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Some Switching Invariant Prime Graphs

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ABSTRACT

We investigate prime labeling for some graphs resulted from switching of a vertex. We discuss switching invariance of some prime graphs and prove that the graphs obtained by switching of a vertex in P_n and $K_{1,n}$ admit prime labeling. Moreover we discuss prime labeling for the graph obtained by switching of vertex in wheel W_n.

Keywords: Prime Labeling; Switching of a Vertex; Switching Invariance

1. Introduction and Definitions

We begin with simple, finite, undirected and non-trivial graph $G=\left(V\left(G\right),E\left(G\right)\right)$, with vertex set $V\left(G\right)$ and edge set $E\left(G\right)$. Throughout this work C_n denotes the cycle with n vertices and P_n denotes the path of n vertices. In wheel $W_n=C_n+K_1$ the vertex corresponding to K_1 is called apex vertex and the vertices corresponding to C_n are called rim vertices where $n\geq 3$. The star $K_{1,n}$ is a graph with one vertex of degree n called apex and n vertices of degree one (pendant vertices). Throughout this paper $\left|V\left(G\right)\right|$ and $\left|E\left(G\right)\right|$ are the cardinality of vertex set and edge set respectively.

For various graph theoretic notations and terminology we follow Gross and Yellen [1] while for number theory we follow Niven and Zuckerman [2]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Vast amount of literature is available in printed as well as in electronic form on different types of graph labeling. More than 1300 research papers have been published so far in last four decades. For a dynamic survey of graph labeling problems along with extensive bibliography we refer to Gallian [3].

Definition 1.2: *A prime labeling* of a graph G is an injective function $f:V\left(G\right)\rightarrow\left\{1,2,3,\cdots,\left|V\left(G\right)\right|\right\}$ such that for every pair of adjacent vertices u and v, $gcd\left(f\left(u\right),f\left(v\right)\right)=1$. The graph which admits a prime labeling is called a *prime graph*.

The notion of a prime labeling was originated by Entringer and was discussed in a paper by Tout *et al.* [4]. Many researchers have studied prime graphs. For e.g. Fu and Huang [5] have proved that P_n and $K_{1,n}$ are prime graphs. Lee *et al* [6] have proved that W_n is a prime graph if and only if n is even. Deretsky *et al.* [7] have proved that C_n is a prime graph.

Definition 1.3: *A vertex switching* G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.

Definition 1.4: A prime graph G is said to be *switching invariant* if for every vertex v of G, the graph G_v obtained by switching the vertex v in G is also a prime graph.

Vaidya and Kanani [8] have established the switching invariance of C_n corresponding to prime labeling while in the present paper we investigate further results on prime graphs.

Bertarnad's Postulate 1.5: For every positive integer $n>1$ there is a prime p such that $n<p<2n$.

2. Some Results on Prime Labeling with Respect to Vertex Switching Operation

Observation 2.1: Every prime graph G of order n has at least one vertex v (corresponding to label 1) such that G_v is a prime graph.

Observation 2.2: Let G be a prime graph of order $n\geq 2$ with a prime labeling. If v is the vertex corresponding to the largest prime less then or equal to n then G_v is a prime graph.

Observation 2.3: Let G be a prime graph of order $n\geq 3$ with a prime labeling and v is any arbitrary vertex having prime label from the set $\left\{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1, \cdots, n\right\}$ then G_v is a prime graph.

Theorem 2.4: P_n is switching invariant.

Proof: Let $v_1, v_2, \cdots, v_n$ be consecutive vertices of P_n. Let G_v be the graph obtained by switching a vertex v of P_n.

For the vertex v we have the following possibilities:

1) $v = v_1$ then in G_{v_1}, v_1 is adjacent to $v_3, v_4, \cdots, v_n$.

Define $f : V\left(G_{v_1}\right) \to \left\{1, 2, 3, \cdots, \left|V\left(G_{v_1}\right)\right|\right\}$ as follows:

$f\left(v_i\right) = i$, where $i = 1, 2, \cdots, n$

Then clearly f is an injection.

