

4-Cycle Decompositions of Graphs

Teresa Sousa

Departamento de Matemática and Centro de Matemática e Aplicações,
Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Lisbon, Portugal
Email: tmjs@fct.unl.pt

Received May 10, 2012; revised June 3, 2012; accepted August 6, 2012

ABSTRACT

In this paper we consider the problem of finding the smallest number ϕ such that any graph G of order n admits a decomposition into edge disjoint copies of C_4 and single edges with at most ϕ elements. We solve this problem for n sufficiently large.

Keywords: Graph Decomposition; 4-Cycle Packing; Graph Packing

1. Introduction

All graphs in this paper are finite, undirected and simple. For notation and terminology not discussed here the reader is referred to [1].

Given two graphs G and H , an H -decomposition of G is a partition of the edge set of G such that each part is either a single edge or forms an H -subgraph, *i.e.*, a graph isomorphic to H . We allow partitions only, that is, every edge of G appears in precisely one part. Let $\phi_H(G)$ be the smallest possible number of parts in an H -decomposition of G . For non-empty H , let $p_H(G)$ be the maximum number of pairwise edge-disjoint H -subgraph that can be packed into G and $e(G)$ the number of edges in G . It is easy to see that

$$\phi_H(G) = e(G) - p_H(G)(e(H) - 1). \quad (1.1)$$

Here we study the function

$$\phi_H(n) = \max \{ \phi_H(G) \mid v(G) = n \},$$

which is the smallest number, such that, any graph G of order n admits an H -decomposition with at most $\phi_H(n)$ elements.

The function $\phi_H(n)$ was first studied by Erdős, Goodman and Pósa [2], who proved that $\phi_{K_3}(n) = t_2(n)$, where K_r denotes the complete graph (clique) of order r and $t_r(n)$ is the maximum size of an r -partite graph on n vertices. A decade later, this result was extended by Bollobás [3], who proved that

$$\phi_{K_r}(n) = t_{r-1}(n), \quad \text{for all } n \geq r \geq 3.$$

Recently, Pikhurko and Sousa [4] studied $\phi_H(n)$ for arbitrary graphs H .

Theorem 1.1. (See Theorem 1.1 from [4]) *Let H be*

any fixed graph of chromatic number $r \geq 3$. Then,

$$\phi_H(n) = t_{r-1}(n) + o(n^2).$$

Let $\text{ex}(n, H)$ denote the maximum number of edges in a graph of order n , that does not contain H as a subgraph. Recall that $\text{ex}(n, K_r) = t_{r-1}(n)$. Pikhurko and Sousa [4] also made the following conjecture.

Conjecture 1. *For any graph H with chromatic number at least 3, there is $n_0 = n_0(H)$ such that $\phi_H(n) = \text{ex}(n, H)$, for all $n \geq n_0$.*

The exact value of the function $\phi_H(n)$ is far from being known. Sousa determined it for a few special edge-critical graphs, namely for clique-extensions of order $r \geq 4$ ($n \geq r$) [5] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [6,7]. Later, Özkahya and Person [8] determined it for all edge-critical graphs with chromatic number $r \geq 3$ and n sufficiently large. They proved the following result.

Theorem 1.2. ([8]) *Let H be any edge-critical graph with chromatic number $r \geq 3$. Then, there exists n_0 such that $\phi_H(n) = \text{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\phi_H(n)$ is the Turán graph $T_{r-1}(n)$.*

Recently, Allen, Böttcher and Person [9] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1.

The case when H is a bipartite graph has been less studied. Pikhurko and Sousa [4] determined $\phi_H(n)$ for any fixed bipartite graph with an $O(1)$ additive error. For a non-empty graph H , let $\text{gcd}(H)$ denote the greatest common divisor of the degrees of H . For example, $\text{gcd}(K_{6,4}) = 2$ while for any tree T with at least 2 vertices we have $\text{gcd}(T) = 1$. They proved the following result.

Theorem 1.3. (See Theorem 1.3 from [4]) *Let H be a bipartite graph with m edges and let $d = \gcd(H)$. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold.*

(1) If $d = 1$, then $\phi_H(n) = \left\lfloor \frac{n(n-1)}{2m} \right\rfloor + C$, where

$C = m - 1$ or $C = m - 2$.

