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Guarding a Koch Fractal Art Gallery

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ABSTRACT

This article presents a generalization of the standard art gallery problem to the case where the sides of the gallery are continuous curves which are limits of polygonal arcs. The allowable limiting processes for such generalized art galleries are defined. We construct an art gallery in which one side is the Koch fractal and the other sides are three sides of a rectangle. The appropriate measure of coverage by guards is not the total number of guards but, rather, the guards-to-side ratio. We compute this ratio for the cases of shallow and deep versions of the Koch fractal art gallery.

Keywords: Art Gallery Theorem; Koch Fractal; Difference Equation

1. Introduction

O'Rourke [1] describes how at a geometry conference in 1973 at Stanford University Victor Klee extemporaneously gave to Vasek Chvátal what has become known as the classical art gallery problem: Determine the minimum number of guards sufficient to cover the interior of any art gallery with n walls.

To make the statement of the problem more precise we introduce some notation and definitions. We say that a set Π in the plane is a polygon if Π is compact, connected, and simply-connected and if the boundary $\partial\Pi$ of Π is a polygonal Jordan curve. A point A in Π sees or covers another point B in Π if the line segment connecting the two points is contained in Π. A finite subset $\Gamma(\Pi)$ of points in Π is a set of guards, or watchmen, if for each point B in Π there is a point A in $\Gamma(\Pi)$ such that A sees B. We define $G(\Pi)$ to be the smallest cardinality of any set of guards for Π. For any natural number n greater than 2, let $P(n)$ be the set of polygons that have exactly n vertices, and set $g(n)=\max\{G(\Pi):\Pi\in P(n)\}$. With these definitions Klee's problem takes the form: For each n, find $g(n)$.

In 1975 Chvátal [2] published a proof that $g(n)\le\left\lfloor\frac{n}{3}\right\rfloor$. For any polygon Π in $P(n)$ we then have $G(\Pi)\le\left\lfloor\frac{n}{3}\right\rfloor$. If we define the guard-to-side ratio, $gsr(\Pi)$, of a polygon as $G(\Pi)$ divided by n, then the last inequality is equivalent to $gsr(\Pi)\le\frac{1}{3}$. It is the purpose of this article to generalize the art gallery problem to galleries whose walls are continuous curves. In Section 2 we indicate how the concept of guard-to-side ratio may be extended to a certain class of art galleries whose walls are limits of polygons. In Section 3 the Koch fractal art gallery is defined as a sequence of approximant art galleries and auxiliary notations are introduced. In Section 4 a system of recursion relations is obtained for the minimum number of watchmen needed to guard the $(n+1)^{st}$ approximant gallery. In Section 5 we calculate the guard-to-side ratio for shallow and deep art galleries in which one of whose walls is the Koch fractal.

2. Generalized Art Galleries

Consider a sequence $\Pi_1,\Pi_2,\Pi_3,\cdots$, of polygons such that the vertex set of each polygon is strictly contained in the vertex set of the next. Assume also that all the polygons are contained in a compact region of the plane. If the limit of the boundaries of the polygons is a Jordan curve with a connected interior, then we denote the union of the Jordan curve and its interior as Π_∞ and write $\Pi_\infty=\lim_{n\to\infty}\Pi_n$. We call Π_∞ a generalized art gallery. We also define the guards-to-side ratio, or the gsr, of Π_∞ by $gsr(\Pi_\infty)=\lim_{n\to\infty}gsr(\Pi_n)$, provided that this limit exists.

As an example suppose that Π_1 is a regular hexagon inscribed in a circle of radius r and that Π_n is the standard sequence of Archimedean polygons generated by Π_1. As is well known Π_∞ is the given circle. Since each Π_n is star-convex, $G(\Pi_n)=1$. Furthermore each Π_n has $6\times2^{n-1}$ sides, and therefore $gsr(\Pi_\infty)=0$. Intuitively for any generalized art gallery the deviation of its gsr from zero is a measure of the irregularity of the gallery's boundary.

3. A Koch Fractal Art Gallery

We consider a rectangle $R = ABCD$ with the vertices labelled consecutively in a clockwise direction. For ease in visualization suppose that edge AB is a horizontal line segment and is the upper horizontal side of R. We perform a "basic process" on edge AB (See **Figure 1**): Let points E and F on AB divide AB into thirds. Construct an equilateral triangle EGF with base EF on AB and such that G lies outside of R. Set K_1 to be the polygon with boundary $AEGFBCDA$.

For ease in refering to parts of the boundary of a polygon, we will adopt the following notation. When V_1 and V_2 are two vertices of a polygon Π, by $\text{arc}(V_1, V_2)$ we will mean the polygonal path along $\partial\Pi$ from V_1 to V_2 going in a clockwise direction. For example $arc(A, B)$ in K_1 is the generator of the Koch curve [3].

