Open Journal of Discrete Mathematics, 2012, 2, 96-98
http://dx.doi.org/10.4236/ojdm.2012.23018 Published Online July 2012 (http://www.SciRP.org/journal/ojdm)

Some New Results on Domination Integrity of Graphs

Samir K. Vaidya[1], Nirang J. Kothari[2]
[1]Saurashtra University, Rajkot, India
[2]B. H. Gardi College of Engineering & Technology, Rajkot, India
Email: samirkvaidya@yahoo.co.in, nirang_kothari@yahoo.com

Received April 10, 2012; revised May 15, 2012; accepted May 30, 2012

ABSTRACT

The domination integrity of a connected graph $G = (V(G), E(G))$ is denoted as $DI(G)$ and defined by $DI(G)$ = $\min\{|S| + m(G-S) : S \text{ is a dominating set}\}$ where $m(G-S)$ is the order of a maximum component of $G-S$. We discuss domination integrity in the context of some graph operations like duplication of an edge by vertex and duplication of vertex by an edge.

Keywords: Integrity; Dominating Set; Domination Integrity

1. Introduction

The vulnerability of communication network measures the resistance of network to the disruption of operation after the failure of certain station or communication links. For any communication network greater degrees of stability or less vulnerability is required. Vulnerability can be measured by certain parameters like connectivity, toughness, integrity, binding number etc. In the analysis of vulnerability of communication network to disruption, following two parameters are of great importance: 1) The size of the largest remaining group within which mutual communication can still occur; 2) The number of elements that are not functioning. In this context Barefoot *et al*. [1] have introduced the concept of integrity of a graph as a new measure of vulnerability of network.

Definition 1.1. *The integrity of a graph G is denoted by* $I(G)$ *and defined by* $I(G) = \min\{|S| + m(G-S) : S \subset V(G)\}$ *where* $m(G-S)$ *is the order of a maximum component of* $G-S$.

Definition 1.2. *An I-set of G is any (proper) subset S of* $V(G)$ *for which* $I(G) = |S| + m(G-S)$.

The connectedness of graph is not essential to define integrity. The integrity of middle graphs is discussed by Mamut and Vumar [2] while integrity of total graphs is discussed by Dundar and Aytac [3]. If *D* is any minimal dominating set and if the order of the largest component of *G* – *D* is small then the removal of *D* will crash the communication network. The decision making process as well as communication between remaining members will also be highly affected. Considering this aspect Sundareswaran and Swaminathan [4] introduced the concept of domination integrity which is defined as follows.

Definition 1.3. The *domination integrity of a connected graph G is denoted as* $DI(G)$ *and defined by* $DI(G) = \min\{|S| + m(G-S) : S \text{ is a dominating set}\}$ *where* $m(G-S)$ *is the order of a maximum component of* $G-S$.

Sundareswarn and Swaminathan [5] have investigated domination integrity of middle graph of some graphs. In the present work, we investigate the domination integrity for the graphs obtained by various graph operations. In other words we have tried to relate expansion of network with measure of vulnerability.

Definition 1.4. *Duplication of a vertex* v_i *by a new edge* $e = v_i'v_i''$ *in graph G produces a new graph* G' *such that* $N(v_i') = \{v_i'', v_i\}$ *and* $N(v_i'') = \{v_i', v_i\}$.

Definition 1.5. *Duplication of an edge* $e = uv$ *by a new vertex* w *in a graph G produces a new graph* G' *such that* $N(w) = \{u, v\}$.

Definition 1.6. *For the dominating set* $S \subseteq V$, *a vertex* $v \in S$ *is called isolate of* S *if* $N(v) \subseteq V - S$.

For all other standard terminology and graph theoretical notation we refer to Hynes *et al*. [6].

2. Main Results

Lemma 2.1. *Let G be a graph obtained by duplication of each edge by vertex of path* P_n *then* $\gamma(G) = \lfloor n/2 \rfloor$.

