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# Reverse Total Signed Vertex Domination in Graphs

**Wensheng Li**
Department of Mathematics and Information Sciences, Langfang Teachers College, Langfang, China
Email: wsli@live.cn

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## ABSTRACT

Let $G=(V,E)$ be a simple graph with vertex set *V* and edge set *E*. A function $f:V\cup E\to\{-1,1\}$ is said to be a reverse total signed vertex dominating function if for every $v\in V$, the sum of function values over $v$ and the elements incident to $v$ is less than zero. In this paper, we present some upper bounds of reverse total signed vertex domination number of a graph and the exact values of reverse total signed vertex domination number of circles, paths and stars are given.

**Keywords:** Reverse Total Signed Vertex Domination; Upper Bounds; Complete Bipartite Graph

## 1. Introduction

In this paper we shall use the terminology of [1]. Let $G=(V,E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)|=n$, $|E(G)|=m$. For every $v\in V$, the open neighborhood of $v$, de- noted by $N_G(v)$, is a set $\{u|uv\in E\}$ and the closed neighborhood of $v$, denoted by $N_G[v]$, is a set $N_G(v)\cup\{v\}$. We write $d_G(v)$ for the degree of a vertex $v\in V(G)$ and the maximum and minimum degree of $G$ are denoted by $\Delta$ and $\delta$, respectively. For every $v\in V$, the edge-closed neighborhood of $v$, denoted by $N_E[v]$, is $N_E[v]=\{x|x=v \text{ or } x \text{ is incident to } v\}$.

Many domination parameters in graphs has been studied richly [2-4] A function $f:V\to\{-1,1\}$ is a signed dominating function if for every vertex

$$v\in V,\ f(N[v])=\sum_{u\in N[v]} f(u)\ge 1.$$

The weight $\omega(f)$ of $f$ is the sum of the function values of all vertices in $G$. The signed domination number $\gamma_s(G)$ of $G$ is the minimum weight of signed dominating functions on $G$. This concept was introduced by Dunbar *et al.* [5] and has been studied by several authors [6-9]. As an extension of the signed domination, we give the definition of the reverse total signed vertex domination in a graph.

**Definition 1.** Let $G=(V,E)$ be a simple graph. A reverse total signed vertex dominating function of $G$ is a function $f:V\cup E\to\{-1,1\}$ such that $f(N_E[v])\le 0$ for all $v\in V$. The reverse total signed vertex domination number of $G$, denoted by $\gamma_{tsv}^0(G)$, is the maximum weight of a reverse total signed vertex dominating function of $G$. A reverse total signed vertex dominating function $f$ is called a $\gamma_{tsv}^0$-function of $G$ if $w(f)=\gamma_{tsv}^0(G)$.

## 2. Properties of Reverse Total Signed Vertex Domination

**Proposition 1** *For any graph* $G$,

$$\gamma_{tsv}^0(G)=(n+m)(\mathrm{mod}\,2).$$

**Proof.** Let $f$ be a $\gamma_{tsv}^0$-function of $G$. Then

$$\gamma_{tsv}^0(G)=f(V)+f(E).$$

Let

$$V_1=\{v\in V(G)|f(v)=1\},$$

$$V_2=\{v\in V(G)|f(v)=-1\},$$

$$E_1=\{e\in E(G)|f(e)=1\},$$

$$E_2=\{e\in E(G)|f(e)=-1\}.$$

Then

$$\begin{aligned}\gamma_{tsv}^0(G)&=|V_1|-|V_2|+|E_1|-|E_2|\\&=|V_1|-(n-|V_1|)+|E_1|-(m-|E_1|).\\&=2|V_1|+2|E_1|-(n+m)\end{aligned}$$

Therefore $\gamma_{tsv}^0(G)=(n+m)(\mathrm{mod}\,2)$.

