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Bounds for Domination Parameters in Cayley Graphs on Dihedral Group

T. Tamizh Chelvam, G. Kalaimurugan
Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, India
Email: tamche59@gmail.com

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ABSTRACT

In this paper, sharp upper bounds for the domination number, total domination number and connected domination number for the Cayley graph $G = Cay(D_{2n}, \Omega)$ constructed on the finite dihedral group D_{2n}, and a specified generating set Ω of D_{2n}. Further efficient dominating sets in $G = Cay(D_{2n}, \Omega)$ are also obtained. More specifically, it is proved that some of the proper subgroups of D_{2n} are efficient domination sets. Using this, an E-chain of Cayley graphs on the dihedral group is also constructed.

Keywords: Cayley Graph; Dihedral Group; Domination; Total Domination; Connected Domination; Efficient Domination

1. Introduction and Notation

Design of interconnection networks is an important integral part of any parallel processing of distributed system. There has been a strong interest recently in using Cayley graphs as a model for developing interconnection networks for large interacting arrays of CPU's. An excellent survey of interconnection networks based on Cayley graphs can be found in [1]. The concept of domination for Cayley graphs has been studied by various authors [2-7]. I. J. Dejter and O. Serra [3] obtained efficient dominating sets for Cayley graphs constructed on a class of groups containing permutation groups. The efficient domination number for vertex transitive graphs has been obtained by Jia Huang and Jun-Ming Xu [4]. A necessary and sufficient condition for the existence of an independent perfect domination set in Cayley graphs has been obtained by J. Lee [5]. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory. T. Tamizh Chelvam and I. Rani [6-8] have obtained bounds for various domination parameters for a class of Circulant graphs.

Let Γ be a finite group. Let Ω be a generating set of Γ satisfying $e \notin \Omega$ and $a \in \Omega$ implies $a^{-1} \in \Omega$. The *Cayley graph* corresponding to Γ is the graph $G = (V, E)$, where $V(G) = \Gamma$ and $E(G)=\{(x, xa): x \in V(G), a \in \Omega\}$ and it is denoted by $G = \mathrm{Cay}(\Gamma, \Omega)$. Let $G= (V, E)$, be a finite, simple and undirected graph. We follow the terminology of [9]. A set $S \subseteq V$ of vertices in a graph G is called a *dominating set* if every vertex $v \in V$ is either an element of S or adjacent to an element of S. A dominating set S is a *minimal dominating set* if no proper subset of S is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G and the corresponding dominating set is called a *γ-set*. A set $S \subseteq V$ is called a *total dominating set* if every vertex $v \in V$ is adjacent to an element u ($\neq v$) of S. The *total domination number* $\gamma_t(G)$ equals the minimum cardinality among all the total dominating sets in G and the corresponding total dominating set is called a *γ_t-set*. A dominating set S is called a *connected dominating set* if the induced subgraph $\langle S\rangle$ is connected. The *connected domination number* $\gamma_c(G)$ of a graph G equals the minimum cardinality of a connected dominating set in G and a corresponding connected dominating set is called a *γ_c-set*. A set $S \subseteq V$ is called an *efficient dominating set* (*E-set*) if for every vertex $v \in V$, $|N[v]\cap S|=1$.

An E-chain is a countable family of nested graphs, each of which has an E-set. We say that a countable family of graphs $\boldsymbol{G} = \{G_i, i \geq 1\}$ with each G_i has an E-set S_i is an *inclusive E-chain* if for every $i \geq 1$, there exists a surjective map f_i: $G_{i+1} \rightarrow G_i$ such that $f_i^{-1}(S_i) \subset S_{i+1}$. And also we define that a finite family of graphs $\boldsymbol{G} = \{G_i, i \geq 0\}$ is an *inductive E-chain* if every G_{i+1} is a spanning subgraph of G_i and each G_i has an E-set S_i. Let $V(G_i)$ be any finite group and if, for each $i \geq 0$, there exists a bijective map ζ_i: $V(G_i) \rightarrow V(G_{i+1})$ such that $\zeta_i(S_i) \subseteq S_{i+1}$ and S_i is the subgroup of $V(G_i)$ then we say that $\boldsymbol{G}$ is an *inductive subgroups E-chain*.