(2) If $d \geq 2$, then

$$\phi_H(n) = \frac{nd}{2m} \left(\left\lfloor \frac{n}{d} \right\rfloor - 1 \right) + \frac{1}{2}n(d-1) + O(1).$$

Moreover, there is a procedure running in polynomial in $\log n$ time which determines $\phi_H(n)$ and describes a family \mathcal{D} of n -sequences such that a graph G of order n satisfies $\phi_H(G) = \phi_H(n)$ if and only if the degree sequence of G belongs to \mathcal{D} . (It will be the case that $|\mathcal{D}| = O(1)$ and each sequence in \mathcal{D} has $n - O(1)$ equal entries, so \mathcal{D} can be described using $O(\log n)$ bits.)

Here we will determine the exact value of $\phi_{C_4}(n)$ for n sufficiently large.

Theorem 1.4. *There is $n_0 = n_0(C_4)$ such that for all $n \geq n_0$ the following statements hold.*

(1) If n is even then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{4} + 1$.

(2) If $n \equiv 1 \pmod{8}$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{14}{8}$.

(3) If $n \equiv 3 \pmod{8}$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}$.

(4) If $n \equiv 5 \pmod{8}$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}$.

(5) If $n \equiv 7 \pmod{8}$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + 2$.

2. Proof of Theorem 1.4

In this section we will prove Theorem 1.4, but first we need to introduce the tools. We start with the following easy result about H -decompositions.

Lemma 5. (Lemma 1.3) *For any non-empty graph H with m edges and any integer n , we have*

$$\phi_H(n) \leq \frac{1}{m} \binom{n}{2} + \frac{m-1}{m} \text{ex}(n, H). \tag{2.1}$$

In particular, if H is a fixed bipartite graph with m edges and $n \rightarrow \infty$, then

$$\phi_H(n) = \left(\frac{1}{m} + O(1) \right) \binom{n}{2}. \tag{2.2}$$

The following result is the well known Erdős-Gallai theorem that gives a necessary and sufficient condition for a finite sequence to be the degree sequence of a

simple graph.

Theorem 2.6. (Erdős-Gallai Theorem [10]) *Let $0 \leq d_1 \leq \dots \leq d_n$ be a sequence of integers. There is a graph with degree sequence d_1, \dots, d_n if and only if*

- (1) $d_1 + \dots + d_n$ is even;
- (2) for each $1 \leq k \leq n$

$$\sum_{i=n-k+1}^n d_i \leq k(k-1) + \sum_{i=1}^{n-k} \min\{d_i, k\}. \tag{2.3}$$

The following results appearing in Alon, Caro and Yuster [11, Theorem 1.1, Corollary 3.4, Lemma 3.5] which follow with some extra work from the powerful decomposition theorem of Gustavsson [12] are essential to the proof of Theorem 1.4.

Lemma 2.7. *For any non-empty graph H with m edges, there are $\gamma > 0$ and N_0 such that the following holds. Let $d = \gcd(H)$. Let G be a graph of order $n \geq N_0$ and of minimum degree $\delta(G) \geq (1-\gamma)n$.*

If $d = 1$, then

$$p_H(G) = \left\lfloor \frac{e(G)}{m} \right\rfloor. \tag{2.4}$$

If $d \geq 2$, let $\alpha_u = d \left\lfloor \frac{\deg(u)}{d} \right\rfloor$ for $u \in V(G)$ and

let X consist of all vertices whose degree is not divisible by d . If $|X| \geq \frac{n}{10d^3}$, then

$$p_H(G) = \left\lfloor \frac{1}{2m} \sum_{u \in V(G)} \alpha_u \right\rfloor. \tag{2.5}$$

If $|X| < \frac{n}{10d^3}$, then

$$p_H(G) \geq \frac{1}{m} \left(e(G) - \frac{n}{5d^2} \right). \tag{2.6}$$

□

One can extract the following result from the proof of Theorem 1.2 from [4].

