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The Equilibrium Distribution of Counting Random Variables

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Abstract

We study the high order equilibrium distributions of a counting random variable. Properties such as moments, the probability generating function, the stop—loss transform and the mean residual lifetime, are derived. Expressions are obtained for higher order equilibrium distribution functions under mixtures and convolutions of a counting distribution. Recursive formulas for higher order equilibrium distribution functions of the $(a,b,0)$-family of distributions are given.

Keywords: Counting Random Variable, Equilibrium Distribution, Stop-Loss Transform, Mean Residual Life, $(a,b,0)$ Family, Recursive Formulas, Probability Generating Function

1. Introduction

Recently, there has been much attention given to higher order equilibrium distributions associated with a given distribution function (d.f.), see e.g., Fagiuoli and Pellerey [1,2], Nanda, Jain and Singh [3], Hesselager, Wang and Willmot [4] and the references therein. Equilibrium distributions arise naturally in ruin theory and play an important in various settings.

The first order equilibrium distribution of a claim size d.f., in classical risk theory, can be interpreted as the distribution of the amount of the first drop below the initial reserve, given there is such a drop (see for instance Bowers *et al.* [5], Chapter 12). Many results on the moments of the time to ruin, the surplus before ruin and the deficit at ruin, heavily depend on the equilibrium distribution of the claim size d.f. [see Lin and Willmot [6,7] for details].

Some classifications of reliability distributions are based on properties of higher order equilibrium distributions. Whence, bounds for the right tail of the total claims distribution and ruin probabilities, can be obtained from the properties of equilibrium distributions associated with the single claim size d.f., see [7-9].

Although much attention has been paid to the equilibrium distributions associated with a given d.f., most results are for continuous random variables. Instead, we discuss higher order equilibrium distributions associated with a discrete probability function (p.f.). Throughout the paper, $\mathbb{N}=\{0,1,2,\cdots\}$ and $\mathbb{N}^+=\{1,2,\cdots\}$.

2. Notation and Definitions

Let X be a non-negative r.v. taking integer values, with probability function (p.f.) $p(x)=P(X=x)$, survival function $\bar{P}(x)=P(X>x)=\sum_{y\ge x+1}p(y)$, $x\in\mathbb{N}$ and n-th moment $\mu_n=E\left[X^n\right]$.

Consider the *equilibrium distribution* of *p*, defined as

$$p_1(x):=\frac{\bar{P}(x)}{\mu_1}=\frac{1}{\mu_1}\sum_{y\ge x+1}p(y),\quad x\in\mathbb{N}.$$

Now, define $\mu_{(n)}:=E\left[X^{(n)}\right]$ to be the *n*-th factorial moment of *X*, where $x^{(n)}=x(x-1)\cdots(x-n+1)$ denotes the *n*-th factorial power of *x* and $x^{(0)}=1$. It is well known in summation calculus (see e.g. Hamming [10], p.182) that $\sum_{k=x}^{y}k^{(n)}=\frac{(y+1)^{(n+1)}-x^{(n+1)}}{n+1}$, for $n\in\mathbb{N}^+$, $x,y\in\mathbb{N}$, and $x\le y$. Hence the *n*-th factorial moment $\mu_{1:(n)}$ of p_1 is given by

$$\mu_{1:(n)}=\sum_{x\ge1}x^{(n)}p_1(x)=\frac{1}{\mu_1}\sum_{x\ge1}x^{(n)}\sum_{y\ge x+1}p(y)$$
$$=\frac{1}{\mu_1}\sum_{y\ge2}p(y)\sum_{x=1}^{y-1}x^{(n)},\quad n\ge1,$$

$$= \frac{1}{\mu_1}\sum_{y\ge 2} p(y)\frac{y^{(n+1)}}{n+1} = \frac{\mu_{(n+1)}}{\mu_1(n+1)}$$
$$= \frac{\mu_{(n+1)}}{\mu_{(1)}(n+1)}, \quad \text{for } \mu_1 = \mu_{(1)}. \tag{1}$$