A graph $\tilde{G}$ is called a *covering* of G with projection $p:\tilde{G} \rightarrow G$ if there is a surjection $p:V(\tilde{G}) \rightarrow V(G)$

such that $p/_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We use the covering function to show the inclusive E-chain.

In this paper, we obtain upper bounds for domination number, total domination number and connected domination number in a Cayley graph $G = Cay(D_{2n}, \Omega)$ constructed on the dihedral group D_{2n}, for $n \geq 3$ and a generating set Ω. Further, we obtain some E-sets in $G = Cay(D_{2n}, \Omega)$. Note that the *dihedral group* D_{2n} with identity *e* is the group generated by two elements *r* and *s* with $o(r) = n, o(s) = 2$ and $rs = sr^{-1}$. From these defining relations, one can take $D_{2n} = \{e, r, r^2, r^3, \cdots, r^{n-1}, s, sr, sr^2, \cdots, sr^{n-1}\}$ and $G = Cay(D_{2n}, \Omega)$, where Ω is a generating set of D_{2n}. Throughout this paper, $n \geq 3$ be an integer, $\Gamma = D_{2n}$, $m = \dfrac{n-1}{2}$ and *k*, *t* be integers such that $1 \leq k \leq m$, $1 \leq t \leq n$. We take the generating set Ω in the form that

$$\Omega = \left\{r^{a_1}, r^{a_2}, \cdots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \cdots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \cdots, sr^{b_t}\right\},$$

where $1 \leq a_1 < a_2 < \cdots < a_k \leq m$ and $0 \leq b_1 < b_2 < \cdots < b_t \leq n-1$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d_1' = b_1, d_j' = b_j - b_{j-1}$ for $2 \leq j \leq t$ and $d = \max_{1 \leq i \leq k, 1 \leq j \leq t}\{d_i, d_j'\}$. Some of the results are listed below for further reference.

Theorem 1 [4] *Let G be a k-regular graph. Then* $\gamma(G) \geq \dfrac{|V(G)|}{k+1}$, *with the equality if and only if G has an efficient dominating set.*

Theorem 2 [5] *Let* $p : \tilde{G} \to G$ *be a covering and let S be a perfect domination set of G. Then* $p^{-1}(S)$ *is a perfect domination set of* $\tilde{G}$. *Moreover, if S is independent, then* $p^{-1}(S)$ *is independent.*

Theorem 3 [10] *Every subgroup of the dihedral group* D_{2n} *is cyclic or dihedral. A complete listing of the subgroups is as follows*:

1) *cyclic subgroups* $\langle r^d \rangle$, *where d divides n, with index* 2*d*.

2) *dihedral subgroups* $\langle r^d, r^i s \rangle$, *where d divides n and* $0 \leq i \leq d-1$ *with index d. Every subgroup of* D_{2n} *occurs exactly once in this listing.*

2. Domination, Total Domination and Connected Domination Numbers

In this section, we obtain upper bounds for the domination number, total domination number and connected domination number of graph $G = Cay(D_{2n}, \Omega)$. Also whenever the equality occurs we give the corresponding sets.

Lemma 4 *Let* $n \geq 3$ *be an integer,* $m = \dfrac{n-1}{2}$ *and k, t are integers such that* $1 \leq k \leq m, 1 \leq t \leq n$. *Let*

$$\Omega = \left\{r^{a_1}, r^{a_2}, \cdots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \cdots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \cdots, sr^{b_t}\right\}$$

and $G = Cay(D_{2n}, \Omega)$. *If* $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d_1' = b_1, d_j' = b_j - b_{j-1}$ for $2 \leq j \leq t$ and $d = \max_{1 \leq i \leq k, 1 \leq j \leq t}\{d_i, d_j'\}$, *then*

$$\gamma(G) \leq 2d \frac{n}{2d + 2a_k + b_t - b_1}.$$

Proof. Let $x = 2a_k + 2d + b_t - b_1$ and $l = \left\lceil \dfrac{n}{x} \right\rceil$. Consider the set

$$S = \left\{r^{ix+g}, sr^{n-(a_k+d-b_1+ix+g)} : 0 \leq i \leq l-1 \text{ and } 0 \leq g \leq d-1\right\}.$$

