1\right)x^i + \sum d\left(P_{n-5}^2,i-1\right)x^i$$

$$d\left(P_n^2,i\right)x^i = x\sum d\left(P_{n-1}^2,i-1\right)x^{i-1} + x\sum d\left(P_{n-2}^2,i-1\right)x^{i-1} + x\sum d\left(P_{n-3}^2,i-1\right)x^{i-1}$$
$$+ x\sum d\left(P_{n-4}^2,i-1\right)x^{i-1} + x\sum d\left(P_{n-5}^2,i-1\right)x^{i-1}$$

$$\sum d\left(P_n^2,i\right)x^i = x\left[\sum d\left(P_{n-1}^2,i-1\right)x^{i-1} + \sum d\left(P_{n-2}^2,i-1\right)x^{i-1} + \sum d\left(P_{n-3}^2,i-1\right)x^{i-1}\right.$$
$$\left. + \sum d\left(P_{n-4}^2,i-1\right)x^{i-1} + \sum d\left(P_{n-5}^2,i-1\right)x^{i-1}\right]$$

$$D\left(P_n^2,x\right) = x\left[D\left(P_{n-1}^2,x\right) + D\left(P_{n-2}^2,x\right) + D\left(P_{n-3}^2,x\right) + D\left(P_{n-4}^2,x\right) + D\left(P_{n-5}^2,x\right)\right]$$

with the initial values

$$D\left(P_1^2,x\right) = x, \ D\left(P_2^2,x\right) = x^2 + 2x,$$

$$D\left(P_3^2,x\right) = x^3 + 3x^2 + 3x$$

$$D\left(P_4^2,x\right) = x^4 + 4x^3 + 6x^2 + 2x,$$

$$D\left(P_5^2,x\right) = x^5 + 5x^4 + 10x^3 + 8x^2 + x.$$

We obtain $d\left(P_n^2,i\right)$ for $1 \le n \le 10$ as shown in **Table 1**.

Table 1. $d\left(P_n^2,i\right)$, the number of dominating set of P_n^2 with cardinality i.

i	1	2	3	4	5	6	7	8	9	10
n										
1	1									
2	2	1								
3	3	3	1							
4	2	6	4	1						
5	1	8	10	5	1					
6	0	9	18	15	6	1				
7	0	8	27	33	21	7	1			
8	0	6	34	60	54	28	8	1		
9	0	3	37	93	114	82	36	9	1	
10	0	1	34	126	206	196	118	45	10	1

In the following Theorem, we obtain some properties of $d\left(P_n^2,i\right)$

Theorem 3.2. The following properties hold for the coefficients of $D\left(P_n^2,x\right)$;

1) $d\left(P_{5n}^2,n\right) = 1$ for every $n \in N$.

2) $d\left(P_n^2,n\right) = 1$ for every $n \in N$

3) $d\left(P_n^2,n-1\right) = n$ for every $n \ge 2$

4) $d\left(P_n^2,n-2\right) = \dfrac{n(n-1)}{2}$ for every $n \ge 3$

5) $d\left(P_n^2,n-3\right) = nC_3 - 2$ for every $n \ge 4$

6) $d\left(P_n^2,n-4\right) = nC_4 - (2n-6)$ for every $n \ge 5$

7) $d\left(P_{5n-1}^2,n\right) = n+1$ for every $n \in N$.

Proof:

1) Since $D\left(P_{5n}^2,n\right) = \{3,8,\cdots,5k-2\}$, we have $d\left(P_{5n}^2,n\right) = 1$.

2) Since $D\left(P_n^2,n\right) = \{[n]\}$, we have the result

3) Since $D\left(P_n^2,n-1\right) = \{[n]-\{x\} \ x \in [n]\}$, we have $d\left(P_n^2,n-1\right) = n$.

4) By induction on n. The result is true for $n = 3$.
L.H.S. $= d\left(P_3^2,1\right) = 3$ (from table)

$$\text{R.H.S} = \left(\frac{3 \times 2}{2}\right) = 3$$

Therefore, the result is true for $n = 4$.

Now suppose that the result is true for all numbers less than "n" and we prove it for n.

By Theorem 3.1,

$d\left(P_n^2, n-2\right) = d\left(P_{n-1}^2, n-3\right) + d\left(P_{n-2}^2, n-3\right)$
$+ d\left(P_{n-3}^2, n-3\right) + d\left(P_{n-4}^2, n-3\right) + d\left(P_{n-5}^2, n-3\right)$

$= \dfrac{(n-1)(n-2)}{2} + (n-2) + 1$

$= \dfrac{n^2 - 3n + 2 + 2n - 4 + 2}{2} = \dfrac{n^2 - n}{2} = \dfrac{n(n-1)}{2}$.

