Open Journal of Discrete Mathematics, 2013, 3, 39-42
http://dx.doi.org/10.4236/ojdm.2013.31008 Published Online January 2013 (http://www.scirp.org/journal/ojdm)

On Cubic Nonsymmetric Cayley Graphs*

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Received October 8, 2012; revised November 8, 2012; accepted November 20, 2012

ABSTRACT

Let $\Gamma = \mathrm{Cay}(G,S)$ be a connected Cayley graph of group *G*, then Γ is called normal if the right regular representation of *G* is a normal subgroup of $\mathrm{Aut}\Gamma$, the full automorphism group of Γ. For the case where *G* is a finite nonabelian simple group and Γ is symmetric cubic Cayley graph, Caiheng Li and Shangjin Xu proved that Γ is normal with only two exceptions. Since then, the normality of nonsymmetric cubic Cayley graph of nonabelian simple group aroused strong interest of people. So far such graphs which have been known are all normal. Then people conjecture that all of such graphs are either normal or the Cayley subset consists of involutions. In this paper we give an negative answer by two counterexamples. As far as we know these are the first examples for the non-normal cubic nonsymmetric Cayley graphs of finite nonabelian simple groups.

Keywords: Cubic Cayley Graph; Nonsymmetric; Non-Normal

1. Introduction

All graphs in this paper are assumed to be finite, simple, connected and undirected.

Let Γ be a graph and denote $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Arc}(\Gamma)$ and $\mathrm{Aut}\Gamma$ the vertex set, edge set, arc set and full automorphism group respectively. Denote $\mathrm{val}(\Gamma)$ the valency of Γ. Then Γ is said to be *X-vertex-transitive*, *X-edge-transitive* and *X-arc-transitive* if $X \le \mathrm{Aut}\Gamma$ acts transitively on $V(\Gamma)$, $E(\Gamma)$, and $\mathrm{Arc}(\Gamma)$ respectively. And further Γ is simply called *vertex-transitive*, *edge-transitive* and *arc-transitive* when $X = \mathrm{Aut}\Gamma$. Sometimes arc-transitive graph is simply called *symmetric graph*.

A graph $\Gamma = \mathrm{Cay}(G,S)$ is a *Cayley graph* of a group G if there is a subset $1 \notin S \subseteq G$ with

$$S = S^{-1} := \left\{ s^{-1} \middle| s \in S \right\},$$

such that $V(\Gamma) = G$ and

$$E\Gamma = \left\{ (g, sg) \middle| g \in G, s \in S \right\}.$$

Bydefinition, Γ has valency $|S|$, and it is connected if and only if $\langle S \rangle = G$. Moreover, G can be viewed as a regular subgroup of $\mathrm{Aut}\Gamma$ by right multiplication action on V(Γ). For convenience, we still denote this regular subgroup by *G*. Then a Cayley graph is vertex-transitive. On the contrary a vertex-transitive graph Γ is a Cayley graph of a group *G* if and only if $\mathrm{Aut}\Gamma$ contains a subgroup that is regular on $V(\Gamma)$ and isomorphic to *G* (see [1, Proposition 16.3]). If *G* is a normal subgroup of $\mathrm{Aut}\Gamma$, then Γ is called a *normal Cayley graph* of *G*. The $\mathrm{Cay}(G,S)$ is said to be *core-free* (with respect to *G*) if *G* is core-free in some

$$X \le \mathrm{Aut}(\mathrm{Cay}(G,S)),$$

that is,

$$\mathrm{Core}_X(G) = \bigcap\nolimits_{x \in X} G^x = 1.$$

Let *X* be an arbitrary finite group with a core-free subgroup *H* and let *D* be a union of several double cosets of H satisfying $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(X,H,D)$ is the graph with vertex set

$$[X:H] := \left\{ Hx \middle| x \in X \right\}$$

such that Hx and Hy are adjacent if and only if $yx^{-1} \in HDH$. Consider the action of *X* on $[X:H]$ by right multiplication on right cosets. Note this action is faithful and preserves the adjacency of the coset graph, thus we identify *X* with a subgroup of $\mathrm{Aut}(\mathrm{Cos}(X,H,D))$. Obviously, $\mathrm{Cos}(X,H,D)$ is connected if and only if $\langle H,D \rangle = X$. The valency of $\mathrm{Cos}(X,H,D)$ is $|D:H|$. Let $\Gamma(H)$ be the set of vertices of $\mathrm{Cos}(X,H,D)$, which are adjacent with *H*. It is easy to check that *H* has