Clearly $|S| = 2dl$ and

$$N[S] = \bigcup_{i=0}^{l-1}\left\{N\left[r^{ix+g}\right] \bigcup N\left[sr^{n-(a_k+d-b_1+ix+g)}\right]\right\},$$

where $0 \leq i \leq l-1$ and $0 \leq g \leq d-1$ We have to prove that $V(G) \subseteq N[S]$. If $v \in V(G)$, then we can write v as either one vertex of the form $v = r^c$ or $v = sr^{n-(c-b_t)}$, where $0 \leq c \leq n-1$. By the division algorithm, $c = xi + j$, where $0 \leq i \leq l-1$ and $0 \leq j \leq x-1$.

Suppose $v = r^c$. We have the following cases:

Case 1. Suppose $0 \leq i \leq l-1$ and $0 \leq j \leq a_k + d - 1$.

Subcase 1.1 If $0 \leq j < a_1$, then by the definition of $d, v \in S \subseteq N[S]$.

Subcase 1.2 If $j = a_m + g$, for some integers *m*, *g* with $1 \leq m \leq k$ and $0 \leq g \leq d-1$ then $v = r^{ix+a_m+g}$ whereas $r^{ix+g} \in S$ and so $v \in N[r^{ix+g}] \subseteq N[S]$.

Case 2. Suppose $0 \leq i \leq l-1$ and $a_k + d \leq j \leq a_k + d + b_t - b_1 + d - 1$. In this case, there exists an integer *h* with $1 \leq h \leq b_t - b_1 + d - 1$ such that $v \cdot sr^h = sr^{n-(a_k+d-b_1+ix)}$.

Subcase 2.1 If $h \in \Omega_2 = \{b_1, b_2, \cdots, b_t\}$, then $v \in N\left(sr^{n-(a_k+d-b_1+ix)}\right) \subseteq N[S]$

Subcase 2.2 Suppose $h = b_m + g$, for some integers *m*, *g* with $1 \leq m \leq t$ and $1 \leq g \leq d-1$. In this case, $v \cdot sr^{b_m} = sr^{n-(a_k+d-b_1+ix+g)}$, which means that $v \in sr^{n-(a_k+d-b_1+ix+g)} \subseteq N[S]$.

Case 3. Suppose $0 \leq i \leq l-2$ and

$$a_k + d + b_t - b_1 + d \leq j \leq a_k + d + b_t - b_1 + d + a_k - 1.$$

In this case, there exists an integer *h* with $1 \leq h \leq a_k$ such that $v \cdot r^h = r^{(i+1)x}$.

Subcase 3.1 If $h \in \Omega_1 = \{a_1, a_2, \cdots, a_k\}$, then $v \in N\left(r^{(i+1)x}\right) \subseteq N[S]$.

Subcase 3.2 Suppose $h = a_m - g$, for some integers *m*, *g*, with $1 \leq m \leq k$ and $1 \leq g \leq d-1$. In this case, $v \cdot r^{a_m} = r^{(i+1)x+g}$, which means that $v \in N\left(r^{(i+1)x+g}\right) \subseteq N[S]$.

Case 4. Suppose $i = l-1$ and

$a_k + d + b_t - b_1 + d \le j \le a_k + d + b_t - b_1 + d + a_k - 1$.

Then there exists an integer h with $1 \le h \le a_k$ such that $v \cdot r^h = r^0$.

Subcase 4.1 If $h \in \Omega_1$, then $v \in N(r^0) \subseteq N[S]$.

Subcase 4.2 Suppose $h = a_m - g$, for some integers m, g with $1 \le m \le k$ and $1 \le g \le d-1$. In this case, $v \cdot r^{a_m} = r^g$ which means that $v \in N(r^g) \subseteq N[S]$.