For an arbitrary edge $e = ab$ of G_{v_1} we claim that $\gcd\left(f\left(a\right), f\left(b\right)\right) = 1$. Because

a) if $e = v_1 v_i$ for some $i \in \{3, 4, \cdots, n\}$ then $gcd\left(f\left(v_1\right), f\left(v_i\right)\right) = gcd\left(1, i\right) = 1$;

b) if $e = v_i v_{i+1}$ for some $i \in \{2, 3, \cdots, n-1\}$ then $gcd\left(f\left(v_i\right), f\left(v_{i+1}\right)\right) = gcd\left(i, i+1\right) = 1$ as i and $i+1$ are consecutive positive integers.

If $v = v_n$ the proof is similar as discussed above.

2) $v = v_i$ for some $i \in \{2, 3, \cdots, n-1\}$. Then in G_{v_i}, v_i is adjacent to all the vertices except v_{i-1} and v_{i+1}.

Define $f : V\left(G_{v_i}\right) \to \left\{1, 2, 3, \cdots, \left|V\left(G_{v_i}\right)\right|\right\}$ as follows:

$$f\left(v_j\right) = \begin{cases} j+1 & \text{if } j = 1, 2, \cdots, i-1; \\ j & \text{if } j = i+1, i+2, \cdots, n; \\ 1 & \text{if } j = i. \end{cases}$$

Then clearly f is an injection.

For an arbitrary edge $e = ab$ of G_{v_i} we claim that $gcd\left(f\left(a\right), f\left(b\right)\right) = 1$.

To prove our claim the following cases are to be considered.

a) If $e = v_i v_j$ for some $j \in \{1, 2, \cdots, i-2\}$ then $gcd\left(f\left(v_i\right), f\left(v_j\right)\right) = gcd\left(1, j+1\right) = 1$;

b) If $e = v_i v_j$ for some $j \in \{i+2, 3, \cdots, n\}$ then $gcd\left(f\left(v_i\right), f\left(v_j\right)\right) = gcd\left(1, j\right) = 1$.

c) If $e = v_j v_{j+1}$ for some $j \in \{1, 2, \cdots, i-2\}$ then $gcd\left(f\left(v_j\right), f\left(v_{j+1}\right)\right) = gcd\left(j+1, j+2\right) = 1$ as j and $j+1$ are consecutive positive integers;

d) If $e = v_j v_{j+1}$ for some $j \in \{i+2, i+3, \cdots w, n\}$ then $gcd\left(f\left(v_j\right), f\left(v_{j+1}\right)\right) = gcd\left(j, j+1\right) = 1$;

Thus in each of the possibilities the graph G_v under consideration admits a prime labeling. *i.e.* G_v is a prime graph.

Thus P_n and the graph obtained by switching of any vertex in P_n are prime graphs. Hence the result.

Illustration 2.5: The prime labeling of the graph obtained by switching a pendant vertex of P_5 is shown in the **Figure 1**.

Illustration 2.6: The prime labeling of the graph obtained by switching a vertex of P_5 which is not a pendant vertex is shown in the **Figure 2**.

Theorem 2.7: $K_{1,n}$ is switching invariant.

Proof: We will separate two cases:

1) Switching of the apex vertex.

2) Switching of any pendant vertex.

<u>Case 1:</u> If v is the apex vertex of $K_{1,n}$ and G_v is the graph obtained by switching the apex vertex v then G_v is the null graph N_n on n vertices, which does not have any edge. Then obviously it is a prime graph.

<u>Case 2:</u> Let v_0 be the apex vertex and $v_1, v_2, \cdots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Let G_{v_n} be the graph obtained by switching the pendant vertex v_n of $K_{1,n}$. So in G_{v_n} every vertex v_i other than v_0 and v_n is adjacent to v_0 and v_n only. By Bertrand's postulate of number theory there exists at least one prime p such that $\frac{n}{2} \le p \le n+1$ then it is possible to define $f : \{v_0, v_1, v_2, \cdots, v_n\} \to \{1, 2, 3, \cdots, n+1\}$ as follows:

$$f\left(v_j\right) = \begin{cases} 1 & \text{if } j = 0; \\ j+1 & \text{if } j = 1, 2, \cdots p-2; \\ j+2 & \text{if } j = p-1, p, p+1, \cdots, n-1; \\ p & \text{if } j = n. \end{cases}$$

In view of the pattern defined above f admits a prime labeling on G_{v_n}. Hence G_{v_n} is a prime graph.