We now proceed to define a sequence K_n of approximant art galleries inductively. For each $n \geq 1$ $K_n = AV_i^n \cdots V_f^n BCD$. The $(n+1)^{st}$ Koch approximant art gallery K_{n+1} is constructed by performing the basic process on each edge of $arc(A, B)$, always choosing new vertices to lie outside of K_n. Since the vertex set of each Koch approximant art gallery is strictly contained in the vertex set of the next one in the sequence, it makes sense to say for example that vertices A and G belong to each K_n for $n \geq 1$. In K_n we call $arc(A, B)$ the front edge of the art gallery and will denote it by L_n. Because the vertices of each K_n are fixed in all subsequent approximant art galleries, the limit of the sequence is well-defined. We label this limit K_∞ and call it a Koch fractal art gallery.

To facilitate the derivation of a system of difference equations for the minimum number of watchmen needed to guard the approximant art galleries, we introduce notations for some of the geometrical features of the approximant art galleries. These notations are illustrated in **Figure 2**. The front edge may be decomposed into the union of three separate arcs: $M_n = arc(A, E)$, $P_n = \text{arc}(E, F)$, and $Q_n = arc(F, B)$. Both M_n and Q_n are geometrically similar to L_{n-1}. Drop a perpendicular from E to DC and label the foot E^*. Similarly let F^* be the foot of the perpendicular from F to DC. Using these two points we may define wing galleries: The "left-hand" gallery is that polygon W_n^L with boundary $M_n \bigcup EE^*DA$, while the "right-hand" gallery W_n^R has boundary $Q_n \bigcup BCF^*F$. We will call the remaining portion of K_n its "central" gallery and denote it by V_n. Thus V_n has boundary $P_n \bigcup FF^*E^*E$.

The "main hall" of K_n is that part of K_n lying within the original rectangle R. By P_j we will denote a side gallery in K_n that is similar to P_j in K_j for some $j < n$ and that also opens onto the main hall. The line λ_n drawn through G in K_n and perpendicular to side CD is the central line of symmetry of K_n. If the side BC in K_n is sufficiently shorter than side AB, no guard placed on λ_n will be able to see completely any P_1 in either of the wing galleries. We'll call such a K_n "shallow" and will denote it by K_n^S. If the side BC in K_n is sufficiently long in comparison to side AB, then there is a point on λ_n from which a guard is able to see all of the P_1 side galleries off of the main hall. We'll call such a K_n "deep" and will denote it by K_n^D.

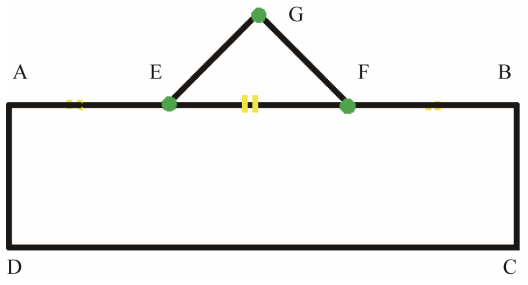

Figure 1. The basic process in forming the first Koch fractal approximant K_1.

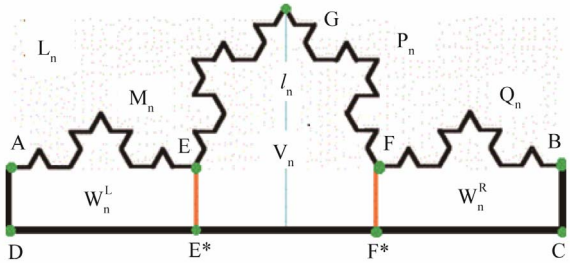

Figure 2. Structure of the Koch approximant gallery K_n. L_n is the front edge of the gallery and is the union of arcs M_n, P_n, and Q_n.

4. Recursion Relations for the Koch Approximant Art Galleries

Let k_n, k_n^L, v_n, and k_n^R denote the smallest number of watchmen sufficient to guard K_n, W_n^L, V_n, and W_n^R respectively. In the following discussion, a symbol with a superscript S will indicate that the quantity being denoted by the symbol is being considered in the case of a shallow art gallery, while a superscript D will indicate that the associated quantity is being considered in the case of a deep art gallery. Our goal is now to obtain recursion relations for k_n^S and k_n^D.

Since W_n^L and W_n^R are similar to K_{n-1}, in a shallow art gallery both k_n^L and k_n^R are equal to k_{n-1}^S. Thus $k_n^S = 2k_{n-1}^S + v_n$. It is straightforward to obtain empirically the values of k_n^S and v_n for small n. For example $k_1^S = 1$ and $v_1 = 1$, $k_2^S = 3$ and $v_2 = 1$, $k_3^S = 8$ and $v_3 = 2$, and finally $k_4^S = 23$ and $v_4 = 7$.

While the method of best placement of guards in K_n^S may be determined empirically for small values of n

without too much difficulty, for larger n the optimal arrangement of guards becomes less clear. Our strategy will be to position those guards with the largest fields of vision first. Such guards will certainly lie on λ_n and we will refer to them as "central guards". Denote the number of central guards in K_n by c_n. Empirically $c_2 = 1$, $c_3 = 2$, and $c_4 = 3$. We obtain a formula for c_n by first assuming that the central guards have been positioned optimally in K_n. We pass from K_n to K_{n+1} by replacing the front edge L_n by L_{n+1}. The guards, who are now on λ_{n+1}, are no longer optimally placed, but small adjustments will restore optimal placements with one exception. Any movement of the central guards so that the guard closest to the front edge is able to view completely into the side gallery centered on G results in too great a loss of visibility within P_{n+1}. Thus to cover P_{n+1} one additional central guard is required. Hence $c_{n+1} = c_n + 1$. Using the empirically determined value of c_2 as an initial condition gives that $c_n = n - 1$.