Proof: Let *G* be a graph obtained by duplicating each edge v_iv_{i+1} of path P_n $(1 \le i \le n-1)$ by vertex u_i. There are three types of vertices in *G*,

1) $d(u_i) = 2\ \forall i$,
2) $d(v_i) = 4$ for $2 \le i \le n-1$ and
3) $d(v_1) = d(v_n) = 2$.

It is obvious that v_i must be in the dominating set as they are the most dominating vertices. From the nature of the graph G it is obvious that out of the vertices v_i and v_{i+1} at least one vertex must belongs to any dominating set S as u_i is adjacent to only v_i and v_{i+1}. Therefore if S is any dominating set then $|S| \geq \left\lfloor \frac{n}{2} \right\rfloor$.

We claim that $S = \{v_2, v_4, \cdots, v_n\}$ when n is even and $S = \{v_2, v_4, \cdots, v_{n-1}\}$ when n is odd are minimal dominating sets.

One can observe that each v_{2i-1} and u_{2i-1} are adjacent to v_{2i} and removal of v_{2i} from set S, u_{2i-1} will not be dominated by any vertex of S Hence $\gamma(G) = \lfloor n/2 \rfloor$.

Theorem 2.2. *Let G be a graph obtained by duplication of each edge by vertex of path P_n then*

$$DI(G) = \begin{cases} 3 & \text{if } n = 2,3 \\ 4 & \text{if } n = 4 \end{cases}$$

Proof: To prove the result, we consider following three cases.

Case-1. When $n = 2$: Let G be a graph obtained by duplication of an edge uv of path P_2 by a vertex w. Then $\gamma(G) = 1$ and $D = \{u\}$ as a γ-set of G and then $m(G - D) = 2$. This implies $DI(G) \leq \gamma(G) + m(G - D) = 3$. If S is any dominating set with $m(G - S) = 1$ then $|S| = 2$ and consequently $DI(G) = 3$. Hence in all the cases $DI(G) = 3$.

Case-2. When $n = 3$: Let G be a graph obtained by duplication of an edge v_1v_2 and v_2v_3 of path P_3 by the vertices u_1 and u_3 respectively. As $D = \{v_2\}$ is the only γ-set of G then $\gamma(G) = 1$ and $m(G - D) = 2$. If S_1 is any dominating set with $m(G - S_1) = 1$ then $|S_1| = 3$ and $|S_1| + m(G - S_1) = 4 > \gamma(G) + m(G - D)$. If S_2 is any dominating set with $m(G - S_2) = 3$ then $|S_2| = 2$ and $|S_2| + m(G - S_2) = 5 > \gamma(G) + m(G - D)$. Moreover $m(G - S_2) > 3$ is not possible, as the order of the largest component of $G - S$ is at most 3. Thus $DI(G) = 3$.

Case-3. When $n = 4$: Let G be a graph obtained by duplication of an edge v_iv_{i+1} of path P_4 by vertex $u_i (1 \leq i \leq 3)$. Then $\gamma(G) = 2$ with $D = \{v_2, v_3\}$ is a γ-set of G and $m(G - D) = 2$. Consequently $\gamma(G) + m(G - D) = 4$. If S_1 is any dominating set with $m(G - S_1) = 1$ then $|S_1| \geq 4$ and $|S_1| + m(G - S_1) \geq 5 > \gamma(G) + m(G - D)$. If S_2 is any dominating set with $m(G - S_2) \geq 3$ then $|S_2| \geq 2$ and $|S_2| + m(G - S_2) \geq 5 > \gamma(G) + m(G - D)$. Hence we have $DI(G) = 4$.

Theorem 2.3. *Let G be a graph obtained by duplication of each edge by vertex of path P_n then*

$$DI(G) = \left\lfloor \frac{n}{2} \right\rfloor + 3. \ (\text{if } n \geq 5)$$

Proof: Let G be a graph obtained by duplicating each edge v_iv_{i+1} of path P_n by vertex u_i where $1 \leq i \leq n-1$. Then from Theorem 2.1 $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$. If n is even then $D = \{v_2, v_4, \cdots, v_n\}$ is a γ-set of G otherwise $D = \{v_2, v_4, \cdots, v_{n-1}\}$ is a γ-set of G Therefore $m(G - D) = 3$ which implies,

$$DI(G) \leq \gamma(G) + m(G - D) = \left\lfloor \frac{n}{2} \right\rfloor + 3 \quad (1)$$

We will show that the number $|S| + m(G - S)$ is minimum. For that we have to take into account the minimality of both $|S|$ and $m(G - S)$. The minimality of $|S|$ is guaranteed as S is γ-set. Now it remains to show that if S is any dominating set other than D then $|S| + m(G - S) \geq 3$.