Thus $K_{1,n}$ and the graph obtained by switching any vertex of $K_{1,n}$ are prime graphs. Hence the result.

Illustration 2.8: The prime labeling of the graph obtained by switching a pendant vertex of $K_{1,6}$ is shown in the **Figure 3**.

Theorem 2.9: Switching the apex vertex in W_n is a prime graph.

Proof: Let $v_1, v_2, \cdots, v_n$ be consecutive rim vertices of W_n and v_0 be the apex vertex of W_n. Let G be the graph obtained by switching the vertex v_0. Thus G

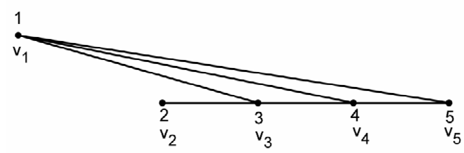

Figure 1. The prime labeling of the graph obtained by switching a pendant vertex of P_5.

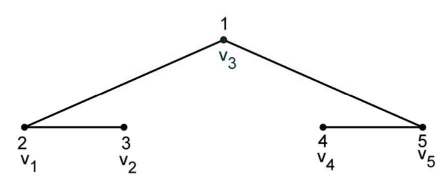

Figure 2. The prime labeling of the graph obtained by switching a vertex of P_5 which is not a pendant vertex.

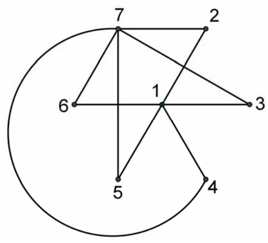

Figure 3. The prime labeling of the graph obtained by switching a pendant vertex of $K_{1,6}$.

is the disjoint union of C_n and K_1.

Define $f: V(G) \to \{1, 2, 3, \cdots, |V(G)|\}$ as follows:

$$f(v_i) = \begin{cases} n+1 & \text{if } i = 0; \\ i & \text{if } i = 1, 2, \cdots n. \end{cases}$$

Then clearly f is an injection.

For an arbitrary edge $e = ab$ of G we claim that $gcd(f(a), f(b)) = 1$.

1) If $e = v_i v_{i+1}$ for some $i \in \{1, 2, 3, \cdots, n\}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(i, i+1) = 1$ as i and $i+1$ are consecutive positive integers.

2) If $e = v_n v_1$ then $gcd(f(v_n), f(v_0)) = gcd(n, 1) = 1$.

Thus in each of the possibilities the graph G admits a prime labeling. *i.e.* G is a prime graph.

Illustration 2.10: The prime labeling of the graph obtained by switching the apex vertex of W_6 is shown in the **Figure 4**.

Theorem 2.11: Switching of a rim vertex of W_n is a prime graph if $n+1$ is a prime number.

Proof: Let $v_1, v_2, \cdots, v_n$ be consecutive rim vertices of W_n and v_0 be the apex vertex of W_n. Let G be the graph obtained by switching the vertex v_n.

Define $f: V(G) \to \{1, 2, 3, \cdots, |V(G)|\}$ as follows:

$f(v_i) = i+1$, $\forall i = 1, 2, \cdots, n$ and $f(v_0) = 1$.

Then clearly f is an injection.

For an arbitrary edge $e = ab$ of G we claim that $gcd(f(a), f(b)) = 1$.

1) If $e = v_i v_{i+1}$ for some $i \in \{1, 2, 3, \cdots, n-2\}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(i+1, i+2) = 1$ as $i+1$ and $i+2$ are consecutive positive integers.

2) If $e = v_0 v_i$ for some $i \in \{1, 2, 3, \cdots, n-1\}$ then $gcd(f(v_0), f(v_i)) = gcd(1, i+1) = 1$.

3) If $e = v_n v_i$ for some $i \in \{2, 3, \cdots, n-2\}$ then $gcd(f(v_n), f(v_i)) = gcd(n+1, i+1) = 1$ as $i+1$ is a positive integer less than the prime number $n+1$.