It is possible to express k_n^D in terms of k_n^S. In each k_n^D, for $n \geq 2$, one central guard placed sufficiently close to side CD will be able to see all the P_1 side galleries in W_n^L and W_n^R. The number of such side galleries is 2^{n-1}. Once the P_1 side galleries in the wings have been covered, the remaining guards can be placed as in k_n^S. Since $k_1 = 1$ for any P_1, we have for $n \geq 1$ that $k_n^D = k_n^S - 2^{n-1} + 1$.

To obtain a formula for v_n we first note that in each K_n^S the central arc in the front edge is the union of two congruent arcs: $P_n = arc(E,G) \cup arc(G,F)$. Since each of $arc(E,G)$ and $arc(G,F)$ is similar to L_{n-1} and since these side galleries open onto the wide central region bounded by P_n, we expect that the number of guards just sufficient to guard each arc should be close to k_{n-1}^D. Certainly the role of the guard closest to side CD in k_{n-1}^D is covered by the central guards on λ_n. So we expect that the count k_{n-1}^D should be adjusted to $k_{n-1}^D - 1$. In each of $arc(E,G)$ and $arc(G,F)$ the frontmost $n-2$ side galleries lie near enough to λ_n that the role of each side gallery's innermost watchman is covered by the central guards. Hence the count $k_{n-1}^D - 1$ should be adjusted to $k_{n-1}^D - 1 - (n-2)$. This gives finally that $v_n = c_n + 2\left(k_{n-1}^D - n + 1\right)$.

This latter recursion relation, when the expression for k_n^D is substituted, becomes $v_n = 2k_{n-1}^S - 2^{n-1} - n + 3$. We now have a coupled system of difference equations for k_{n+1}^S and v_{n+1}, namely, for $n \geq 0$

$$k_{n+1}^S = 2k_n^S + v_{n+1} \tag{1}$$

$$v_{n+1} = 2k_n^S - 2^n - n + 2 \tag{2}$$

In addition, in view of the empirical data, we may take as an initial condition $k_0^S = 0$.

5. Calculation of $gsr\left(K_\infty^S\right)$ and $gsr\left(K_\infty^D\right)$

The system of difference equations we have obtained is amenable to standard techniques. Substitution of the expression for v_{n+1} into Equation (1) yields

$$k_{n+1}^S = 4k_n^S + 2 - 2^n - n. \tag{3}$$

This is a first order equation of the form $y_{n+1} = p_n y_n + q_n$. The general solution is given by Mickens [4] as

$$y_n = \sum_{j=0}^{n-1}\left(q_j \prod_{k=j+1}^{n-1} p_k\right). \tag{4}$$

In the case at hand for $n = 0, 1, \cdots$, $p_n = 4$ and $q_n = 2 - 2^n - n$. Substitution of these values into Equation (4) produces the solution

$$k_n^S = 2^{n-1} + \left(\frac{2 \times 4^{n-1} + 3n - 5}{9}\right). \tag{5}$$

Thus the sequence of numbers of watchmen needed to guard successive shallow Koch approximant art galleries is 0, 1, 2, 3, 8, 23, 74, $\cdots$ Since each K_n has $4^n + 3$ sides, it follows that for a shallow Koch fractal art gallery the guards-to-side ratio is given by $gsr\left(K_\infty^S\right) = \frac{1}{18}$. By utilizing the relationship between the minimum number of guards for shallow and deep galleries, we obtain, for $n \geq 1$, the general formula for k_n^D is

$$k_n^D = \left(\frac{2 \times 4^{n-1} + 3n + 4}{9}\right), \tag{6}$$

and the sequence of numbers of watchmen needed to guard successive deep Koch approximant art galleries is 1, 2, 5, 16, 59, $\cdots$ The guards-to-side ratio for a deep Koch fractal art gallery follows immediately:

$gsr\left(K_\infty^D\right) = \frac{1}{18}$.

6. Conclusion

In this article we have presented a generalization of the standard art gallery problem to the case where the sides of the gallery are continuous curves which are limits of polygonal arcs. In such cases the appropriate measure of coverage by guards is not the total number of guards but, rather, the guards-to-side ratio. This ratio has been computed for both shallow and deep versions of a Koch fractal art gallery and has been found to be $\frac{1}{18}$. Obtaining a formula for the number of watchmen needed to guard approximant art galleries in the cases intermediate between the shallow and the deep limits is an open question. However, since the *gsr*'s of the Koch fractal art

gallery in the extreme cases are equal, it is reasonable that the common value will also be the gsr for the intermediate cases.

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