Lemma 2.8. *Let H be a bipartite graph with m edges and let $\gcd(H) = d \geq 2$. Then, there is $n_0 = n_0(H)$ such that if G is a graph of order $n \geq n_0$ with $\phi_H(G) = \phi_H(n)$ then the following holds:*

(1) Let d_1, \dots, d_n be the degree sequence of G , then

$$\phi_H(G) = \frac{1}{2} \sum_{i=1}^n d_i - (m-1) \left\lfloor \frac{1}{2m} \sum_{i=1}^n \left\lfloor \frac{d_i}{d} \right\rfloor d \right\rfloor. \tag{2.7}$$

(2) Let $n = qd + r$ with $0 \leq r \leq d-1$ and $d_i = q_i d + r_i$ with $0 \leq r_i \leq d-1$. Then, for $1 \leq i \leq n$ exactly one of the following holds:

- (a) $d_i = qd - 1$;
- (b) $i \in C_1 = \{i \in [n] \mid r_i = d-1 \text{ and } d_i < qd-1\}$;
- (c) $i \in C_2 = \{i \in [n] \mid d_i = n-1\}$ if $n-1 \neq R$ and

$C_2 = \emptyset$ otherwise.

Furthermore, $|C_1| \leq \frac{2m}{d} - 1$ and $|C_2| \leq 2m - 1$.

In the following we briefly sketch the proof of Lemma 2.8 by giving the argument form [4]. We refrain from doing all the calculations.

Sketch of the proof of Lemma 2.8. Let $\gamma(C_4)$ and N_0 be given by Lemma 2.7. Assume that γ is sufficiently small and that $n_0 \geq N_0$ is sufficiently large to satisfy all the inequalities we will encounter. Let $n \geq n_0$ and let G be any graph of order n with $\phi_{C_4}(G) = \phi_{C_4}(n)$.

Let $G_n = G$. Repeat the following at most $\lfloor n/\log n \rfloor$ times: If the current graph G_i has a vertex x_i of degree at most $(1-\gamma/2)i$, let $G_{i-1} = G_i - x_i$ and decrease i by 1.

Suppose we stopped after s repetitions. Then, either $\delta(G_{n-s}) \geq (1-\gamma/2)(n-s)$ or $s = \lfloor n/\log n \rfloor$. Let us show that the later cannot happen. Otherwise, we have

$$e(G) \leq \binom{n-s}{2} + \left(1 - \frac{\gamma}{2}\right) \sum_{i=n-s+1}^n i < \binom{n}{2} - \frac{\gamma n^2}{4 \log n}. \quad (2.8)$$

Let t satisfy $K_{t,t} \supset H$. Using the fact that $\text{ex}(n, K_{t,t}) = O(n^{2-1/t})$, (2.1) and (2.8) we obtain

$$\begin{aligned} \phi_H(G) &\leq \frac{1}{m} \left(\binom{n}{2} - \frac{\gamma n^2}{4 \log n} \right) + \frac{m-1}{m} cn^{2-1/t} \\ &< \frac{1}{m} \binom{n}{2} \leq \phi_H(K_n), \end{aligned}$$

which contradicts our assumption on G . Therefore, $s < \lfloor n/\log n \rfloor$ and we have $\delta(G_{n-s}) \geq (1-\gamma/2)(n-s)$.

Let $\alpha = 2\gamma$. We will have another pass over the vertices x_n, \dots, x_{n-s+1} , each time decomposing the edges incident to x_i by H -subgraphs and single edges. It will be the case that each time we remove the edges incident to the current vertex x_i , the degree of any other vertex drops by at most $3h^4$, where $h = v(H)$. Here is a formal description. Initially, let $G'_n = G$ and $i = n$. If in the current graph G'_i we have $\deg_{G'_i}(x_i) \leq \alpha n$, then we remove all G'_i -edges incident to x_i as single edges and let $G'_{i-1} = G'_i - x_i$.

Suppose that $\deg_{G'_i}(x_i) > \alpha n$. Then, the set

$$X_i = \{y \in V(G_{n-s}) \mid x_i y \in E(G'_i)\},$$

has at least $\alpha n - s + 1$ vertices. The minimum degree of $G[X_i]$ is

$$\delta(G[X_i]) \geq |X_i| - s - \frac{\gamma n}{2} - s \times 3h^4 \geq \frac{2}{3}|X_i|.$$

Let $y \in V(H)$, $A = \Gamma_H(y)$ and $a = |A|$. Another

result from [4] (Lemma 3.1) states that there is a constant C , such that, all but at most C vertices of $G[X_i]$ can be covered by edge disjoint copies of $H-y$ each of them having vertex disjoint sets A . Therefore, all but at most C edges between x_i and X_i can be decomposed into copies of H . All other edges incident to x_i are removed as single edges. Let G'_{i-1} consist of the remaining edges of $G'_i - x_i$ (that is, those edges that do not belong to an H -subgraph of the above x_i -decomposition). This finishes the description of the case $\deg_{G'_i}(x_i) > \alpha n$.

