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A Note on Weakly-*α*-*I*-Functions and Weakly-*α*-*I*-Paracompact Spaces*

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ABSTRACT

This paper introduces the new notion of weakly-*α*-*I*-functions and weakly-*α*-*I*-paracompact spaces in the ideal topological space. And it obtains that some properties of them.

Keywords: Weakly-*α*-*I*-Open Set; Weakly-*α*-*I*-Functions; Weakly-*α*-*I*-Paracompact Spaces

1. Introduction

Throughout this paper, $Cl(A)$ and $Int(A)$ denote the closure and interior of A, respectively. Let (X,τ) be a topological space and let I be an ideal of subsets of X. An ideal topological space is a topological space (X,τ) with an ideal I on X, and is denoted by (X,τ,I). For a subset $A\subset X$,

$$A^*(I)=\{x\in X|U\cap A\notin I \text{ for each neighborhood of } x\}$$

is called the local function of A with respect to I and τ [1]. It is well known that $Cl^*(A)=A\cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

Let S be a subset of a topological space (X,τ). The complement of a semi-open set is said to be semi-closed [2]. The intersection of all semi-closed sets containing S, denoted by $sCl(S)$ is called the semi-closure [3] of S. The semi-interior of S, denoted by $sInt(S)$, is defined by the union of all semi-open sets contained in S.

In recent years, E. Hatir and T. Noiri have extended the study to *α*-*I*-open and semi-*I*-open sets. In this paper, we introduce the new sets which are called weakly-*α*-*I*-open and weakly-*α*-*I*-functions, then obtain some properties of them.

First we recall some definitions used in the sequel.

Definition 1.1. [4] A subset A of an ideal topological space (X,τ,I) is said to be weakly-*α*-*I*-open, if

$$A\subset sCl\left(Int\left(Cl^*\left(Int(A)\right)\right)\right).$$

Definition 1.2. [4] A subset S of an ideal topological space (X,τ,I) is said to be a weakly-*α*-*I*-closed set if its complement is a weakly-*α*-*I*-open set.

Theorem 1.1. [4] Let (X,τ,I) be an ideal topological space. Then all weakly-*α*-*I*-open sets constitute a topology of X. Then

(1) $\varnothing$ and X are weakly-*α*-*I*-open sets.

(2) The finite intersection of weakly-*α*-*I*-open sets are weakly-*α*-*I*-open sets.

(3) If A_α is weakly-*α*-*I*-open for each $\alpha\in\Delta$, then $\cup A_\alpha$ is weakly-*α*-*I*-open.

Theorem 1.2. [4] Let (X,τ,I) be an ideal topological space and $A\subset U\in\tau$. Then A is weakly-*α*-*I*-open if and only if A is weakly-*α*-*I*-open in $\left(U,\tau|_U,I|_U\right)$.

2. Weakly-*α*-*I*-Functions

Definition 2.1. A function $f:(X,\tau,I)\to(Y,\sigma)$ is said to be weakly-*α*-*I*-continuous , if $f^{-1}(V)$ is weakly-*α*-*I*-open in (X,τ,I) , for any $V\in(Y,\sigma)$.

Definition 2.2. A mapping $f:(X,\tau,I)\to(Y,\sigma,J)$ is said weakly-*α*-*I*-open (resp weakly-*α*-*I*-closed) if for any U is weakly-*α*-*I*-open (resp. U is weakly-*α*-*I*-closed) $f(U)$ is weakly-*α*-*I*-open (resp. $f(U)$ is weakly-*α*-*I*-closed).

Theorem 2.1. For a function $f:(X,\tau,I)\to(Y,\sigma)$, the followings are equivalent:

(1) f is weakly-*α*-*I*-continuous.

(2) For any $x\in X$ and each $V\in\sigma$ containing $f(x)$, there exists weakly-*α*-*I*-open U containing x such that $f(U)\subset V$.

(3) The inverse image of each closed set in is weakly-*α*-*I*-closed.

Proof: $(1)\Leftrightarrow(3)$ it is obviously.

$(1)\Rightarrow(2)$ For each $x\in X$ and each weakly-*α*-*I*--open $V\in(Y,\sigma)$ containing $f(x)$. Since f is weakly-*α*-*I*-continuous, then $f^{-1}(V)$ is a weakly-*α*-*I*-open set containing x in X. Let $U=f^{-1}(V)$, then $f(U)\subset V$.