Suppose $v = sr^{n-(c-b_t)}$. We have the following cases:

Case 1. Suppose $0 \le i \le l-1$ and $0 \le j \le b_t - b_1 + d - 1$. In this case, there exists an integer h with $0 \le h \le b_t - b_1 + d - 1$ such that $v.sr^h = r^{ix}$

Subcase 1.1 If $h \in \Omega_2$, then $v \in N(r^{ix}) \subseteq N[S]$.

Subcase 1.2 Suppose $h = b_m + g$, for some integers m, g with $1 \le m \le t$ and $1 \le g \le d-1$. In this case, $v \cdot sr^{b_m} = r^{ix+g}$, which means that $v \in N(r^{ix+g}) \subseteq N[S]$.

Case 2. Suppose $0 \le i \le l-1$ and $b_t - b_1 + d \le j \le b_t - b_1 + d + a_k - 1$. In this case, there exists an integer h with $1 \le h \le a_k$ such that $v \cdot r^h = sr^{n-(a_k+d-b_1+ix)}$.

Subcase 2.1 If $h \in \Omega_1$ then $v \in N\left(sr^{n-(a_k+d-b_1+ix)}\right) \subseteq N[S]$.

Subcase 2.2 Suppose $h = a_m - g$, for some integers m, g with $1 \le m \le k$ and $1 \le g \le d-1$. In this case, $v \cdot r^{a_m} = sr^{n-(a_k+d-b_1+ix+g)}$, which means that $v \in sr^{n-(a_k+d-b_1+ix+g)} \subseteq N[S]$.

Case 3. Suppose $0 \le i \le l-1$ and $b_t - b_1 + d + a_k \le j \le b_t - b_1 + 2d + 2a_k - 1$. In this case, there exists an integer h with $0 \le h \le a_k + d - 1$ such that $v \cdot r^h = sr^{n-(a_k+d-b_1+xi)}$.

Subcase 3.1 If $0 \le h < a_1$, then by the definition of $d, v \in S \subseteq N[S]$.

Subcase 3.2 Suppose $h = a_m + g$, for some integers m, g with $1 \le m \le k$ and $0 \le g \le d-1$. In this case, $v \cdot r^{a_m} = sr^{n-(a_k+d-b_1+ix+g)}$, which means that $v \in sr^{n-(a_k+d-b_1+ix+g)} \subseteq N[S]$.

Thus S is a dominating set of G.

The following lemma provides an upper bound for the total domination number in $G = Cay(D_{2n}, \Omega)$.

Lemma 5 *Let* $n \ge 3$ *be an integer,* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$ *and* k, t *be integers such that* $1 \le k \le m, 1 \le t \le n$. *Let*

$\Omega = \left\{r^{a_1}, r^{a_2}, \cdots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \cdots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \cdots, sr^{b_t}\right\}$,

and $G = Cay(D_{2n}, \Omega)$. *If* $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \le i \le k$, $d'_1 = b_{11}, d'_j = b_j - b_{j-1}$ for $2 \le j \le t$ and $d = \max_{1\le i\le k, 1\le j\le t}\{d_i, d'_j\}$, *then* $\gamma_t(G) \le 2d \frac{n}{d+2a_k}$.

Proof. Let $x = d + 2a_k$ and $l = \left\lceil \frac{n}{x} \right\rceil$. Consider the set

$$S_t = \left\{r^{ix+g}, sr^{n-(ix+g-b_1)} : 0 \le i \le l- \text{ and } 0 \le g \le d-1\right\}.$$

Clearly $|S_t| = 2dl$. We have to prove that $V(G) \subseteq N(S_t)$ If $v \in V(G)$, then we can write v as either one vertex of the form $v = r^c$ or $v = sr^{n-(c-b_1)}$, where $0 \le c \le n-1$. By the division algorithm, $c = xi + j$, where $0 \le i \le l-1$ and $0 \le j \le x-1$. We have the following cases:

Case 1. Suppose $0 \le i \le l-1$ and $0 \le j < a_1$. For some integer g with $0 \le g \le d-1$ and by the definition of d, if $v = r^c$, then $v \in N\left(sr^{n-(ix+g-b_1)}\right) \subseteq N(S_t)$ or if $v = sr^{n-(c-b_1)}$, then $v \in N(r^{ix+g}) \subseteq N(S_t)$.