*The project sponsored by the scientific research foundation of Guangxi University (Grant No. XBZ110328), the fund of Yunnan University (No. 2012CG015), the National Natural Science Foundation of China (No. 11126343) and (No. 11171292), and NSF of Guangxi (No. 2012-GXNSFBA053010).
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n orbits on $\Gamma(H)$ if and only if D is the union of n double cosets of H. Further, the properties stated in the following lemma are well-known, its proof can be found in [2-4].

Proposition 1.1 *Let* $\mathrm{Cos}(X,H,D)$ *be defined as above*.

1) If $\mathrm{Cos}(X,H,D)$ is a X-symmetric graph of valency at least 3, then there exists an element $g \in X \setminus H$ satisfying $g^2 \in H$ and $\langle H, g\rangle = X$. Furthermore, we may choose g to be a 2-element;

2) Let $\Gamma = \mathrm{Cay}(G,S)$ be a Cayley graph and $G \le X \le \mathrm{Aut}(\Gamma)$. Let $H = X_\alpha$ be the stabilizer of $\alpha \in V(\Gamma)$ in X. We have $\Gamma \cong \mathrm{Cos}(X,H,HSH)$;

3) Let $\mathrm{Cos}(X,H,D)$ be a coset graph and G be a complement of H in X. Denote $S = G \cap D$. Then the Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ is isomorphic to $\mathrm{Cos}(X,H,D)$, and hence $|S| = |D:H|$. In particular, S contains an involution of G if the valency of Γ is odd.

Tutte [5,6] proved that every finite connected cubic symmetric graph is s-regular for some $s \le 5$. Since Tutte's seminal work, the study of s-arc-transitive graphs, aiming at constructing and characterizing such graphs, has received considerable attention in the literature, see [7-12] for example, and now there is an extensive body of knowledge on such graphs. Fang, Li, Wang and Xu [13] proved that for most finite nonabelian simple groups, the corresponding connected cubic Cayley graphs are normal. Caiheng Li [14] and Shangjin Xu [15] proved that every cubic symmetric Cayley graph of finite nonabelian simple group is normal except two 5-arc transitive graphs of the alternating group A_{47} (up to isomorphic). Then it arises a natural problem: whether each of the cubic non-symmetric Cayley graph of finite nonabelian simple group is normal? This problem has become the topics of greatest concern after the results of Li and Xu. Based on past experience, people conjure that if there exist some normal graphs, then the Cayley subsets of them must be consist of involutions. However, there have no any answer to the problem until now.

To answer this problem, by studying cubic nonsymmetric Cayley graphs, we give a negative answer. In the present paper, we give two non-normal examples which subsets are not consist of involutions. It's worth noting that these are the first examples for the non-normal cubic nonsymmetric Cayley graphs of finite nonabelian simple groups.

In the rest of this section, we assume that $\Gamma = \mathrm{Cay}(G,S)$ is a cubic nonsymmetric Cayley graph with $G \le X \le \mathrm{Aut}\Gamma$. Denote H the vertex stabilizer of X on $1 \in V\Gamma$. Note Γ is cubic nonsymmetric, then H must be 2-group. Let N be the maximal one among normal subgroups of X contained in G, that is, $N = \mathrm{Core}_X(G)$ is the core of G in X.

2. Main Results

In this section, we construct some examples of cubic nonnormal nonsymmetric Cayley graphs on finite nonableian simple groups.

Example 2.1 *Let* G *be the alternating group* A_{15} and *set* $S = \{x_1, x_2, x_2^{-1}\}$, *where*

$$x_1 = (5\ \ 7)(6\ \ 8)(9\ \ 16)(10\ \ 13)(11\ \ 14)(12\ \ 15),$$
$$x_2 = (2\ \ 11\ \ 4\ \ 8\ \ 13\ \ 6)(3\ \ 14\ \ 6)$$
$$(5\ \ 12\ \ 10)(9\ \ 15).$$

Let $\Gamma = \mathrm{Cay}(G,S)$, then Γ is cubic nonsymmetric connected Cayley graph, which is not normal.