Consider the sets $S = \{x_n, \dots, x_{n-s+1}\}$,

$S_1 = \{x_i \in S \mid \deg_{G'_i}(x_i) \leq \alpha n\}$, and $S_2 = S/S_1$. Let their sizes be s , s_1 , and s_2 respectively, so $s = s_1 + s_2$.

Let F be the graph with vertex set $V(G_{n-s}) \cup S_2$, consisting of the edges coming from the removed H -subgraphs when we processed the vertices in S_2 . We have

$$\phi_H(G) \leq \phi_H(G'_{n-s}) + \frac{e(F)}{m} + s_1 \alpha n + s_2 C + \binom{s}{2}. \quad (2.9)$$

We know that $\phi_H(G'_{n-s}) = e(G'_{n-s}) - p_H(G'_{n-s}(m-1))$, furthermore, $\delta(G'_{n-s}) \geq (1-\gamma)(n-s)$. Thus, $p_H(G'_{n-s})$ can be estimated using Lemma 2.7.

If (2.6) holds, some calculations show that there exists a graph G^* such that $\phi_H(G) < \phi_H(G^*)$, which contradicts the optimality of G .

Therefore, (2.5) must hold. It follows that $p_H(G)$ and thus $\phi_H(G)$, depends only on the degree sequence d_1, \dots, d_n of G . Namely, the packing number

$\ell = p_H(G)$ equals $\left\lfloor \frac{1}{2m} \sum_{i=1}^n r_i \right\rfloor$, where $r_i = d \lfloor d_i/d \rfloor$ is the largest multiple of d not exceeding d_i .

Therefore,

$$\phi_H(G) = \frac{1}{2} \sum_{i=1}^n d_i - (m-1) \left\lfloor \frac{1}{2m} \sum_{i=1}^n \left\lfloor \frac{d_i}{d} \right\rfloor d \right\rfloor, \quad (2.10)$$

where d_1, \dots, d_n is the degree sequence of G .

To conclude the proof we need to estimate the values that the degrees of G can attain. To do that we need to prove an upper bound on $\phi_H(G)$ by estimating ϕ_{\max} , the maximum of

$$\phi(d_1, \dots, d_n) = \frac{1}{2} \sum_{i=1}^n d_i - (m-1) \left\lfloor \frac{1}{2m} \sum_{i=1}^n \left\lfloor \frac{d_i}{d} \right\rfloor d \right\rfloor, \quad (2.11)$$

over all (not necessarily graphical) sequences d_1, \dots, d_n of integers with $0 \leq d_i \leq n-1$.

Let d_1, \dots, d_n be an optimal sequence attaining the value ϕ_{\max} . For $i = 1, \dots, n$ let $d_i = q_i d + r_i$ with

$$0 \leq r_i \leq d-1. \text{ Then, } \ell = \left\lfloor \frac{(q_1 + \dots + q_n)d}{2m} \right\rfloor.$$

Let $n = qd + r$ with $0 \leq r \leq d - 1$ and $q = \lfloor n/d \rfloor$. Define $R = qd - 1$ to be the maximum integer which is at most $n - 1$ and is congruent to $d - 1$ modulo d . Let $C_1 = \{i \in [n] \mid r_i = d - 1 \text{ and } d_i < R\}$ and $C_2 = \{i \in [n] \mid d_i = n - 1\}$ if $n - 1 \neq R$ and $C_2 = \emptyset$ otherwise.

Since d_1, \dots, d_n is an optimal sequence, we have that if $r_i \neq d - 1$ then $d_i = n - 1$ for all $i \in [n]$. To conclude the proof it remains to show that $|C_1| \leq \frac{2m}{d} - 1$ and $|C_2| \leq 2m - 1$. Suppose first that $|C_1| \geq \frac{2m}{d} =: t$.