Similarly, the probability generating function (p.g.f.) of the equilibrium distribution p_1 is given by

$$\hat{p}_1(s) = \sum_{x\ge 0} s^x p_1(x) = \frac{1-\hat{p}(s)}{\mu_1(1-s)}, \quad -1 \le s \le 1,$$

with $\hat{p}_1(1) = 1$ and its survival function is

$$\bar{P}_1(x) = \sum_{y\ge x+1} p_1(y) = \frac{1}{\mu_1}\sum_{y\ge x+1}\sum_{k\ge y+1} p(k)$$
$$= \frac{1}{\mu_1}\sum_{k\ge x+1} p(k)\left[k-(x+1)\right]. \tag{2}$$

Now define the equilibrium distribution of p_1, or equivalently, the *second order equilibrium distribution* of p:

$$p_2(x) = \frac{\bar{P}_1(x)}{\mu_{1:1}}$$
$$= \frac{1}{\mu_{1:1}\mu_1}\sum_{y\ge x+1}\left[y-(x+1)\right]p(y), \; x\in\mathbb{N},$$

where $\mu_{1:1}$ is the first order moment of p_1. The factorial moments of p_2 are obtained as in (1) to be

$$\mu_{2:(n)} = \frac{\mu_{1:(n+1)}}{\mu_{1:1}(n+1)} = \frac{\mu_{1:(n+1)}}{\mu_{1:(1)}(n+1)}.$$

Then the p.g.f. of $p_2(x)$ is given by

$$\hat{p}_2(s) = \sum_{x\ge 0} s^x p_2(x) = \frac{1-\hat{p}_1(s)}{\mu_{1:1}(1-s)}, \quad -1 < s < 1,$$

with $\hat{p}_2(1) = 1$, and the corresponding survival function $\bar{P}_2(x) = \sum_{k\ge x+1} p_2(k)$.

Define similarly the subsequent equilibrium distributions of p, from the third order $p_3(x) = 1/\mu_{2:1}\bar{P}_2(x)$ up to the n-th order $p_n(x) = 1/(\mu_{n-1:1})\bar{P}_{n-1}(x)$ for $x\in\mathbb{N}$, where the following theorem gives an expression for $\bar{P}_n(x)$ and $p_{n+1}(x)$.

Theorem 1 The survival function $\bar{P}_n(x)$, of the n-th order equilibrium distribution p_n can be expressed as

$$\bar{P}_n(x) = \frac{1}{n!\prod_{l=0}^{n-1}\mu_{l:1}}\sum_{y\ge x+1} p(y)\left[y-(x+1)\right]^{(n)}, \quad x\in\mathbb{N}, \tag{3}$$

$$= \frac{1}{\mu_{(n)}}\sum_{y\ge x+1} p(y)\left[y-(x+1)\right]^{(n)}, \tag{4}$$

and accordingly

$$p_{n+1}(x) = \frac{(n+1)}{\mu_{(n+1)}}\sum_{y\ge x+1} p(y)\left[y-(x+1)\right]^{(n)}, \tag{5}$$

where $\mu_{l:1}$ is the mean of l-th order equilibrium distribution and $\mu_{0:1} = \mu_1$ is the mean of p (or 0-th order equilibrium distribution).

Proof: (2) shows that (3) holds for $n=1$. By induction, assume that (3) holds for any n, then

$$\bar{P}_{n+1}(x) = \sum_{t\ge x+1} p_{n+1}(t) = \sum_{t\ge x+1}\frac{1}{\mu_{n:1}}\bar{P}_n(t)$$
$$= \frac{1}{n!\prod_{l=0}^{n}\mu_{l:1}}\sum_{y\ge x+2} p(y)\sum_{t=x+1}^{y-1}\left[y-(t+1)\right]^{(n)}$$
$$= \frac{1}{n!\prod_{l=0}^{n}\mu_{l:1}}\sum_{y\ge x+2} p(y)\sum_{k=0}^{y-(x+2)} k^{(n)}$$
$$= \frac{1}{(n+1)!\prod_{l=0}^{n}\mu_{l:1}}\sum_{y\ge x+1} p(y)\left[y-(x+1)\right]^{(n+1)},$$

verifies (3) also for $n+1$. Further, since $\bar{P}_n(-1) = 1$, we conclude from (3) that

$$n!\prod_{l=0}^{n-1}\mu_{l:1} = \mu_{(n)}, \quad n\in\mathbb{N}^+. \tag{6}$$

Hence $\bar{P}_n(x)$ is also given by (4). To prove (5), use $p_{n+1}(x) = \frac{\bar{P}_n(x)}{\mu_{n:1}}$ and (6). □