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$(2)\Rightarrow(1)$ For any $x\in X$ and each $V\in\sigma$ containing $f(x)$, there exists weakly-*α*-*I*-open U_x containing x such that $f(U_x)\subset V$ and $U_x\subset f^{-1}(V)$.
We have $f^{-1}(V)=\bigcup_{x\in f^{-1}(V)} U_x$, where U_x is weakly-*α*-*I*-open for any $x\in f^{-1}(G)$. Thus $f^{-1}(V)$ is weakly-*α*-*I*-open by Theorem 1.1. And conclude that f is weakly-*α*-*I*-continuous.

Theorem 2.2. Let $f:(X,\tau,I)\to(Y,\sigma)$ be a function and $\{U_\alpha \mid \alpha\in\Delta\}$ an open cover of X. Then f is weakly-*α*-*I*-continuous if and only if the restriction $f_{|U_\alpha}:\left(U_\alpha,\tau_{|U_\alpha},I_{|U_\alpha}\right)\to(Y,\sigma)$ is weakly-*α*-*I*-continuous for each $\alpha\in\Delta$.

Proof: Necessity. Let V be any open set of (Y,σ). Since f is weakly-*α*-*I*-continuous, $f^{-1}(V)$ is a weakly-*α*-*I*-open set of (X,τ,I). Since $U_\alpha\in\tau$, from Theorem 1.2 $U_\alpha\cap f^{-1}(V)$ is weakly-*α*-*I*-open in $\left(U_\alpha,\tau_{|U_\alpha},I_{|U_\alpha}\right)$.
On the other hand,
$\left(f_{|U_\alpha}\right)^{-1}(V)=U_\alpha\cap f^{-1}(V)$ and $\left(f_{|U_\alpha}\right)^{-1}(V)$ is weakly-*α*-*I*-open in $\left(U_\alpha,\tau_{|U_\alpha},I_{|U_\alpha}\right)$. This shows that $\left(f_{|U_\alpha}\right)$ is weakly-*α*-*I*-continuous for each $\alpha\in\Delta$.

Sufficiency. Let V be any open set of (Y,σ). Since $f_{|U_\alpha}$ is weakly-*α*-*I*-continuous for each $\alpha\in\Delta$.
$\left(f_{|U_\alpha}\right)^{-1}(V)$ is weakly-*α*-*I*-open of $\left(U_\alpha,\tau_{|U_\alpha},I_{|U_\alpha}\right)$ and hence by Theorem 1.2. $\left(f_{|U_\alpha}\right)^{-1}(V)$ is weakly open in (X,τ,I) for each $\alpha\in\Delta$. Moreover, we have

$$f^{-1}(V)=\left(\bigcup_{\alpha\in\Delta}U_\alpha\right)\cap f^{-1}(V)=\bigcup_{\alpha\in\Delta}\left(U_\alpha\cap f^{-1}(V)\right)=\bigcup_{\alpha\in\Delta}\left(f_{|U_\alpha}\right)^{-1}(V).$$

Therefore $f^{-1}(V)$ is a weakly open in (X,τ,I) by Theorem 1.1. This shows that f is weakly-*α*-*I*-continuous.

Theorem 2.3. A function f is weakly-*α*-*I*-continuous if and only if the graph function $g:X\to X\times Y$ defined by $g(x)=(x,f(x))$ for each $x\in X$, is weakly-*α*-*I*-continuous.

Proof: Necessity. Suppose that f is weakly-*α*-*I*-continuous. Let $x\in X$ and W be any open set of $X\times Y$ containing $g(x)$. Then there exists a basic open set $U\times V$ such that $g(x)=(x,f(x))\in U\times V\subset W$. Since f is weakly-*α*-*I*-continuous, there exists a weakly-*α*-*I*-open set U_0 of X containing x such that $f(U_0)\subset V$. By Theorem 1.1. $U_0\cap U$ is weakly-*α*-*I*-open and $g(U_0\cap U)\subset U\times V\subset W$. This show that g is weakly-*α*-*I*-continuous.