Case 2. Suppose $0 \le i \le l-1$ and $a_1 \le j < a_k + d - 1$. We can write $j = a_m + g$, for some integers m, g with $1 \le m \le k$ and $0 \le g \le d-1$. If $v = r^c$, then $v = r^{ix+g+a_m}$ whereas $r^{ix+g} \in S_t$ and so $v \in N(r^{ix+g}) \subseteq N(S_t)$ or if $v = sr^{n-(c-b_1)}$, then $v = sr^{n-(ix+g+a_m-b_1)}$ whereas $sr^{n-(ix+g-b_1)} \in S_t$ and so $v \in N\left(sr^{n-(ix+g-b_1)}\right) \subseteq N(S_t)$.

Case 3. Suppose $0 \le i \le l-2$ and $d + a_k \le j < d + 2a_k$. In this case, there exists an integer h with $1 \le h \le a_k$ such that $v.r^h = r^{(i+1)x}$ or $v \cdot r^h = sr^{n-((i+1)x-b_1)}$.

Subcase 3.1 Suppose $h \in \Omega_1 = \{a_1, a_2, \cdots, a_k\}$ and if $v = r^c$, then $v \in \left(r^{(i+1)x}\right) \subseteq N(S_t)$ or if $v = sr^{n-(c-b_1)}$, then $v \in N\left(sr^{n-((i+1)x-b_1)}\right) \subseteq N(S_t)$.

Subcase 3.2 Suppose $h = a_m - g$, for some integers m, g with $1 \le m \le k$ and $1 \le g \le d-1$. In this case, if $v = r^c$, then $v \cdot r^{a_m} = r^{(i+1)x+g}$, which means that $v \in N\left(r^{(i+1)x+g}\right) \subseteq N(S_t)$ or if $v = sr^{n-(c-b_1)}$, then $v \cdot r^{a_m} = sr^{n-((i+1)x+g-b_1)}$, which implies that $v \in N\left(sr^{n-((i+1)x+g-b_1)}\right) \subseteq N(S_t)$.

Case 4. Suppose $i = l-1$ and $d + a_k \le j < d + 2a_k$. Then there exists an integer h with $1 \le h \le a_k$ such that $v \cdot r^h = r^0$ or $v \cdot r^h = sr^{n-b_1}$.

Subcase 4.1 When $h \in \Omega_1$, and if $v = r^c$, then $v \in N(r^0) \subseteq N(S_t)$ or if $v = sr^{n-(c-b_1)}$, then $v \in N(sr^{n-b_1}) \subseteq N(S_t)$.

Subcase 4.2 Suppose $h = a_m - g$, for some integers m, g with $1 \le m \le k$ and $1 \le g \le d-1$. In this case, if $v = r^c$ and $v \cdot r^{a_m} = r^g$, which means that $v \in N(r^g) \subseteq N(S_t)$ or if $v = sr^{n-(c-b_1)}$, then $v \cdot r^{a_m} = sr^{n-(g-b_1)}$, which means that $v \in N\left(sr^{n-(g-b_1)}\right) \subseteq N(S_t)$.

Thus S_t is a total dominating set of G.

$\gamma_t \le |S_t| = 2dl$.

Now we obtain an upper bound for the connected domination number.

Lemma 6 *Let* $n \ge 3$ *be an integer,* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$ *and* *k, t be integers such that* $1 \le k \le m, 1 \le t \le n$ *. Let*
$\Omega =$
$\left\{ r^{a_1}, r^{a_2}, \cdots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \cdots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \cdots, sr^{b_t} \right\}$,
and $G = Cay(D_{2n}, \Omega)$ *. If* $d_1 = a_1 = 1, d_i = a_i - a_{i-1}$ for $2 \le i \le k$, $d_1' = b_{11}, d_j' = b_j - b_{j-1}$ for $2 \le j \le t$ and $d = \max_{1 \le i \le k, 1 \le j \le t} \{d_i, d_j'\}$, *then* $\gamma_c(G) \le 2d \frac{n}{a_{k+d-1}}$.