Example 1: If X is geometrically distributed with $p(x) = (1-\theta)\theta^x$ and survival function $\bar{P}(x) = \theta^{x+1}$, for $x\in\mathbb{N}$ and $\theta\in(0,1)$ then

$$\bar{P}_m(x) = \frac{1}{\mu_{(m)}}\sum_{y\ge x+1}(1-\theta)\theta^y\left[y-(x+1)\right]^{(m)}$$
$$= \frac{1}{\mu_{(m)}}(1-\theta)\theta^{x+1}\sum_{y\ge 0}\theta^y y^{(m)} = \theta^{x+1},$$

where the last equality holds true as

$$\mu_{(m)} = \sum_{y\ge 0}(1-\theta)\theta^y y^{(m)},$$

by definition.

This shows that any order equilibrium distribution of the geometric distribution is identical to the original distribution.

Example 2: Let X be a discrete uniform with

$$p(x) = \frac{1}{m+1}, \quad x = 0,1,2,\cdots,m.$$

As $\mu_{(n)} = \sum_{x=0}^{m}\frac{x^{(n)}}{(m+1)} = \frac{(m+1)^{(n+1)}}{(m+1)(n+1)}$, then for $n \le m$,

$$
\begin{aligned}
p_n(x) &= \frac{n}{\mu_{(n)}} \sum_{y=x+1}^{m} \frac{1}{(m+1)} \left[y-(x+1)\right]^{(n-1)} \\
&= \frac{n(n+1)}{(m+1)^{(n+1)}} \sum_{y=0}^{m-(x+1)} y^{(n-1)} \\
&= \frac{(n+1)(m-x)^{(n)}}{(m+1)^{(n+1)}}, \quad 0 \le x \le m-n,
\end{aligned}
$$

while for $n > m$, $p_n(x) \equiv 0$.

3. Properties of the Equilibrium Distribution

In deriving the properties of the higher order equilibrium distributions of p, the following lemmas will be needed.

Lemma 1 The relationships between raw and factorial moments are given by

$$
\mu_{(n)} = \sum_{k=1}^{n} S_k^n \mu_k, \text{ and } \mu_n = \sum_{k=1}^{n} s_k^n \mu_{(k)}, \quad n \in \mathbb{N}^+, \quad (7)
$$

where S_k^n, s_k^n, $k = 1, 2, \cdots, n$, called the first and the second Stirling numbers respectively, are given recursively by

$$
S_{n+1}^{n+1} = S_n^n = \cdots = S_1^1 = 1, \; S_k^{n+1} = S_{k-1}^n - nS_k^n,
$$

with $S_0^n = 0$,

$$
s_{n+1}^{n+1} = s_n^n = \cdots = s_1^1 = 1, \; s_k^{n+1} = s_{k-1}^n + ks_k^n,
$$

with $s_0^n = 0$ and $1 \le k \le n$.

Proof: See p.160 in [10]. □

Lemma 2 For $n \in \mathbb{N}$ and $y \in \mathbb{N}^+$,

$$
\sum_{x=0}^{y-1} x^{(n)} s^x = n! \frac{s^n (1-s^y)}{(1-s)^{n+1}} - \sum_{k=1}^{n} \frac{n!}{k!} \frac{s^{n-k}}{(1-s)^{n-k+1}} y^{(k)} s^y, \quad (8)
$$

$$
\sum_{x \ge 0} x^{(n)} s^x = n! \frac{s^n}{(1-s)^{n+1}}, \quad s \in (0,1). \quad (9)
$$

Proof: Let $I_n = \sum_{x=0}^{y-1} x^{(n)} s^x$. It is easy to show that $I_n = n\frac{s}{1-s} I_{n-1} - \frac{1}{1-s} y^{(n)} s^y$, while $I_0 = \frac{1-s^y}{1-s}$. Then (8) is verified by mathematical induction. To prove (9), simply let $y \to \infty$ in (8).

