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On Cycle Related Graphs with Constant Metric Dimension

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ABSTRACT

If G is a connected graph, the distance $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \cdots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), \cdots, d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a resolving set or locating set for G. A resolving set of minimum cardinality is called a basis for G and this cardinality is the metric dimension of G, denoted by $dim(G)$. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of G in $\mathcal{G}$. In this paper, we show that dragon graph denoted by $T_{n,m}$ and the graph obtained from prism denoted by $2C_k + \{x_k y_k\}$ have constant metric dimension.

Keywords: Metric Dimension; Basis; Resolving Set; Dragon Graph

1. Notation and Preliminary Results

If G is a connected graph, the *distance* $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \cdots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* of the v with respect to W is denoted by $r(v|W)$ is the k-tuple $(d(v,w_1), d(v,w_2), \cdots, d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or *locating set* for G [1]. A resolving set of minimum cardinality is called a *metric basis* for G and its cardinality is the *metric dimension* of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1-14]).

For a given ordered set of vertices $W = \{w_1, w_2, \cdots, w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 f and only if $v = w_i$. Thus, to show that W is a resolving set it sufficient to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \setminus W$.

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [2] and studied independently by Harary *et al.* [3]. Applications of this invariant to the navigation of robots in networks are discussed in [4] and applications to chemistry in [1] while applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures are given in [5].

By denoting $G + H$ the join of G and H, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Caceres *et al.* [6] found the metric dimension of fan f_n and Tomescu *et al.* [7] found the metric dimension of *Jahangir graph* J_{2n}. Also Tomescu *et al.* [8] the partition and connected dimension of wheels.

Chartrand *et al.* proved:

Theorem 1: [1] *A graph G has metric dimension 1 if and only if G is a path.*

Hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2. Since *prisms* D_n are the trivalent plane graphs obtained by the cartesian product of the path P_2 with a cycle C_n, hence they constitute a family of 3-*regular graphs* with constant metric dimension. Also Javaid *et al.* proved in [9] that the plane graph *antiprism* A_n constitutes a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$.

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Let $2C_n + \{x_n y_n\}$ be a family of graphs of order $2n$ obtained from a prism D_n as shown in **Figure 1** and **Figure 2** respectively, by deleting the spokes $x_i y_i$ for $i \in \{1, 2, \cdots, n-1\}$. We prove the following.

Theorem 2: *Let* $G = 2C_n + \{x_n y_n\}$ *with* $|V(G)| = 2n$, *then* $dim(G) = 2$ *for* $n \geq 3$.

Let C_n be a cycle with vertex set $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and P_{m+1} be a path with vertex set $V(P_{m+1}) = \{v_n = u_0, u_1, \cdots, u_m\}$. Dragon graph $T_{n,m}$ as shown in **Figure 3**, is a graph of order $n+m$ obtained by identifying v_n of C_n with u_0 of P_{m+1}. We prove the following.

Theorem 3: *For all* $n \geq 3, m \geq 2$ $dim(T_{n,m}) = 2$.

2. Proofs

***Proof of the Theorem* 2:** By Theorem 1, $dim(G) \geq 2$. We only need to show that $W = \{y_1, x_1\}$ is a resolving set for G, which is obviously of minimal cardinality.

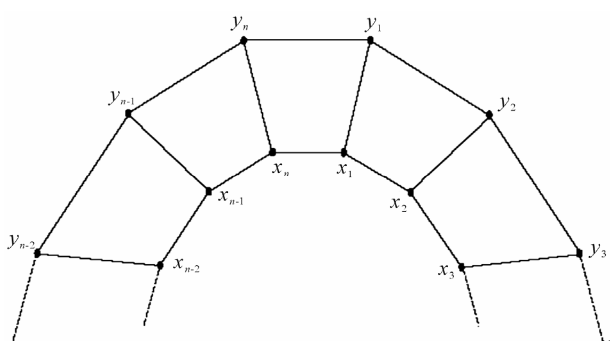

Figure 1. Prism D_n.