**Propositon 2** *For any graph* $G$, $\gamma_{tsv}^0(G)\le m$.

**Proof.** Let $f$ be a $\gamma_{tsv}^0$-function of $G$. Then for every $v\in V(G)$, $f(N_E[v])\le 0$ and we have

 

$$0 \ge \sum_{v\in V(G)} f\left(N_E[v]\right) = \sum_{v\in V(G)} f(v) + 2\sum_{uv\in E(G)} f(uv) = f(V) + 2f(E).$$

Thus $\gamma_{tsv}^0(G) = f(V) + f(E) \le -f(E) \le m$.

**Propositon 3** *For any graph* $G$, $\gamma_{tsv}^0(G) \le \lfloor n\Delta/2 \rfloor$.

**Proof.** Let $f$ be a $\gamma_{tsv}^0$-function of $G$. $V_1$, $V_2$, $E_1$ and $E_2$ are defined as Proposition 2. Then

$$\gamma_{tsv}^0(G) = f(V) + f(E) = |V_1| - |V_2| + |E_1| - |E_2|.$$

We define two induced graphs $G_1$ and $G_2$ of $G$ as follows:

$$V(G_1) = V(G_2) = V(G),\quad E(G_1) = E_1,\quad E(G_2) = E_2.$$

Then for every $v \in V(G_1)$,

$$f\left(N_E[v]\right) = f(v) + d_{G_1}(v) - d_{G_2}(v) \le 0$$

and $d_{G_1}(v) - d_{G_2}(v) \le -1$. For every $v \in V(G_2)$, we have

$$f\left(N_E[v]\right) = f(v) + d_{G_1}(v) - d_{G_2}(v) \le 0$$

and $d_{G_1}(v) - d_{G_2}(v) \le 1$. Thus

$$\begin{aligned} f(E) &= |E(G_1)| - |E(G_2)| \\ &= \frac{1}{2}\sum_{v\in V(G)} d_{G_1}(v) - \frac{1}{2}\sum_{v\in V(G)} d_{G_2}(v) \\ &= \frac{1}{2}\sum_{v\in V(G)}\left(d_{G_1}(v) - d_{G_2}(v)\right) \\ &= \frac{1}{2}\Big(\sum_{v\in V_1}\left(d_{G_1}(v) - d_{G_2}(v)\right) \\ &\quad + \sum_{v\in V_2}\left(d_{G_1}(v) - d_{G_2}(v)\right)\Big) \\ &\le \frac{1}{2}\left(|V_2| - |V_1|\right). \end{aligned}$$

Therefore

$$\begin{aligned} \gamma_{tsv}^0(G) &= |V_1| - |V_2| + f(E) \\ &\le |V_1| - |V_2| + \frac{1}{2}\left(|V_2| - |V_1|\right) \\ &= \frac{1}{2}\left(2|V_1| - n\right). \end{aligned}$$

Since

$$\begin{aligned} 0 &\ge \sum_{v\in V(G)} f\left(N_E[v]\right) \\ &= \sum_{v\in V(G)} f(v) + 2\sum_{uv\in E(G)} f(uv) \\ &= f(V) + 2f(E) = |V_1| - |V_2| + 2f(E) \\ &= |V_1| - |V_2| + \sum_{v\in V(G)}\left(d_{G_1}(v) - d_{G_2}(v)\right) \\ &\ge |V_1| - |V_2| + n(0 - \Delta) \\ &= 2|V_1| - n - n\Delta \end{aligned},$$

we have $|V_1| \le \dfrac{n + n\Delta}{2}$. Therefore $\gamma_{tsv}^0(G) \le \lfloor n\Delta/2 \rfloor$.