Lemma 3 For y, m and $n \in \mathbb{N}$,

$$
\sum_{x=0}^{y} x^{(m)} (y-x)^{(n)} = \frac{m!n!}{(m+n+1)!} (y+1)^{(n+m+1)}. \quad (10)
$$

Proof: Since $\sum_{x=0}^{y} x^{(m)} = \frac{(y+1)^{m+1}}{(m+1)}$, then (10) holds when $n = 0$ and $m \in \mathbb{N}$. Assuming that it also holds for an arbitrary $n = k$ and $m \in \mathbb{N}$, then for $n = k+1$ the left-hand-side (LHS) of (10) becomes

$$
\begin{aligned}
\text{LHS} &= \sum_{x=0}^{y} x^{(m)} (y-x)^{(k)} (y-x-k) \\
&= y\sum_{x=0}^{y} x^{(m)} (y-x)^{(k)} - \sum_{x=0}^{y} (x-m) x^{(m)} (y-x)^{(k)} \\
&\quad -(m+k)\sum_{x=0}^{y} x^{(m)} (y-x)^{(k)} \\
&= \frac{y(y+1)^{(m+k+1)} m!k!}{(m+k+1)!} - \frac{(y+1)^{(m+k+2)} k!(m+1)!}{(m+k+2)!} \\
&\quad - \frac{(m+k)(y+1)^{(m+k+1)} k!m!}{(m+k+1)!} \\
&= (y+1)^{(m+k+1)} \left[y-(m+k)\right] \frac{m!k!}{(m+k+1)!} \\
&\quad - \frac{k!(m+1)!}{(m+k+2)!} (y+1)^{(m+k+2)} \\
&= \frac{m!(k+1)!}{(m+k+2)!} (y+1)^{(m+k+2)}.
\end{aligned}
$$

□

Remark: (10) is a discrete version of the formula

$$
\begin{aligned}
\int_0^y x^m (y-x)^n \, \mathrm{d}x &= y^{m+n+1} \beta(m+1, n+1) \\
&= y^{n+m+1} \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)}.
\end{aligned}
$$

Let $\hat{p}_{n+1}(s) = \sum_{x \ge 0} s^x p_{n+1}(x)$ be the p.g.f. of p_{n+1}. The following theorem gives an expression for $\hat{p}_{n+1}(s)$.

Theorem 2

$$
\begin{aligned}
\hat{p}_{n+1}(s) &= \frac{(-1)^n (n+1)!}{\mu_{(n+1)}} \frac{\left[1-\hat{p}(s)\right]}{(1-s)^{n+1}} \\
&\quad + \frac{(n+1)!}{\mu_{(n+1)}} \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k!} \frac{\mu_{(k)}}{(1-s)^{n-k+1}}.
\end{aligned}
\quad (11)
$$

Proof: Since

$$
p_{n+1}(x) = \frac{n+1}{\mu_{(n+1)}} \sum_{y \ge x+1} p(y) \left[y-(x+1)\right]^{(n)},
$$

$$
\begin{aligned}
\hat{p}_{n+1}(s) &= \sum_{x \ge 0} s^x p_{n+1}(x) \\
&= \frac{n+1}{\mu_{(n+1)}} \sum_{x \ge 0} s^x \sum_{y \ge x+1} p(y) \left[y-(x+1)\right]^{(n)} \\
&= \frac{n+1}{\mu_{(n+1)}} \sum_{y \ge 1} p(y) \sum_{x=0}^{y-1} s^x \left[y-(x+1)\right]^{(n)}
\end{aligned}
$$

$$=\frac{n+1}{\mu_{(n+1)}}\sum_{y\ge 1}p(y)s^{y-1}\sum_{x=0}^{y-1}s^{-x}x^{(n)}$$

$$=\frac{n+1}{\mu_{(n+1)}}\sum_{y\ge 1}p(y)s^{y-1}\left[(-1)^n n!\frac{(1-s^y)}{(1-s)^{n+1}}\left(\frac{1}{s}\right)^{y-1}\right.$$

$$\left.+\sum_{k=1}^{n}\frac{(-1)^{n-k}n!}{k!}\frac{y^{(k)}}{(1-s)^{n-k+1}}\left(\frac{1}{s}\right)^{y-1}\right],\text{by Lemma 2},$$

$$=\frac{(-1)^n(n+1)!}{\mu_{(n+1)}}\frac{[1-\hat{p}(s)]}{(1-s)^{n+1}}$$

$$+\frac{(n+1)!}{\mu_{(n+1)}}\sum_{k=1}^{n}\frac{(-1)^{n-k}}{k!}\frac{\mu_{(k)}}{(1-s)^{n-k+1}}.$$