Consider the new sequence of integers

$$d_i^* = \begin{cases} d_i + d, & \text{if } i \in C_1, \\ d_i, & \text{if } i \notin C_1. \end{cases}$$

Then, $\ell^* = \ell + 1$ and $\phi^* = \phi_{\max} + 1$ which contradicts our assumption on ϕ_{\max} .

Now suppose that $|C_2| \geq 2m$ and consider the new sequence of integers d_1^*, \dots, d_n^* obtained from d_1, \dots, d_n by replacing $2m$ values of $n - 1$ by R . Then, $\ell^* = \ell - d$ and $\phi^* \geq \phi_{\max} + m - d > \phi_{\max}$, which contradicts our assumption on ϕ_{\max} and the proof is concluded. \square

We now have all the tools needed to prove Theorem 1.4.

Proof of Theorem 1.4. Let n_0 be given by Lemma 2.8. Let G be a graph of order $n \geq n_0$ with $\phi_{C_4}(G) = \phi_{C_4}(n)$ and degree sequence d_1, \dots, d_n . For $i = 1, \dots, n$ let $d_i = 2q_i + r_i$ with $0 \leq r_i \leq 1$. Let, $R = 2(\lfloor n/2 \rfloor - 1) + 1$ and let the sets C_1 and C_2 be as in Lemma 2.8.

Let $n = 2q + r$ with $0 \leq r \leq 1$ and $q = \lfloor n/2 \rfloor$. From (2.7) we obtain

$$\begin{aligned} \phi_{C_4}(n) &= n(q - 1) + \frac{n}{2} + \frac{1}{2}|C_2|r - \sum_{i \in C_1} (q - 1 - q_i) \\ &\quad - 3 \left[\frac{1}{4}n(q - 1) + \frac{1}{4}|C_2| - \frac{1}{4} \sum_{i \in C_1} (q - 1 - q_i) \right] \end{aligned} \quad (2.12)$$

In what follows let $\alpha = |C_2|$ and $\beta = \sum_{i \in C_1} (q - 1 - q_i)$.

We consider first the case when n is even. Then $C_2 = \emptyset$ and we have

$$\begin{aligned} \phi_{C_4}(n) &= n(q - 1) + \frac{n}{2} - \beta - 3 \left[\frac{1}{4}n(q - 1) - \frac{\beta}{4} \right] \\ &= n(q - 1) + \frac{n}{2} - \frac{3q(q - 1)}{2} - \beta - 3 \left[-\frac{\beta}{4} \right] \end{aligned} \quad (2.13)$$

Claim 1. Let d_1, \dots, d_n be the degree sequence of a graph. Then,

$$-\beta - 3 \left\lfloor -\frac{\beta}{4} \right\rfloor \leq 1.$$

Proof. Routine calculations show that for $\beta \neq 1$ we have $-\beta - 3 \left\lfloor -\frac{\beta}{4} \right\rfloor \leq 1$. Suppose $\beta = 1$. Then C_1 has exactly one element, thus the sequence $(d_i)_{i=1, \dots, n}$ has exactly one element equal to $n - 3$ and all the others equal to $n - 1$. But this is not a degree sequence of a graph since condition (2.3) of Theorem 2.6 does not hold for $k = n - 2$. \square

Therefore, using the estimate of Claim 1 in (2.13) it follows that

$$\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{4} + 1.$$

To prove the lower bound consider the graph L_5 obtained from K_n after the deletion of the edges of a C_5 . Using (1.1) and (2.5) we show that

$$\phi_{C_4}(L_5) = \frac{n^2}{8} + \frac{n}{4} + 1.$$

We now consider the case when n is an odd number.

Case 1: Let $n = 8t + 1$ and $q = 4t$.