Thus in each of the possibilities the graph G under consideration admits a prime labeling. *i.e.* G is a prime graph.

Illustration 2.12: The prime labeling of the graph obtained by switching a rim vertex of W_6 is shown in the **Figure 5**.

Proposition 2.13: The graph obtained by switching of a rim vertex in W_7 is not a prime graph.

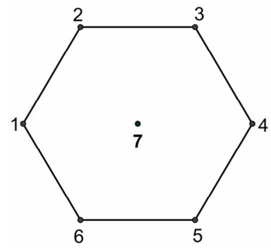

Figure 4. The prime labeling of the graph obtained by switching the apex vertex of W_6.

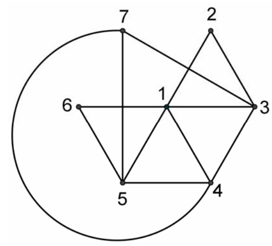

Figure 5. The prime labeling of the graph obtained by switching a rim vertex of W_6.

Proof: Let $v_1, v_2, \cdots, v_7$ be consecutive rim vertices of W_7 and v_0 be the apex vertex of W_7. Let G be the graph obtained by switching the vertex v_7.

If possible let $f: V(G) \to \{1, 2, 3, \cdots, 8\}$ be a prime labeling. As v_7 is adjacent to four vertices v_2, v_3, v_4, v_5 and v_0 is adjacent to six vertices $v_1, v_2, v_3, v_4, v_5, v_6$, the labels of v_7 and v_0 can not be even. Moreover we have to distribute four even labels among six vertices. Therefore at least two adjacent vertices from $v_1, v_2, v_3, v_4, v_5, v_6$ will receive the even labels which contradicts the fact that f is a prime labeling.

We noticed that it is not easy to discuss the prime labeling of a graph obtained by switching any rim vertex of W_n when $n+1$ is a composite number. However we prove a following result and pose a conjecture.

Theorem 2.14: Switching of a rim vertex in W_n is a not a prime graph if $n+1$ is an even integer greater than 9.

Proof: Let $v_1, v_2, \cdots, v_n$ be consecutive rim vertices and v_0 be the apex vertex of W_n. Let G be the graph obtained by switching the vertex v_n. If possible assume that there exists a prime labeling $f: V(G) \to \{1, 2, \cdots, n+1\}$ on G.

Observe that in $\{1, 2, \cdots, n+1\}$ the number of even integers is equal to the number of odd integers in $\{1, 2, \cdots, n+1\}$ which is $\frac{n+1}{2}$.

If a vertex v is adjacent to at least $n-3$ vertices then it cannot be labeled with even integer otherwise each of its $n-3$ neighbours should receive the labels with odd integers. Consequently there should be at least $n-3$ odd integers in $\{1,2,\cdots,n+1\}$ and at the most $n+1-(n-3)=4$ even integers. However, there are at least 5 even integers in $\{1,2,3,\cdots,n\}$, a contradiction.

Consequently each of the vertices v_0 and v_n have at least $n-3$ neighbours must be labeled with an odd integer. Therefore the remaining vertices $v_1,v_2,\cdots,v_{n-1}$ forms a path P_{n-1} in G and these vertices will receive $\frac{n+1}{2}-2=\frac{n-3}{2}$ odd labels and $\frac{n+1}{2}$ even labels. If L and M denote the number of vertices with even labels and number of vertices with odd labels respectively in P_{n-1} then $L-M=\frac{n+1}{2}-\frac{n-3}{2}=2$. Hence there must be two adjacent vertices in P_{n-1} which will receive even labels which contradicts our assumption that f is a prime labeling of G.

Conjecture 2.15: The graph obtained by switching of a rim vertex in W_n is a not a prime graph if $n+1$ is a composite odd integer greater than 9.

3. Concluding Remarks

The study of prime numbers is of great importance as prime numbers are scattered and there are arbitrarily large gaps in the sequence of prime numbers. If these characteristics are studied in the frame work of graph theory then it is more challenging and exciting as well.

Here we investigate several results on prime labeling of graphs and pose a conjecture. This discussion becomes more relevant as it is carried out in the context of a graph operation namely switching of a vertex. We have studied switching invariant behaviour of some standard graphs.

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