□

Theorem 3

$$\mu_{n:(m)}=\frac{n!m!}{(m+n)!}\frac{\mu_{(n+m)}}{\mu_{(n)}},\quad m,n\in\mathbb{N},\tag{12}$$

$$\mu_{n:m}=\frac{n!}{\mu_{(n)}}\sum_{k=1}^{m}\frac{k!s_k^m}{(n+k)!}\mu_{(n+k)},\quad m\in\mathbb{N}^+,n\in\mathbb{N},\tag{13}$$

where $\mu_{n:m}$ is the m-th moment of the distribution p_n.

Proof:

$$\mu_{n:(m)}=\sum_{x\ge 0}x^{(m)}p_n(x)$$

$$=\frac{n}{\mu_{(n)}}\sum_{y\ge 1}p(y)\sum_{x=0}^{y-1}x^{(m)}\left[y-(x+1)\right]^{(n-1)}$$

$$=\frac{n}{\mu_{(n)}}\sum_{y\ge 1}p(y)y^{(n+m)}\frac{m!(n-1)!}{(n+m)!},\text{by Lemma 3},$$

$$=\frac{n!m!}{(m+n)!}\frac{\mu_{(n+m)}}{\mu_{(n)}}.$$

To prove (13), use $\mu_{n:m}=\sum_{k=1}^{m}s_k^m\mu_{n:(k)}$, as stated in Lemma 1, and (12). □

Consider now the *stop-loss transform* $\pi(x)=E(X-x)_+$ of the r.v. X (where the notation $(a)_+=aI[a>0]$). For $n\in\mathbb{N}^+$ and $x\in\mathbb{N}$, denote by $E[X-x]_+^n$ the n-th stop-loss transform of X (with probability function p) and by $E[X-x]_+^{(n)}$ its n-th factorial stop-loss transform. Theorem 1 and Lemma 1 show that $E[X-x]_+^{(n)}=\mu_{(n)}\bar{P}_n(x-1)$ and

$$E[X-x]_+^n=\sum_{k=1}^{n}\mu_{(k)}s_k^n\bar{P}_k(x-1).$$

Let X_m be a random variable following the probability function p_m. Define $E[X_m-x]_+^{(n)}$ to be the n-th factorial stop-loss transform of p_m and $E[X_m-x]_+^n$ to be the n-th stop-loss transform of p_m. The following theorem holds.

Theorem 4 For $n\in\mathbb{N}^+$ **and** $m,x\in\mathbb{N}$,

$$\begin{aligned}E[X_m-x]_+^{(n)}&=\frac{m!n!}{(m+n)!}\frac{E[X-x]_+^{(n+m)}}{\mu_{(m)}}\\&=\frac{m!n!}{(m+n)!}\frac{\mu_{(n+m)}}{\mu_{(m)}}\bar{P}_{n+m}(x-1),\end{aligned}\tag{14}$$

$$\begin{aligned}E[X_m-x]_+^n&=\frac{m!}{\mu_{(m)}}\sum_{k=1}^{n}\frac{k!s_k^n}{(m+k)!}E[X-x]_+^{(m+k)}\\&=\frac{m!}{\mu_{(m)}}\sum_{k=1}^{n}\frac{k!s_k^n}{(m+k)!}\mu_{(m+k)}\bar{P}_{m+k}(x-1)\end{aligned}\tag{15}$$

Proof: The argument is similar to that in the above proof. □

Now define $r_m(x)=E[X_m-x|X_m>x]$, $x\in\mathbb{N}$, $m\in\mathbb{N}^+$, to be the mean residual lifetime (MRL) of p_m, and $h_m(x)=\dfrac{P(X_m=x)}{P(X_m\ge x)}$ to be the hazard rate function of p_m. Then the following result holds.