**Propositon 4** *For any star* $K_{1,n}$, $\gamma_{tsv}^0(K_{1,n}) = 1$.

**Proof.** Let $f$ be a $\gamma_{tsv}^0$-function. Let

$$V(K_{1,n}) = \{v_0, v_1, v_2, \cdots, v_n\},$$

$$E(K_{1,n}) = \{v_0v_1, v_0v_2, v_0v_3, \cdots, v_0v_n\},$$

where $v_0$ is the center of $K_{1,n}$. Since for every $v \in V(K_{1,n})$, $f\left(N_E[v]\right) \le 0$, we have

$$\begin{aligned} \gamma_{tsv}^0(K_{1,n}) &= f(V) + f(E) \\ &= \sum_{i=1}^{n} f\left(N_E[v_i]\right) + f(v_0). \\ &\le 0 + f(v_0) \le 1 \end{aligned}$$

On the other hand, consider the function

$$g:\ V(K_{1,n}) \cup E(K_{1,n}) \to \{-1, 1\},$$

such that

$$g(v_i) = 1\ (0 \le i \le n),\quad g(v_0v_j) = -1\ (1 \le j \le n).$$

Then $g$ is a reverse total signed vertex dominating function on $K_{1,n}$ and

$$w(g) = g(V) + g(E) = 1 + n - n = 1.$$

Thus $\gamma_{tsv}^0(K_{1,n}) \ge w(g) = 1$, which implies that $\gamma_{tsv}^0(K_{1,n}) = 1$.

**Propositon 5** *For any circle* $C_n$, $\gamma_{tsv}^0(C_n) = 0$.

**Proof.** Let $f$ be a $\gamma_{tsv}^0$-function of $C_n$. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$, $E(C_n) = \{v_1v_2, v_2v_3, \cdots, v_nv_1\}$. Since for every $v \in C_n$, $\|N_E[v]\| = 3$, we have

$$f\left(N_E[v]\right) \le -1.$$

Thus

$$\begin{aligned} -n &\ge \sum_{v\in V(G)} f\left(N_E[v]\right) \\ &= \sum_{v\in V(G)} f(v) + 2\sum_{uv\in E(G)} f(uv). \\ &= f(V) + 2f(E) \end{aligned}$$

Therefore $\gamma_{tsv}^0(C_n) = f(V) + f(E) \le -n - f(E) \le 0$.

On the other hand, consider the mapping

$$g:\ V(C_n) \cup E(C_n) \to \{-1, 1\},$$

such that

$$g(v_i) = 1\ (0 \le i \le n),\quad g(e_i) = -1\ (1 \le i \le n).$$

Then $g$ is a reverse total signed vertex dominating function on $C_n$ and $w(g) = 0$. Therefore

$$\gamma_{tsv}^0(C_n) \ge w(g) = 0,$$

which implies $\gamma_{tsv}^0(C_n) = 0$.

 

**Propositon 6** *For any complete bipartite graph* $K_{2,n}$ $(n \geq 2)$, $\gamma_{tsv}^{0}(K_{2,n}) = 2-n$.

**Proof.** Let $f$ be a $\gamma_{tsv}^{0}$ -function. Let

$$U = \{u_1, u_2\},\ V = \{v_1, v_2, \cdots, v_{n-1}, v_n\},$$

$$V(K_{2,n}) = U_1 \cup V_1$$

and

$$E(K_{2,n}) = \{u_1 v_i, u_2 v_i \mid 1 \leq i \leq n\}.$$

Since for every $v \in V$, $|N_E[v]| = 3$, we have $f(N_E[v]) \leq -1$. Therefore

$$\begin{aligned}\gamma_{tsv}^{0}(K_{2,n}) &= f(V) + f(E) \\ &= \sum_{v \in V} f(N_E[v]) + f(u_1) + f(u_2). \\ &\leq 2-n\end{aligned}$$

On the other hand, consider the mapping

$$g : V(K_{2,n}) \cup E(K_{2,n}) \to \{-1, 1\}$$

such that $g(u_1) = g(u_2) = 1$, $g(v_j) = 1$ for $1 \leq j \leq n$, $g(u_i v_j) = -1$ for $i \in \{1,2\}$ and $1 \leq j \leq n$. Then $g$ is a reverse total signed vertex dominating function on $K_{2,n}$ and $w(g) = 2-n$. Therefore $\gamma_{tsv}^{0}(K_{2,n}) \geq w(g) = 2-n$, which implies $\gamma_{tsv}^{0}(K_{2,n}) = 2-n$.

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