From (2.12) we obtain

$$\begin{aligned} \phi_{C_4}(n) &= \frac{n}{2}(n - 3) + \frac{n}{2} - 3(8t^2 - t) \\ &\quad + \frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta - 1}{4} \right\rfloor \end{aligned} \quad (2.14)$$

Claim 2. Let d_1, \dots, d_n be the degree sequence of a graph. Then,

$$\frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta - 1}{4} \right\rfloor \leq \frac{5}{2}.$$

Proof. Routine calculations show that the result follows if $\alpha \neq 0$ or $\beta \neq 0$. If $\alpha = 0$ and $\beta = 0$ then $d_i = n - 2$ for all $1 \leq i \leq n$. This is not a degree sequence of a graph since $\sum_{i=1}^n d_i$ is not even. \square

Therefore, using the estimate of Claim 2 in (2.14) we prove that

$$\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{8} + \frac{14}{8}.$$

As for the lower bound consider the graph L^* with all vertices of degree $n - 2$ except one of degree $n - 3$. Using (1.1) and (2.5) we show that

$$\phi_{C_4}(L^*) = \frac{n^2}{8} + \frac{n}{8} + \frac{14}{8}.$$

Case 2: Let $n = 8t + 3$ and $q = 4t + 1$.

From (2.12) we obtain

$$\begin{aligned} \phi_{C_4}(n) &= \frac{n}{2}(n-3) + \frac{n}{2} - 3(8t^2 + 3t) \\ &\quad + \frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta}{4} \right\rfloor. \end{aligned} \tag{2.15}$$

Claim 3. Let d_1, \dots, d_n be the degree sequence of a graph. Then,

$$\frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta}{4} \right\rfloor \leq \frac{3}{2}.$$

Proof. It follows from routine calculations for all values of α and β except when $\alpha = 0$ and $\beta = 1$. Suppose that $\alpha = 0$ and $\beta = 1$. Then $C_2 = \emptyset$ and C_1 has exactly one element, thus the sequence $(d_i)_{i=1, \dots, n}$ has exactly one element equal to $n-2$ and all the others equal to $n-1$. But this is not a degree sequence of a graph since $\sum d_i$ is not even. \square

Therefore, using the estimate of Claim 3 in (2.15) we prove that

$$\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}.$$

As for the lower bound consider the graph L with degree sequence $d_1 = d_2 = n-4$, $d_3 = \dots = d_{n-1} = n-2$ and $d_n = n-1$ (the existence of L can be proved directly or by Erdős-Gallai theorem, Theorem 2.6). Using (1.1) and (2.5) we show that

$$\phi_{C_4}(L) = \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}.$$

Case 3: Let $n = 8t + 5$ and $q = 4t + 2$. From (2.12) we obtain

$$\begin{aligned} \phi_{C_4}(n) &= \frac{n}{2}(n-3) + \frac{n}{2} - 3(8t^2 + 7t) \\ &\quad + \frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 5}{4} \right\rfloor. \end{aligned} \tag{2.16}$$

Claim 4. Let d_1, \dots, d_n be the degree sequence of a graph. Then,

$$\frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 5}{4} \right\rfloor \leq -\frac{5}{2}.$$

Proof. Routine calculations show that

$\alpha/2 - \beta - 3 \left\lfloor \frac{\alpha - \beta + 5}{4} \right\rfloor \leq -5/2$ for all values of α and β except for $\alpha = 2$ and $\beta = 0$ or $\alpha = 0$ and $\beta = 2$.

Suppose first that $\alpha = 2$ and $\beta = 0$. Then the sequence $(d_i)_{i=1, \dots, n}$ has two elements equal to $n-1$ and all the others equal to $n-2$. This is not a degree sequence of a graph since $\sum_{i=1}^n d_i$ is not even. \square

Suppose now that $\alpha = 0$ and $\beta = 2$. If $|C_1| = 2$ then the sequence has two elements equal to $n-4$ and

all the others equal to $n-2$ and this is not a degree sequence of a graph since $\sum d_i$ is not even. Finally, if $|C_1| = 1$ then we have one element equal to $n-6$ and all the others equal to $n-2$. Again, this is not a degree sequence of a graph since $\sum d_i$ is not even. \square

Therefore, using the estimate of Claim 4 in (2.16) we prove that

$$\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}.$$

As for the lower bound consider the graph $K_n - I$ obtained from K_n by deleting the edges of a maximum matching. Using (1.1) and (2.5) we show that

$$\phi_{C_4}(K_n - I) = \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}.$$

Case 4: Let $n = 8t + 7$ and $q = 4t + 3$. From (2.12) we obtain

$$\begin{aligned} \phi_{C_4}(n) &= \frac{n}{2}(n-3) + \frac{n}{2} - 3(8t^2 + 11t) \\ &\quad + \frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 14}{4} \right\rfloor. \end{aligned} \tag{2.17}$$