Theorem 5 For $x\in\mathbb{N}$ and $m\in\mathbb{N}^+$,

$$r_m(x)=1+\frac{\mu_{(m+1)}}{(m+1)\mu_{(m)}}\frac{\bar{P}_{m+1}(x)}{\bar{P}_m(x)}\tag{16}$$

$$h_m(x)=\frac{1}{r_{m-1}(x)}.\tag{17}$$

Proof:

$$r_m(x)=E[X_m-x|X_m>x]=\frac{\sum_{k\ge x+1}(k-x)p_m(k)}{\bar{P}_m(x)}$$

$$=\frac{\sum_{k\ge x+1}[k-(x+1)]p_m(k)+\sum_{k\ge x+1}p_m(k)}{\bar{P}_m(x)}$$

$$=1+\frac{E[X_m-(x+1)]^{(1)}}{\bar{P}_m(x)}$$

$$=1+\frac{\mu_{(m+1)}}{(m+1)\mu_{(m)}}\frac{\bar{P}_{m+1}(x)}{\bar{P}_m(x)},\quad\text{by (14).}$$

while

$$h_m(x)=\frac{p_m(x)}{p_m(x)+\bar{P}_m(x)}=\frac{1}{1+\dfrac{\bar{P}_m(x)}{p_m(x)}}$$

$$=\frac{1}{1+\dfrac{m\mu_{(m-1)}}{\mu_{(m)}}\dfrac{\bar{P}_m(x)}{\bar{P}_{m-1}(x)}}=\frac{1}{r_{m-1}(x)},\text{by (16)}.$$

This proves the conclusion. □

4. Equilibrium Distribution and Convolutions

This section studies the equilibrium distribution of the n-th fold convolution of a counting distribution.

The following lemma shows that the usual formulas

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

and

$$(x_1+x_2+\cdots+x_m)^n = \sum_{l_1+\cdots+l_m=n}\frac{n!}{l_1!\cdots l_m!}x_1^{l_1}\cdots x_m^{l_m},$$

for $n\in\mathbb{N}$, also hold for factorial integer powers.

Lemma 4 For $n\in\mathbb{N}$

$$(x+y)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}x^{(k)}y^{(n-k)}, \tag{18}$$

$$(x_1+x_2+\cdots+x_m)^{(n)} = \sum_{l_1+l_2+\cdots+l_m=n}\frac{n!}{l_1!l_2!\cdots l_m!}x_1^{(l_1)}x_2^{(l_2)}\cdots x_m^{(l_m)}. \tag{19}$$

Proof: Clearly (18) holds for $n=1$. Assume it holds for an arbitrary n, then

$$\begin{aligned}(x+y)^{(n+1)} &= (x+y)^{(n)}(x+y-n)\\ &= \sum_{k=0}^{n}\binom{n}{k}x^{(k)}y^{(n-k)}\left[(x-k)+(y-n+k)\right]\\ &= \sum_{k=0}^{n}\binom{n}{k}x^{(k+1)}y^{(n-k)}+\sum_{k=0}^{n}\binom{n}{k}x^{(k)}y^{(n+1-k)}\\ &= \sum_{k=1}^{n+1}\binom{n}{k-1}x^{(k)}y^{(n+1-k)}+\sum_{k=0}^{n}\binom{n}{k}x^{(k)}y^{(n+1-k)}\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}x^{(k)}y^{(n+1-k)}\end{aligned}$$

and (18) holds by induction. A similar argument proves (19).□

Next we will discuss the high order equilibrium distributions of the convolution of p with itself.

Let $p^{*n}(x)=\sum_{k=0}^{x}p(k)p^{*n-1}(x-k)$ be the n-th fold convolution of p, with $p^{*1}=p$, and p_m^{*n} be the m-th order equilibrium distribution of p^{*n}. Consider $\mu_{(k)}^{*n}$, the k-th factorial moment of p^{*n}, then

$$\begin{aligned}\mu_{(k)}^{*n} &= E\left[(X_1+\cdots+X_n)^{(k)}\right]\\ &= E\left[\sum_{l_1+\cdots+l_n=k}\frac{k!}{l_1!\cdots l_n!}X_1^{(l_1)}\cdots X_n^{(l_n)}\right]\\ &= \sum_{l_1+l_2+\cdots+l_n=k}\frac{k!}{l_1!l_2!\cdots l_n!}\mu_{(l_1)}\mu_{(l_2)}\cdots\mu_{(l_n)},\end{aligned}$$