Let $\Sigma = \mathrm{Cos}(X,H,D)$ be a connected graph, where $D = Hx_1H \cup Hx_2H$ and $H = \langle a,b,c\rangle$ with

$$a = (1\ \ 2\ \ 3\ \ 4)(5\ \ 6\ \ 7\ \ 8)$$
$$(9\ \ 10\ \ 11\ \ 12)(13\ \ 14\ \ 15\ \ 16)$$
$$b = (1\ \ 16)(2\ \ 15)(3\ \ 14)(4\ \ 13)$$
$$(5\ \ 12)(6\ \ 11)(7\ \ 10)(8\ \ 9)$$
$$c = (1\ \ 5)(12\ \ 16)(2\ \ 6)(15\ \ 11)$$
$$(3\ \ 7)(14\ \ 10)(4\ \ 8)(13\ \ 9)$$

It is easy to see $\langle a,b\rangle = D_8$ with 1 and 5 being in different orbits, which follows $c \notin \langle a,b\rangle$. On the other hand, $a^c = a$ and $b^c = b$ lead to

$$H = \langle a,b\rangle \times \langle c\rangle \cong D_8 \times \mathbb{Z}_2.$$

Simple computation shows $|Hx_1H : H| = 1$ and $|Hx_2H : H| = 2$, *i.e.*, $|D:H| = 3$. Then the valency of Σ is 3, and moreover Σ is nonsymmetric since $|H| = 16$. Since Σ is connected, so $\langle a,b,c,x_1,x_2\rangle = X$. Let $\Omega = \{1,2,3,\cdots,16\}$. Clearly $\langle x_1, x_2\rangle$ acts transitively on $\Omega - \{1\}$, which follows X acts 2-transitively on Ω, and hence primitively, on Ω. Let $\rho = x_2\left(x_2^2\right)^{x_1} x_2^4$. Then $\rho \in X$ and X contains a 5-cycle $\rho^{12} = (5\ \ 12\ \ 11\ \ 10\ \ 13)$. Noting that every generator of X is even permutation, $X \cong A_{16}$ by [16, Theorem 3.3E]. Then the stabilizer $X_1 \cong A_{15} \cong G$. On the other hand H acts regularly on Ω leads to that X_1 acts regularly on $[X:H]$. Simple computation shows $X_1 \cap D = S$. Hence $\Sigma \cong \Gamma$ by Proposition 1.1, and furthermore Γ is connected by the connectivity of Σ. Namely $G = \langle x_1, x_2\rangle$, which leads to $G = X_1$. However $x_1^b = (18)(25)(36)(47)(911)(1012)$, which changes 1. Thus $x_1^b \notin G$, *i.e.*, G is not normal in X.

Example 2.2 *Let* G *be the alternating group* A_{31}. *Set* $S = \{x_1, x_2, x_2^{-1}\}$, *where*

$$
\begin{aligned}
x_1 = &(2\ \ 6)(4\ \ 8)(9\ \ 15)(10\ \ 12)(11\ \ 13)\\
&(14\ \ 16)(17\ \ 32)(18\ \ 25)(19\ \ 30)(20\ \ 27)\\
&(21\ \ 28)(22\ \ 29)(23\ \ 26)(24\ \ 31),\\
x_2 = &(2\ \ 10\ \ 12\ \ 4)(5\ \ 26\ \ 13\ \ 28)\\
&(6\ \ 17\ \ 8\ \ 25)(7\ \ 18\ \ 15\ \ 20)\\
&(14\ \ 19\ \ 16\ \ 27)(21\ \ 31)\\
&(22\ \ 24\ \ 32\ \ 30)(3\ \ 9).
\end{aligned}
$$

Let $\Gamma = \mathrm{Cay}(G,S)$, then Γ is cubic nonsymmetric connected Cayley graph, which is not normal.