Sufficiency. Suppose that g is weakly-*α*-*I*-continuous. Let $x\in X$ and V be any open set of Y containing $f(x)$. Then $X\times V$ is open in $X\times Y$ and by the weakly*α*-*I*-continuous of g ,there exists a weakly-*α*-*I*-open set U containing x such that $g(U)\subset X\times V$. Therefore we obtain $f(U)\subset V$. This shows that f is weakly-*α*-*I*-continuous

Theorem 2.4. Let $f:(X,\tau,I)\to(Y,\sigma,J)$ is weakly-*α*-*I*-open (resp weakly-*α*-*I*-closed) mapping. If $y\in Y$ and U is a weakly-*α*-*I*-closed (resp weakly-*α*-*I*-open) set of X containing $f^{-1}(y)$, then there exists a weakly-*α*-*I*-closed (resp weakly-*α*-*I*-open) subset V of Y containing y such that $f^{-1}(V)\subset U$.

Proof: Suppose that f is weakly-*α*-*I*-closed mapping. Given $y\in Y$ and U is a weakly-*α*-*I*-open subset of X containing $f^{-1}(y)$, then $X-U$ is a weakly-*α*-*I*-closed set. Since f is weakly-*α*-*I*-closed, $f(X-U)$ is weakly-*α*-*I*-closed. Hence $V=Y-f(X-U)$ is weakly-*α*-*I*-open. It follows from $f^{-1}(y)\subset U$ that
$\{y\}\cap f(X-U)=\varnothing$.
Therefore $y\in Y-f(X-U)=V$ and

$$f^{-1}(V)=f^{-1}\left(Y-f(X-U)\right)=X-f^{-1}f(X-U)\subset U.$$

Similar argument holds for a weakly-*α*-*I*-open mapping.

3. Weakly-*α*-*I*-Paracompact Spaces

Definition 3.1. A space X is said to be weakly-*α*-*I*-Hausdorff, if for each pair of distinct points x and y in X, there exist disjoint weakly-*α*-*I*-open sets U and V in X such that $x\in U$ and $y\in V$, and $U\cap V=\varnothing$.

Definition 3.2. A space X is said to be weakly-*α*-*I*-regular space, if for every $x\in X$ and every weakly-*α*-*I*-closed set $F\subset X$ such that $x\notin F$, there exist weakly-*α*-*I*-open sets U_1, U_2 such that $x\in U_1$, $F\subset U_2$ and $U_1\cap U_2=\varnothing$.

Definition 3.3. A space X is said to be weakly-*α*-*I*-normal space, if for every pair of disjoint weakly-*α*-*I*-closed sets A, $B\subset X$, there exist weakly-*α*-*I*-open sets U, V such that $A\subset U$, $B\subset V$ and $U\cap V=\varnothing$.

Definition 3.4. An ideal topological space (X,τ,I) is said to be a weakly-*α*-*I*-compact space if every weakly-*α*-*I*-open cover of X has a finite subcover.

Definition 3.5. An ideal topological space (X,τ,I) is said to be weakly-*α*-*I*-paracompact space, if every weakly-*α*-*I*-open cover of X has a locally finite weakly-*α*-*I*-open refinement.

Definition 3.6. Mapping $f:(X,\tau,I)\to(Y,\sigma,J)$ is said to be weakly-*α*-*I*-perfect, if f is weakly-*α*-*I*-closed and for any $y\in Y$, $f^{-1}(y)$ is a weakly-*α*-*I*-compact subset of X.

Theorem 3.1. An ideal topological space (X,τ,I) is a weakly-α-I-compact space if and only if every family of weakly-α-I-closed sets of X satisfying the finite intersection property has nonempty intersection.

Proof: Necessity. If $\mathcal{F}$ is any family of weakly-α-I-closed sets which has finite intersection property, and $\bigcap_{F\in\mathcal{F}} F=\varnothing$, then $X-\bigcap_{F\in\mathcal{F}} F=\bigcup_{F\in\mathcal{F}}(X-F)=X$.

Thus $X-F$ is a weakly-α-I-open set. Since X is a weakly-α-I-compact space, hence there exist finite sets $F_1,F_2,\cdots,F_n$, such that $\bigcup_{i=1}^{n}(X-F_i)=X$. So $X-\bigcap_{i=1}^{n}F_i=X$, and $\bigcap_{i=1}^{n}F_i=\varnothing$, a contradiction.