where $X_1, X_2,\cdots,X_n$ are i.i.d. with common p.f. p, which can be computed recursively by

$$\begin{aligned}\mu_{(k)}^{*n} &= E\left\{\left[(X_1+X_2+\cdots+X_{n-1})+X_n\right]^{(k)}\right\}\\ &= \sum_{l=0}^{k}\binom{k}{l}E\left[X_1+X_2+\cdots+X_{n-1}\right]^{(l)}E\left[X_n^{(k-l)}\right]\\ &= \sum_{l=0}^{k}\binom{k}{l}\mu_{(l)}^{*(n-1)}\mu_{(k-l)}.\end{aligned}$$

The following Theorem gives an expression for $p_m^{*n}(x)$.

Theorem 6 For $m\in\mathbb{N}^+$, $n\in\{2,3,\cdots\}$ and $x\in\mathbb{N}$,

$$p_m^{*n}(x) = \frac{\mu_{(m)}^{*(n-1)}}{\mu_{(m)}^{*n}}p*p_m^{*(n-1)}(x) + \frac{1}{\mu_{(m)}^{*n}}\sum_{l=1}^{m}\binom{m}{l}\mu_{(l)}\mu_{(m-l)}^{*(n-1)}p_l(x). \tag{20}$$

Proof:

$$\begin{aligned}p_m^{*n}(x) &= \frac{m}{\mu_{(m)}^{*n}}\sum_{y\ge x+1}p^{*n}(y)\left[y-(x+1)\right]^{(m-1)}\\ &= \frac{m}{\mu_{(m)}^{*n}}\sum_{y\ge x+1}\sum_{k=0}^{y}p(k)p^{*(n-1)}(y-k)\left[y-(x+1)\right]^{(m-1)}\\ &= \frac{m}{\mu_{(m)}^{*n}}\sum_{k=0}^{x}p(k)\sum_{y\ge x+1}p^{*(n-1)}(y-k)\left[y-(x+1)\right]^{(m-1)}\\ &\quad+\frac{m}{\mu_{(m)}^{*n}}\sum_{k\ge x+1}p(k)\sum_{y\ge k}p^{*(n-1)}(y-k)\left[y-(x+1)\right]^{(m-1)}\\ &= \frac{m}{\mu_{(m)}^{*n}}\sum_{k\ge 0}p(k)\sum_{t\ge x-k+1}p^{*(n-1)}(t)\left[t-(x-k+1)\right]^{(m-1)}\\ &\quad+\frac{m}{\mu_{(m)}^{*n}}\sum_{k\ge x+1}p(k)\sum_{y\ge 0}p^{*(n-1)}(y)\left\{y+\left[k-(x+1)\right]\right\}^{(m-1)}\\ &= \frac{\mu_{(m)}^{*(n-1)}}{\mu_{(m)}^{*n}}\sum_{k=0}^{x}p(k)p_m^{*(n-1)}(x-k)+\frac{m}{\mu_{(m)}^{*n}}\sum_{k\ge x+1}p(k)\\ &\quad\times\sum_{y\ge 0}p^{*(n-1)}(y)\sum_{l=0}^{m-1}\binom{m-1}{l}\left[k-(x+1)\right]^{(l)}y^{(m-1-l)}\\ &= \frac{\mu_{(m)}^{*(n-1)}}{\mu_{(m)}^{*n}}p*p_m^{*(n-1)}(x)+\frac{m}{\mu_{(m)}^{*n}}\sum_{k\ge x+1}p(k)\\ &\quad\times\sum_{l=0}^{m-1}\binom{m-1}{l}\left[k-(x+1)\right]^{(l)}\mu_{(m-1-l)}^{*(n-1)}\\ &= \frac{\mu_{(m)}^{*(n-1)}}{\mu_{(m)}^{*n}}p*p_m^{*(n-1)}(x)+\frac{m}{\mu_{(m)}^{*n}}\sum_{l=0}^{m-1}\binom{m-1}{l}\mu_{(m-1-l)}^{*(n-1)}\\ &\quad\times\sum_{k\ge x+1}p(k)\left[k-(x+1)\right]^{(l)}\end{aligned}$$