5) By induction on n. The result is true for $n = 4$.
L.H.S $= d\left(P_4^2, 1\right) = 2$ (from table)

R.H.S $= d\left(P_4^2, 1\right) = \dfrac{4 \times 3 \times 2}{6} - 2$

Therefore the result is true for $n = 1$. Now suppose the result is for all natural numbers less than n.
By Theorem 3.1,

$d\left(P_n^2, n-3\right) = d\left(P_{n-1}^2, n-4\right) + d\left(P_{n-2}^2, n-4\right) + d\left(P_{n-3}^2, n-4\right) + d\left(P_{n-4}^2, n-4\right) + d\left(P_{n-5}^2, n-4\right)$

$= \dfrac{(n-1)(n-2)(n-3)}{6} - 2 + \dfrac{(n-1)(n-2)}{2} + (n-3) + 1$

$= \dfrac{1}{6}\left[(n-1)(n-2)(n-3) + 3(n-2)(n-3) + 6(n-3) - 12 + 6\right]$

$= \dfrac{1}{6}\left[(n-3)\left[(n-2)(n+2) + 6\right] - 6\right] = \dfrac{1}{6}\left[(n-3)\left[n^2 - 4 + 6\right] - 6\right]$

$= \dfrac{1}{6}\left[n^3 + 2n - 3n^2 - 6 - 6\right] = \dfrac{1}{6}\left[n^3 - 3n^2 + 2n - 12\right]$

6) By induction on n. Let $n = 5$
L.H.S $= d\left(P_5^2, 1\right) = 1$ (from table)

R.H.S $= \dfrac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} - (2n - 6) = \dfrac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4} - (2 \times 5 - 6) = 5 - 4 = 1$

Therefore the result is true for $n = 1$.
Now suppose that the result is true for all natural numbers less than or equal to n.

$d\left(P_n^2, n-4\right) = d\left(P_{n-1}^2, n-5\right) + d\left(P_{n-2}^2, n-5\right) + d\left(P_{n-3}^2, n-5\right) + d\left(P_{n-4}^2, n-5\right) + d\left(P_{n-5}^2, n-5\right)$

$= \dfrac{n(n-1)(n-2)(n-3)(n-4)}{24}(2(n-1) - 6) + \dfrac{(n-2)(n-3)(n-4)}{6}2 + \dfrac{(n-3)(n-4)}{2} + (n-4) + 1$

$= \dfrac{1}{24}\left[(n-1)(n-2)(n-3)(n-4) - 12(n-2)(n-3)(n-4) + 24(n-4) + 4(n-2)(n-3)(n-4) + 48n + 192 - 48 + 24\right]$

$= \dfrac{1}{24}\left[(n-2)(n-3)(n-4)[n-1+4] + 12(n-4)[n-3+2] - 48n + 168\right]$

$= \dfrac{1}{24}\left[(n-2)(n-3)(n-4)(n+3) + 12(n-4)(n-1) - 48n + 168\right]$

$= \dfrac{1}{24}\left[(n-4)\left[(n-2)(n-3)(n+3) + 12(n-1) - 48n + 168\right]\right]$

$= \dfrac{1}{24}\left[(n-4)\left[(n-2)(n^2 - 9) + 12n - 12 - 48n\right]\right] = \dfrac{1}{24}\left[(n-4)\left(n^3 - 9n - 2n^2 + 18\right) - 36n + 156\right]$

$= \dfrac{1}{24}\left[n^4 - 9n^2 - 2n^3 + 18n - 4n^3 + 36n + 8n^2 + 72 - 36n + 156\right] = \dfrac{1}{24}\left[n^4 - 6n^3 + 11n^2 - 54n + 144\right]$

7) From the table it is true.
Theorem 3.3.

1) $\sum_{i=n}^{5n} d\left(P_i^2, n\right) = 5 \sum_{i=2}^{5n-5} d\left(P_i^2, n-1\right)$

2) For every $j \geq \left\lceil \dfrac{n}{5} \right\rceil$,

$d\left(P_{n+1}^2, j+1\right) - d\left(P_n^2, j+1\right) = d\left(P_n^2, j\right) - d\left(P_{n-5}^2, j\right)$

3) If $S_n = \sum_{i=\left\lceil \frac{n}{5} \right\rceil}^{n} d\left(P_n^2, j\right)$, then for every $n \geq 6$,

$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5}$ with initial values S_1

$= 1$, $S_2 = 3$, $S_3 = 7$, $S_4 = 13$ and $S_5 = 25$.