Let $\Sigma = \mathrm{Cos}(X,H,D)$ be a connect vertex-transitive graph, where $D = Hx_1H \cup Hx_2H$ and $H = \langle a,b,c,d \rangle$ with

$$
\begin{aligned}
a = &(1\ \ 2\ \ 3\ \ 4)(5\ \ 6\ \ 7\ \ 8)\\
&(9\ \ 10\ \ 11\ \ 12)(13\ \ 14\ \ 15\ \ 16)\\
&(17\ \ 18\ \ 19\ \ 20)(21\ \ 22\ \ 23\ \ 24)\\
&(25\ \ 26\ \ 27\ \ 28)(29\ \ 30\ \ 31\ \ 32),\\
b = &(1\ \ 32)(2\ \ 31)(3\ \ 30)(4\ \ 29)(5\ \ 28)(6\ \ 27)\\
&(7\ \ 26)(8\ \ 25)(9\ \ 24)(10\ \ 23)(11\ \ 22)\\
&(12\ \ 21)(13\ \ 20)(14\ \ 19)(15\ \ 18)(16\ \ 17),\\
c = &(1\ \ 5)(32\ \ 28)(2\ \ 6)(31\ \ 27)(3\ \ 7)(30\ \ 26)\\
&(4\ \ 8)(29\ \ 25)(9\ \ 13)(24\ \ 20)(10\ \ 14)\\
&(23\ \ 19)(11\ \ 15)(22\ \ 18)(12\ \ 16)(21\ \ 17),\\
d = &(1\ \ 9)(32\ \ 24)(2\ \ 10)(31\ \ 23)(3\ \ 11)(30\ \ 22)\\
&(4\ \ 12)(29\ \ 21)(5\ \ 13)(28\ \ 20)(6\ \ 14)\\
&(27\ \ 19)(7\ \ 15)(26\ \ 18)(8\ \ 16)(25\ \ 17).
\end{aligned}
$$

It is trivial for us to get $\langle a,b \rangle \cong D_8$ with 1, 5 and 9 being in different orbits, then $c,d \notin \langle a,b \rangle$. By simple checking, we find $c^d = c, b^d = b$ and $a^d = a$. It follows

$$H = \langle a,b \rangle \times \langle c \rangle \times \langle d \rangle \cong D_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 .$$

Note that $|Hx_1H : H| = 1$ and $|Hx_2H : H| = 2$, then $|D:H| = 3$. Namely the valency of Σ is 3. However $|H| = 2^5$, thus Σ is nonsymmetric. Notice that Σ is connected, *i.e.*,

$$X = \langle a,b,c,d,x_1,x_2 \rangle .$$

Set $\Omega = \{1,2,3,\cdots,32\}$. Clearly $\langle x_1,x_2 \rangle$ acts transitively on $\Omega - \{1\}$, and then X acts 2-transitively on Ω, and hence primitively, on Ω. Let $\rho = x_2^{x_1} x_2 \left(x_2^2\right)^{x_1}$. Then $\rho \in X$ and X contains a 17-cycle

$$\rho^{24} = (28283021202918312625236121335) .$$

Note that all generators of X are even permutations, then $X \cong \mathrm{A}_{32}$ by [16, Theorem 3.3E]. Then the stabilizer $X_1 \cong \mathrm{A}_{31} \cong G$. Noting H acts regularly on Ω, X_1 acts regularly on $[X:H]$. It is shown, by computing, that $X_1 \cap D = S$. That is $\Sigma \cong \Gamma$ by Proposition 1.1. And moreover the connectivity of Σ leads to Γ is also connected. Hence $G = \langle x_1,x_2,x_3 \rangle$, *i.e.*, $G = X_1$. However

$$
\begin{aligned}
x_1^a = &(1\ \ 5)(3\ \ 7)(9\ \ 11)(10\ \ 16)\\
&(12\ \ 14)(13\ \ 15)(17\ \ 28)\\
&(18\ \ 29)(19\ \ 26)(20\ \ 31)\\
&(21\ \ 32)(22\ \ 25)(23\ \ 30)(24\ \ 27),
\end{aligned}
$$

which changes 1. Thus $x_1^a \notin G$, *i.e.*, G is not normal in *X*.

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