If $m(G - S) = 2$ then $|S| \geq n - 2 \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 = \gamma(G) + 1$ and $|S| + m(G - S) \geq (\gamma(G) + 1) + 2 = \gamma(G) + 3$.

If $m(G - S) = 1$ then $|S| \geq n \geq \left\lfloor \frac{n}{2} \right\rfloor + 2 = \gamma(G) + 2$ which implies that

$$|S| + m(G - S) \geq (\gamma(G) + 2) + 1 = \gamma(G) + 3.$$

If $m(G - S) \geq 3$ then trivially $|S| + m(G - S) \geq \gamma(G) + 3$. Hence for any dominating set S

$$|S| + m(G - S) \geq \gamma(G) + 3 \quad (2)$$

From (1) and (2) we have $DI(G) = \left\lfloor \frac{n}{2} \right\rfloor + 3$ (if $n \geq 5$).

Theorem 2.4. *Let G be a graph obtained by duplication of each vertex by an edge of path P_n or cycle C_n then $\gamma(G) = n$.*

Proof: Let G be a graph obtained by duplication of vertices $\{v_1, v_2, \cdots, v_n\}$ of path P_n or cycle C_n by an edge $u_{2i-1}u_{2i}$ $(1 \leq i \leq n)$. Then from the construction of graph G it is obvious that from the vertices v_i, u_{2i} and u_{2i-1} at least one vertex must belong to any dominating set S. Consequently if S is any dominating set then $|S| \geq n$.

We claim that set $S = \{v_1, v_2, \cdots, v_n\}$ is a minimal dominating set. Because each v_i is adjacent to u_{2i} and u_{2i-1}. If v_i is removed from set S then u_{2i} and u_{2i-1}. will not be dominated by any vertex. Thus S is a minimal dominating set with minimum cardinality. Hence $\gamma(G) = n$.

Theorem 2.5. *Let G be a graph obtained by duplication of each vertex of path P_n or cycle C_n by an edge then $DI(G) = n + 2$.*

Proof: Let G be a graph obtained by duplication of each vertex v_i of path P_n or cycle C_n by an edge $u_{2i-1}u_{2i}$ $(1 \leq i \leq n)$. Then from Theorem 2.4 we have $\gamma(G) = n$. Let $D = \{v_1, v_2, \cdots, v_n\}$ be a γ-set of graph G. Then $m(G - D) = 2$. Therefore,

$$DI(G) \le \gamma(G) + m(G-D) = n+2 \quad (1)$$

We will show that the number $|S| + m(G-S)$ is minimum. For that we have to take into account the minimality of both $|S|$ and $m(G-S)$. The minimality of $|S|$ is guaranteed as S is γ-set. Now it remains to show that if S is any dominating set other than D then $|S| + m(G-S) \ge 2$. If $m(G-S) = 1$ then $|S| \ge 2n > n+1$, consequently $|S| + m(G-S) \ge n+2$. If $m(G-S) \ge 2$ then trivially $|S| + m(G-S) \ge n+2$. Hence for any dominating set S,

$$|S| + m(G-S) \ge n+2 \quad (2)$$

From (1) and (2) we have $DI(G) = n+2$.

Proposition 2.6 [6]. A dominating set S is a minimal dominating set if and only if for each vertex $u \in S$, one of the following two conditions holds:

1) u is an isolate of S.

2) There exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.