Proof:

1) Proof by induction on n

First suppose that $n = 2$ then,

$$\sum_{i}^{10} d\left(P_i^2, 2\right) = 45 = 5\sum_{i=2}^{5} d\left(P_i^2, 1\right)$$

$$\sum_{i=k}^{5k} d\left(P_i^2, k\right) = \sum_{i=k}^{5k} d\left(P_{i-1}^2, k-1\right) + \sum_{i=k}^{5k} d\left(P_{i-2}^2, k-1\right) + \sum_{i=k}^{5k} d\left(P_{i-3}^2, k-1\right) + \sum_{i=k}^{5k} d\left(P_{i-4}^2, k-1\right) + \sum_{i=k}^{5k} d\left(P_{i-5}^2, k-1\right)$$

$$= 5\sum_{i=k-1}^{5(k-1)} d\left(P_{i-1}^2, k-2\right) + 5\sum_{i=k-1}^{5(k-1)} d\left(P_{i-2}^2, k-2\right) + 5\sum_{i=k-1}^{5(k-1)} d\left(P_{i-3}^2, k-2\right) + 5\sum_{i=k-1}^{5(k-1)} d\left(P_{i-4}^2, k-2\right) + 5\sum_{i=k-1}^{5(k-1)} d\left(P_{i-5}^2, k-2\right)$$

$$= 5\sum_{i=k-1}^{5k-5} d\left(P_i^2, k-1\right)$$

We have the result.

2) By Theorem 3.1, We have

$$d\left(P_{n+1}^2, j+1\right) - d\left(P_n^2, j+1\right) = d\left(P_n^2, j\right) + d\left(P_{n-1}^2, j\right) + d\left(P_{n-2}^2, j\right) + d\left(P_{n-3}^2, j\right)$$
$$+ d\left(P_{n-4}^2, j\right) + d\left(P_{n-1}^2, j\right) - d\left(P_{n-1}^2, j\right) - d\left(P_{n-2}^2, j\right) - d\left(P_{n-3}^2, j\right) - d\left(P_{n-4}^2, j\right) - d\left(P_{n-5}^2, j\right)$$
$$d\left(P_{n+1}^2, j+1\right) - d\left(P_n^2, j+1\right) = d\left(P_n^2, j\right) - d\left(P_{n-5}^2, j\right)$$

Therefore we have the result

3) By theorem (3.1), we have

$$S_n = \sum_{i=\left\lceil \frac{n}{5} \right\rceil}^{n} d\left(P_n^2, j\right)$$

$$S_n = \sum_{i=\left\lceil \frac{n}{5} \right\rceil}^{n} \left[d\left(P_{n-1}^2, j-1\right) + d\left(P_{n-2}^2, j-1\right) + d\left(P_{n-3}^2, j-1\right) + d\left(P_{n-4}^2, j-1\right) + d\left(P_{n-5}^2, j-1\right) \right]$$

$$= \sum_{i=\left\lceil \frac{n}{5} \right\rceil-1}^{n-1} d\left(P_{n-1}^2, j-1\right) + \sum_{i=\left\lceil \frac{n}{5} \right\rceil-1}^{n-1} d\left(P_{n-2}^2, j-1\right) + \sum_{i=\left\lceil \frac{n}{5} \right\rceil-1}^{n-1} d\left(P_{n-3}^2, j-1\right) + \sum_{i=\left\lceil \frac{n}{5} \right\rceil-1}^{n-1} d\left(P_{n-4}^2, j-1\right) + \sum_{i=\left\lceil \frac{n}{5} \right\rceil-1}^{n-1} d\left(P_{n-5}^2, j-1\right)$$

$$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + S_{n-5}.$$

4. Conclusions

In [2], the domination polynomial of path was studied and obtained the very important property,

$$d\left(P_n^2, i\right) = d\left(P_{n-1}^2, i-1\right) + d\left(P_{n-2}^2, i-1\right) + d\left(P_{n-3}^2, i-1\right).$$

It is interesting that we have derived an analogues relation for the square of path of the form

$$d\left(P_n^2, i\right) = d\left(P_{n-1}^2, i-1\right) + d\left(P_{n-2}^2, i-1\right) + d\left(P_{n-3}^2, i-1\right)$$
$$+ d\left(P_{n-4}^2, i-1\right) + d\left(P_{n-5}^2, i-1\right)$$

One can characterise the roots of the polynomial $D\left(P_n^2, x\right)$ and identify whether they are real or complex.

Another interesting character r to be investigated is whether $D\left(P_n^2, x\right)$ is log concave are not.

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