Proof. Let $x = a_k + d - 1$ and $l = \left\lceil \frac{n}{x} \right\rceil$. Consider the set

$$S_t = \left\{ r^{ix+g}, sr^{n-(ix+g-b_1)} : 0 \le i \le l-1 \text{ and } 0 \le g \le d-1 \right\}.$$

In the notation of Lemma 5, $a_1 = 1$ and $x = a_k + d - 1$ and S_c is a total dominating set. Since $r \in \Omega$ and for each i with $0 \le i \le l-1$, we have paths $r^{ix}, r^{ix+1}, \cdots, r^{ix+(d-1)}$ and $sr^{n-(ix-b_1)}, sr^{n-(ix+1-b_1)}, \cdots, sr^{n-(ix+d-1-b_1)}$. Also note that $r^{ix+(d-1)}$ and $r^{ix+(d-1)+a_k} = r^{(i+1)x}$, $sr^{n-(ix+d-1-b_1)}$ and $sr^{n-(ix+d-1+a_k-b_1)} = sr^{n-((i+1)x-b_1)}$ are connected. Hence the induced subgraph $\langle S_c \rangle$ is connected.

3. Subgroups as Efficient Domination Sets

In this section, we obtain some E-sets in $G = Cay(D_{2n}, \Omega)$. Moreover we have identified certain subgroups of D_{2n} which are also efficient domination sets in G.

Theorem 7 *Let* $n \ge 3$ *be an integer,* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$ *and* *k, t be integers such that* $1 \le k \le m$, $1 \le t \le n$ *and d is an integer such that* $d(2k+t+1)$ *divides n. Let*

$$\Omega = \left\{ r^d, r^2 d, \cdots, r^k d, r^{(n-kd)}, r^{n-(k-1)d}, \cdots, r^{(n-d)}, sr^d, sr^2 d, \cdots, sr^t d \right\}$$

and $G = Cay(D_{2n}, \Omega)$. *Then* $\gamma(G) = \frac{2n}{2k+t+1}$. *In this case,* G *has an E-set.*

Proof. Let $l = \frac{2n}{d(2k+t+1)}$ and $x = d(2k+t+1)$. In the notation of Lemma 4, d_i's and d_i''s are same, $a_i = id$ for all $1 \le i \le k$ and $b_j = jd$ for all $1 \le j \le t$. Let $x = d(2k+t+1)$ and $l = \left\lceil \frac{n}{x} \right\rceil$. By Lemma 4,

$$S = \left\{ r^{ix+g}, sr^{n-(kd+ix+g)} : 0 \le i \le l-1, 0 \le g \le d-1 \right\}$$

is a dominating set and hence $\gamma(G) \le \frac{2n}{2k+t+1}$. Since G is $2k+t$ regular, by Theorem 1, one can conclude that S is an E-set in G.

Remark 8 Note that Theorem 3 identifies all subgroups of the dihedral group D_{2n}. Now we us identify some of the subgroups as efficient dominating sets.

Theorem 9 *Let* $n \ge 3$ *be an integer,* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$ *and* *k, t be integers such that* $1 \le k \le m$, $1 \le t \le n$ *and* $2k+t+1$ *divides* n. *Let* $H = \langle r^a, sr^{n-b} \rangle$ be a subgroup of the dihedral group D_{2n}, *where* $a = 2k+t+1$ *and* b, $0 \le b \le k-1$ *Then, there exists a generating set* Ω *of* D_{2n} *such that H is an efficient dominating set for the Cayley graph* $G = Cay(D_{2n}, \Omega)$.