Sufficiency. If $\mathcal{A}$ is any weakly-α-I-open cover for X, then $\mathcal{F}=\{X-A:A\in\mathcal{A}\}$ is a weakly-α-I-closed family that satisfies

$\bigcap_{F\in\mathcal{F}} F=\bigcap_{A\in\mathcal{A}}(X-A)=X-\bigcup_{A\in\mathcal{A}}A=\varnothing$. So $\mathcal{F}$ dose not satisfy finite intersection property, which means $\mathcal{F}$ has a finite subfamily $\{F_1,F_2,\cdots,F_n\}$ which has intersection empty. Suppose $F_i=X-A_i, i=1,2,\cdots,n$, where $A_i\in\mathcal{A}$. We have $\bigcap_{i=1}^{n}F_i=\bigcap_{i=1}^{n}(X-A_i)=X-\bigcup_{i=1}^{n}A_i=\varnothing$. So $\{A_1,A_2,\cdots,A_n\}$ is a finite cover of $\mathcal{A}$.

Theorem 3.2. Weakly-α-I-compactness is an inverse invariant of weakly-α-I-perfect mapping.

Proof: Let $f:X\to Y$ be a weakly-α-I-perfect mapping onto a weakly-α-I-compact space Y. Given a weakly-α-I-open cover $\{U_s\}_{s\in S}$ of the space X. Since f is a weakly-α-I-perfect mapping and for every $y\in Y$ choose a finite set $S_{(y)}\subset S$, such that $f^{-1}(y)\subset\bigcup_{s\in S_{(y)}}U_s=U_y$. U_y is a weakly-α-I-open from Theorem 1.1. And from Theorem 2.4 there exists a weakly-α-I-open set V_y containing y such that $f^{-1}(y)\subset f^{-1}(V_y)\subset U_y$. Since Y is a weakly-α-I-compact space, the weakly-$\alpha-I-$open cover $\{V_y\}_{y\in Y}$ of Y has a finite subcover $\{V_{y_i}:i=1,\cdots,k\}$ such that $Y\subset\bigcup_{i=1}^{k}V_{y_i}$. Therefore

$X\subset f^{-1}\left(\bigcup_{i=1}^{k}V_{y_i}\right)\subset\bigcup_{i=1}^{k}U_{y_i}=\bigcup_{i=1}^{k}\left(\bigcup_{s\in S_{(y_i)}}U_s\right)$, which means $X\subset\bigcup_{i=1}^{k}\left(\bigcup_{s\in S_{(y_i)}}U_s\right)$. Because the family of these $\{U_s:s\in S_{(y_i)},i=i,\cdots,k\}$ is a finite subfamily of $\{U_s\}_{s\in S}$, and we conclude that X is a weakly-α-I-compact space.

Lemma 3.1. Let X be a weakly-α-I-paracompact space and A, B a pair of weakly-α-I-closed sets of X. If for every $x\in B$ there exist weakly-α-I-open sets U_x, V_x such that $A\subset U_x$, $x\in V_x$ and $U_x\bigcap V_x=\varnothing$. Then there also exist weakly-α-I-open sets U, V such that $A\subset U$, $B\subset V$ and $U\bigcap V=\varnothing$.

Proof: $\{X-B\}\bigcup\{V_x\}_{x\in B}$ is a weakly-α-I-open cover of the weakly-α-I-paracompact space X, so that it has a locally finite weakly-α-I-open refinement $\{W_s\}_{s\in S}$. Let $S_1=\{s\in S:\exists x\in B, W_s\subset V_x,\}$, then for any $s\in S_1$, $A\bigcap\overline{W_s}=\varnothing$ and $B\subset\bigcup_{s\in S_1}W_s$. Since $\{W_s\}_{s\in S}$ is a locally finite family, it is a closure preserving family. So $\bigcup_{s\in S_1}\overline{W_s}=\overline{\bigcup_{s\in S_1}W_s}$, then the set $U=X-\overline{\bigcup_{s\in S_1}W_s}$ is open. Therefore U is a weakly-α-I-open set. $V=\bigcup_{s\in S_1}W_s$ is also a weakly-α-I-open set from theorem1.1. and $U\bigcap V=\varnothing$.

Theorem 3.3. Assuming X is a weakly-α-I-paracompact space, if one-point sets of X are weakly-α-I-closed sets, then X is weakly-α-I-normal space.

Proof: Substituting one-point sets for A in Lemma 4.1., and if one-point sets are weakly-α-I-closed sets, we see that every weakly-α-I-paracompact space is weakly-α-I-regular. Applying Lemma 3.1. again we have the conclusion.

4. Summary

Combining the topological structure with other mathematical features has provided many interesting topics in the development of general topology. And this paper has done much work on the ideal topological space. On certain extent, it promotes the development of topology.

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