Claim 5. Let d_1, \dots, d_n be the degree sequence of a graph. Then,

$$\frac{1}{2}\alpha - \beta - 3 \left\lfloor \frac{\alpha - \beta + 14}{4} \right\rfloor \leq -\frac{17}{2}.$$

Proof. It follows directly from simple calculations. \square

Therefore, using the estimate of Claim 5 in (2.17) we prove that

$$\phi_{C_4}(n) \leq \frac{n^2}{8} + \frac{n}{8} + 2.$$

Furthermore, using (1.1) and (2.5) we have

$$\phi_{C_4}(K_n - I) = \frac{n^2}{8} + \frac{n}{8} + 2,$$

so the equality follows and the proof is now complete. \square

3. Acknowledgements

The author would like to thank Oleg Pikhurko for helpful comments and discussions. The author acknowledges the support from FCT—Fundação para a Ciência e a Tecnologia (Portugal), through Projects PTDC/MAT/113207/2009 and PEst-OE/MAT/UI0297/2011 (CMA).

REFERENCES

- [1] B. Bollobás, “Modern Graph Theory,” Springer-Verlag, New York, 1998. [doi:10.1007/978-1-4612-0619-4](https://doi.org/10.1007/978-1-4612-0619-4)
- [2] P. Erdős, A. W. Goodman and L. Pósa, “The Representation of a Graph by Set Intersections,” *Canadian Journal*

- of Mathematics*, Vol. 18, No. 1, 1966, pp. 106-112.
[doi:10.4153/CJM-1966-014-3](https://doi.org/10.4153/CJM-1966-014-3)
- [3] B. Bollobás, “On Complete Subgraphs of Different Orders,” *Mathematical Proceedings of the Cambridge Philosophical Society*, Vol. 79, No. 1, 1976, pp. 19-24.
[doi:10.1017/S0305004100052063](https://doi.org/10.1017/S0305004100052063)
- [4] O. Pikhurko and T. Sousa, “Minimum H-Decompositions of Graphs,” *Journal of Combinatorial Theory Series B*, Vol. 97, No. 6, 2007, pp. 1041-1055.
[doi:10.1016/j.jctb.2007.03.002](https://doi.org/10.1016/j.jctb.2007.03.002)
- [5] T. Sousa, “Decompositions of Graphs into a Given Clique-Extension,” *ARS. Combinatoria*, Vol. 100, 2011, pp. 465-472.
- [6] T. Sousa, “Decompositions of Graphs into 5-Cycles and Other Small Graphs,” *Electronic Journal of Combinatorics*, Vol. 12, 2005, 7p.
- [7] T. Sousa, “Decompositions of Graphs into Cycles of Length Seven and Single Edges,” *ARS. Combinatoria*, to appear.
- [8] L. Özkahya and Y. Person, “Minimum H-Decompositions of Graphs: Edge-Critical Case,” *Journal of Combinatorial Theory Series B*, Vol. 102, No. 102, 2012, pp. 715-725.
[doi:10.1016/j.jctb.2011.10.004](https://doi.org/10.1016/j.jctb.2011.10.004)
- [9] P. Allen, J. Böttcher and Y. Person, “An Improved Error Term for Minimum H-Decompositions of Graphs,” arXiv: 1109.2571v1, 2011.
- [10] P. Erdős and T. Gallai, “Graphs with Prescribed Degree of Vertices,” *Matematikai Lapok*, Vol. 11, 1960, pp. 264-274.
- [11] N. Alon, Y. Caro and R. Yuster, “Packing and Covering Dense Graphs,” *Journal of Combinatorial Designs*, Vol. 6, No. 6, 1998, pp. 451-472.
[doi:10.1002/\(SICI\)1520-6610\(1998\)6:6<451::AID-JCD6>3.0.CO;2-E](https://doi.org/10.1002/(SICI)1520-6610(1998)6:6<451::AID-JCD6>3.0.CO;2-E)
- [12] T. Gustavsson, “Decompositions of Large Graphs and Digraphs with High Minimum Degree,” Ph.D. Thesis, University of Stockholm, Stockholm, 1991.