Proof. Let

$$\Omega = \left\{ r, r^2, \cdots, r^k, r^{n-k}, r^{n-(k-1)}, \cdots, r^{n-1}, sr, sr^2, \cdots, sr^t \right\},$$

$l = \frac{n}{2k+t+1}$ and $x = 2k+t+1$. By taking $d = 1$ in Theorem 7,

$$S = \left\{ r^0, r^x, \cdots, r^{(l-1)x}, sr^{n-k}, sr^{n-(k+x)}, \cdots, sr^{n-(k+(l-1)x)} \right\}$$

is an efficient dominating set of G.

Remark 10 Under the assumptions of Theorem 9, $S.x$ is an efficient dominating set for the Cayley graph $G = Cay(D_{2n}, \Omega)$ for all $x \in D_{2n}$.

4. E-Chains in Cayley Graphs

Theorem 7 and 9 provide a tool to produce E-sets and visualize some of the subgroups as E-sets in $Cay(D_{2n}, \Omega)$. We use this tool to obtain an inclusive E-chain and inductive subgroups E-chain of Cayley graphs on the dihedral group.

Theorem 11 *Let* $n \ge 3$ *be an integer,* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$ *and k be an integers such that* $1 \le k \le m$,
$G_0 = Cay(D_{2n}, D_{2n} - \{e\})$,
$\Omega_i = \left\{ r, r^2, \cdots, r^k, r^{n-kd}, r^{n-(k-1)}, \cdots, r^{n-1}, sr, sr^2, \cdots, sr^{n-b_i} \right\}$
and $G_i = Cay(D_{2n}, \Omega_i)$ $(i \ge 1)$ *Assume that* $|\Omega_i| + 1$ *divides* n *and* $|\Omega_{i+1}| + 1$ *divides* $|\Omega_i| + 1$. *Then the finite family of graphs* $\mathbb{G} = \{G_i, i \ge 0\}$ *is inductive subgroups E-chain.*

Proof. Let $\lambda_i = |\Omega_i| + 1$. By the assumption λ_{i+1}. divides λ_i. Define the map $\zeta_i : V(G_i) \to V(G_{i+1})$ by $\zeta_i(v) = v$ for all $v \in G_i$. By Theorem 9, G_i has an efficient dominating set and it is of the form

$$S_i = \left\{ r^0 = e, r^{\lambda_i}, r^{2\lambda_i}, \cdots, r^{\left(\frac{n}{\lambda_i} - 1\right)\lambda_i}, sr^{n-(k+\lambda_i)}, sr^{n-k}, sr^{n-(k+2\lambda_i)}, \cdots, sr^{n-\left(k+\left(\frac{n}{\lambda_i}-1\right)\lambda_i\right)} \right\}$$

and also S_i's are subgroups. It implies that $\zeta_i(S_i) \subseteq S_{i+1}$ for every $i \geq 1$. Hence the family of graphs $\mathbb{G} = \{G_i, i \geq 0\}$ is inductive subgroups E-chain.

The construction of an inclusive E-chain of Cayley graphs is based on the following lemma.

Lemma 12 *Let* $n \geq 3$ *be an integer,* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$, *k, t be integers such that* $1 \leq k \leq m$, $1 \leq t \leq n$ *and d is an integer such that* $d(2k+t+1)$ *divides* n. *For* $i \geq 1$, *let*

$$\Omega_i = \left\{ r^d, r^2 d, \cdots, r^k d, r^{2^i n-d}, r^{2^i n-2d}, \cdots, r^{2^i n-kd}, sr^d, sr^2 d, \cdots, sr^t d \right\}$$

and $G_i = Cay\left(D_{2^i n}, \Omega_i\right)$. Then G_{i+1} is a covering of G_i.