Lemma 2.7. *Let G be a graph obtained by duplication of each vertex of wheel W_n by an edge then $\gamma(G) = n$.*

Proof: Let G be a graph obtained by duplication of rim vertices as well as apex vertex altogether of wheel $W_n = C_n + K_1$ by edges $u_{2i}u_{2i-1}$ and uu' respectively. Then each rim vertex v_i will dominate the vertices u_{2i-1}, u_{2i} and apex vertex c_1. For $S = \{v_1, v_2, \cdots, v_n\}$ there exists a vertex $u_1 \in V - S$ such that $N(u_1) \cap S$ is a singleton set. Then from Proposition 2.6 S will be a minimal dominating set of G. If S_1 is any dominating set then we claim that $|S_1| \ge n$. Because

1) If all the elements of S_1 are only of the type v_i then $|S_1| = |S| = n$.

2) If elements of S_1 are combination of v_i and u_i then $|S_1| \ge n$.

3) If S_1 contains any of first two types together with the apex vertex then $|S_1| \ge n$.

4) If S_1 contains u_i and apex vertex then $|S_1| \ge n$.

Thus we have $\gamma(G) = n$.

Theorem 2.8. *Let G be a graph obtained by duplication of each vertex of wheel W_n by an edge then $DI(G) = n+3$.*

Proof: Let G be a graph obtained by duplication of apex vertex v of wheel W_n by an edge uu' and the rim vertices $\{v_1, v_2, \cdots, v_n\}$ of wheel W_n by an edge $u_{2i-1}u_{2i}$ $(1 \le i \le n)$. Then from Lemma 2.7 we have $\gamma(G) = n$. Let $D = \{v_1, v_2, \cdots, v_n\}$ be a γ-set of graph G. Then $m(G-D) = 3$. Therefore

$$DI(G) \le \gamma(G) + m(G-D) = n+3 \quad (1)$$

We will show that the number $|S| + m(G-S)$ is minimum. For that we have to take into account the minimality of both $|S|$ and $m(G-S)$. The minimality of $|S|$ is guaranteed as S is γ-set. Now it remains to show that if S is any dominating set other than D then

$$|S| + m(G-S) \ge 3.$$

If $m(G-S) = 2$ then $|S| \ge n+1$, consequently

$$|S| + m(G-S) \ge n+3.$$

If $m(G-S) = 1$ then $|S| \ge 2n+1 > n+2$, consequently

$$|S| + m(G-S) > n+3.$$

If $m(G-S) \ge 3$ then trivially $|S| + m(G-S) \ge n+3$. Thus for any dominating set S,

$$|S| + m(G-S) \ge n+3 \quad (2)$$

Hence from (1) and (2) $DI(G) = n+3$.

3. Concluding Remarks

We have investigated domination integrity of three special graph families. This work relates to network expansion and measure of vulnerability. We conclude that expansion of network will provide the reason for increase of vulnerability. To investigate similar results for different graph families obtained by various graph operations is an open area of research.

4. Acknowledgements

The authors are highly thankful to the anonymous referee for valuable comments and constructive suggestions.

REFERENCES

[1] C. A. Barefoot, R. Entringer and H. C. Swart, "Vulnerability in Graphs—A Comparative Survey," *Journal of Combinatorial Mathematics and Combinatorial Computing*, Vol. 1, 1987, pp. 13-22.

[2] A. Mamut and E. Vumar, "A Note on the Integrity of Middle Graphs," *Lecture Notes in Computer Science*, Vol. 4381, 2007, pp. 130-134.

[3] P. Dundar and A. Aytac, "Integrity of Total Graphs via Certain Parameters," *Mathematical Notes*, Vol. 75, No. 5, 2004, pp. 665-672.

[4] R. Sundareswaran and V. Swaminathan, "Domination Integrity in Graphs," *Proceedings of International Conference on Mathematical and Experimental Physics*, Prague, 3-8 August 2009, pp. 46-57.

[5] R. Sundareswaran and V. Swaminathan, "Domination Integrity of Middle Graphs," In: T. Chelvam, S. Somasundaram and R. Kala, Eds., A*lgebra*, *Graph Theory and Their Applications*, Narosa Publishing House, New Delhi, 2010, pp. 88-92.

[6] T. Haynes, S. Hedetniemi and P. Slater, "Fundamentals of Domination in Graphs," Marcel Dekker, Inc., New York, 1998.