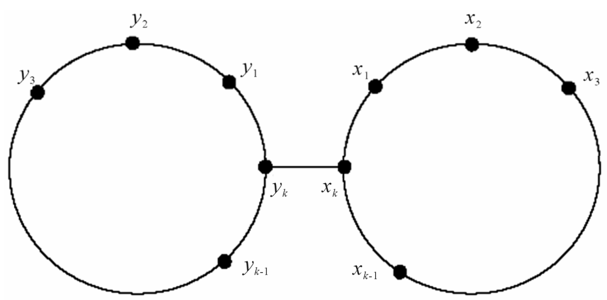

Figure 2. Graph $2C_k + \{x_k y_k\}$.

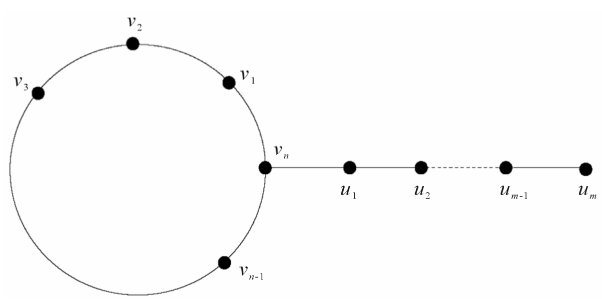

Figure 3. Dragon graph.

Case (a) When $n = 2k$ for $k \in \mathbb{N}$. Representations of all vertices from $V(G) \setminus \{y_1, x_1\}$ are as follows,

$$r(x_{i+1}|W) = \begin{cases} (i, i+3), & 1 \leq i \leq k-1; \\ (n-i, n-i+1), & k \leq i \leq n-1; \\ (i+3, i), & 1 \leq i \leq k-1; \\ (n-i+1, n-i), & k \leq i \leq n-1. \end{cases}$$

It is easy to check that all the above representations are distinct. For example, suppose that $(s+3, s) = (n-j, n-j+1)$ for some fixed s and j. Then $s = n-j-3$ and $s = n-j+1$, a contradiction.

Case (b) When $n = 2k+1$ for $k \in \mathbb{N}$. Representations of vertices from $V(G) \setminus \{y_1, x_1\}$ are as follows,

$$r(y_{i+1}|W) = \begin{cases} (i, i+3), & 1 \leq i \leq k-1; \\ (k, k+2), & \\ (n-i, n-i+1), & k+1 \leq i \leq n-1; \\ (i+3, i), & 1 \leq i \leq k-1; \\ (k+2, k), & \\ (n-i+1, n-i), & k+1 \leq i \leq n-1. \end{cases}$$

All the above representations are also distinct.

***Proof of the Theorem* 3:** By Theorem 1, $dim(T_{n,m}) \geq 2$. We only need to show that there is a resolving set W of cardinality 2.

Case (a) When $n = 2k$ for $k \in \mathbb{N}$. The set $W = \{v_k, v_{k+1}\}$ is a resolving set for the graph $T_{n,m}$. Representations of all vertices from $V(G) \setminus W$ are as follows,

$$r(v_i|W) = \begin{cases} (k-i, k-i+1), & 1 \leq i \leq k-1; \\ (i-k, i-k-1), & k+2 \leq i \leq n; \end{cases}$$

and

$$r(u_i|W) = (k+i, k+i-1), \quad 1 \leq i \leq m.$$

It is easy to check that all the representations are distinct. For example, suppose that $(k+s, k+s-1) = (j-k, j-k-1)$ for some fixed s and j. Then $j = 2k+s > n$ because $1 \leq s$, a contradiction.

Case (b) When $n = 2k+1$ for $k \in \mathbb{N}$. The set $W = \{v_1, v_{n-1}\}$ is a resolving set for the graph $T_{n,m}$. Representations of all vertices from $V(G) \setminus W$ are as follows,

$$r(v_i\,|W) = \begin{cases} (i-1, i+1), & 2 \leq i \leq k-1; \\ (i-1, n-i-1), & k \leq i \leq k+1; \\ (n-i+1, n-i-1), & k+2 \leq i \leq n-2; \end{cases}$$

$$r(v_n|W) = (1,1);$$

and

$$r(u_i|W) = (i+1, i+1), \quad 1 \leq i \leq m.$$

All the above representations are distinct.

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