Proof. Define the surjective map $f_i : V(G_{i+1}) \to V(G_i)$ by $f_i(r^j) = r^{j \bmod 2^i n}$ and $f_i(sr^j) = sr^{j \bmod 2^i n}$ for all *j*, where $0 \leq j \leq 2^{i+1}(n-1)$. Note that f_i is a group homomorphism from $D_{2^{i+1} n}$ onto $D_{2^i n}$. Let $\tilde{u}, \tilde{v} \in G_{i+1}$. Suppose $\tilde{u}$ and $\tilde{v}$ are adjacent in G_{i+1}. Then, there exists r^k with $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ or sr^t with $1 \leq t \leq n-1$ such that $\tilde{u} = \tilde{v}.r^k$ or $\tilde{u} = \tilde{v}.sr^k$. Since f_i is a group homomorphism and $f_i(r^k) = r^{k \bmod 2^i n} = r^k$, $f_i(sr^t) = sr^{t \bmod 2^i n} = sr^t$, we have $f_i(\tilde{u}) = f_i(\tilde{v}) \cdot r^k$ or $f_i(\tilde{u}) = f_i(\tilde{v}) \cdot sr^t$ and so $f_i(\tilde{u})$ and $f_i(\tilde{v})$ are adjacent in G_i. Consider the map $f_i/_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ for any vertex $\tilde{v} \in V(G_{i+1})$ and $v \in V(G_i)$. Claim $f_i/_{N(\tilde{v})}$ is bijection. Any element x in $N(\tilde{v})$ as either one vertex of the form $x = r^e$ or $x = sr^e$, where $0 \leq j \leq 2^{i+1}(n-1)$. Let $x, y \in N(\tilde{v})$. Then we have following three cases:

Case 1. Let $x = r^{e_1}$ and $y = r^{e_2}$ with $e_1 \neq e_2$. Suppose $f_i(x) = f_i(y)$, *i.e.* $r^{e_1 \bmod 2^i n} = r^{e_2 \bmod 2^i n} \Rightarrow r^{(e_1 - e_2) \bmod 2^i n} = e$. *i.e.* $o(r) = (e_1 - e_2) mod 2^i n < n$, which is a contradiction to $o(r) = n$. Therefore $f_i(x) \neq f_i(y)$.

Case 2. Let $x = r^{e_1}$ and $y = sr^{e_2}$. Suppose $f_i(x) = f_i(y)$, *i.e.* $r^{e_1 \bmod 2^i n} = sr^{e_2 \bmod 2^i n}$ This means $r^{(e_1 - e_2) \bmod 2^i n} = e$ or $s = sr^{(e_1 - e_2) \bmod 2^i n} = e$, which is a contradiction. Therefore $f_i(x) \neq f_i(y)$.

Case 3. Let $x = sr^{e_1}$ and $y = sr^{e_2}$ with $e_1 \neq e_2$. Suppose $f_i(x) = f_i(y)$, *i.e.* $sr^{e_1 \bmod 2^i n} = sr^{e_2 \bmod 2^i n} \Rightarrow r^{(e_1 - e_2) \bmod 2^i n} = e$. *i.e.* $o(r) = (e_1 - e_2) \bmod 2^i n < n$ which is a contradiction. Therefore $f_i(x) \neq f_i(y)$. Hence distinct elements of $N(\tilde{v})$ are distinctly mapped onto $N(v)$ and so $f_i/_{N(\tilde{v})}$ is a required bijection.

Theorem 13 *Let* $n \geq 3$ *be an integer,* $m = \left\lfloor \frac{n-1}{2} \right\rfloor$, *k, t, be integers such that* $1 \leq k \leq m$, $1 \leq t \leq n$ *and d is an integer such that* $d(2k+t+1)$ *divides* n. *For* $i \geq 1$ *let*

$$\Omega_i = \left\{ r^d, r^2 d, \cdots, r^k d, r^{2^i n-d}, r^{2^i n-2d}, \cdots, r^{2^i n-kd}, sr^d, sr^2 d, \cdots, sr^t d \right\}$$

and $G_i = Cay\left(D_{2^i n}, \Omega_i\right)$. *Let* S_i *be an efficient dominating set for* G_i. *Then the finite family of graphs* $\mathbb{G} = \{G_i, i \geq 1\}$ *is an inclusive E-chain.*

Proof. Since by above Lemma, G_{i+1} is a covering of G_i, $(i \geq 1)$. Since by Theorem 2, $f_i^{-1}(S_i) \subset S_{i+1}$. Hence the finite family of graphs $\mathbb{G} = \{G_i, i \geq 1\}$ is an inclusive E-chain.

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