$$= \frac{\mu_{(m)}^{*(n-1)}}{\mu_{(m)}^{*n}} p * p_m^{*(n-1)}(x) + \frac{m}{\mu_{(m)}^{*n}} \sum_{l=0}^{m-1} \binom{m-1}{l}$$

$$\times \mu_{(m-1-l)}^{*(n-1)} \frac{\mu_{(l+1)}}{l+1} p_{l+1}(x)$$

$$= \frac{\mu_{(m)}^{*(n-1)}}{\mu_{(m)}^{*n}} p * p_m^{*(n-1)}(x) + \frac{1}{\mu_{(m)}^{*n}} \sum_{l=1}^{m} \binom{m}{l}$$

$$\times \mu_{(l)} \mu_{(m-l)}^{*(n-1)} p_l(x).$$

This completes the proof. □

Remark: Theorem 6 gives a recursive formula for the high order equilibrium distributions of the convolution p_m^{*n}. First obtain $p_l(x)$ and $\mu_{(l)}^{*k}$, for $l = 0,1,\cdots,m$ and $k = 1,2,\cdots,n$. Then compute the starting p.f. $p_m(x)$, followed by the convolutions $p_m^{*2}(x)\cdots$ up to $p_m^{*n}(x)$.

Example 3 Consider $X \sim$ negative binomial (α,θ), for $\alpha \ge 2$ and $\theta \in (0,1)$, that is

$$p(x) = \binom{x+\alpha-1}{x} (1-\theta)^\alpha \theta^x, \quad \text{for } x \in \mathbb{N}.$$

Since the $NB(\alpha,\theta)$ distribution can be viewed as the α -th convolution of a geometric distribution $g(x) = (1-\theta)\theta^x$, then $p(x) = g^{*\alpha}(x)$ and the above theorem can be used to compute $p_m(x) = g_m^{*\alpha}(x)$ recursively.

Here the *k*-th factorial moment of $g(x)$ is

$$\mu_{(k)} = (1-\theta)\sum_{x\ge 0} x^{(k)}\theta^x = k!\left[\frac{\theta}{1-\theta}\right]^k,$$

by Lemma 1. Now $\bar{P}_k(x) = \theta^{x+1}$ and

$$\mu_{(k)}^{*\alpha} = \sum_{l_1+l_2+\cdots+l_\alpha = k} \frac{k!}{l_1! l_2! \cdots l_\alpha!} \mu_{(l_1)} \mu_{(l_2)} \cdots \mu_{(l_\alpha)}$$

$$= k!\left(\frac{\theta}{1-\theta}\right)^k \sum_{l_1+l_2+\cdots+l_\alpha = k} 1$$

$$= k!\left(\frac{\theta}{1-\theta}\right)^k \frac{(k+\alpha-1)^{(\alpha-1)}}{(\alpha-1)!}.$$

After some simplifications, we have

$$g_m^{*\alpha}(x) = \frac{\mu_{(m)}^{*(\alpha-1)}}{\mu_{(m)}^{*\alpha}} g * g_m^{*(\alpha-1)}(x)$$

$$+ \frac{1}{\mu_{(m)}^{*\alpha}} \sum_{l=1}^{m} \binom{m}{l} \mu_{(l)} \mu_{(m-l)}^{*(\alpha-1)} p_l(x)$$

$$= \frac{\alpha-1}{m+\alpha-1} g * g_m^{*(\alpha-1)}(x)$$

$$+ \frac{m}{m+\alpha-1} (1-\theta)\theta^x$$

$$= z g * g_m^{*(\alpha-1)}(x) + (1-z) g(x),$$

where $z = \dfrac{\alpha-1}{m+\alpha-1}$. This shows that the *m*-th order equilibrium distribution of a negative binomial is a mixture of the distributions $g * g_m^{*(\alpha-1)}$ and g, where the mixing factor $z = \dfrac{\alpha-1}{m+\alpha-1}$.

5. Equilibrium Distribution of a Mixture

This section discusses the equilibrium distribution of a mixed p.f.. For $x \in \mathbb{N}$. let $p(x|\theta)$, be the conditional distribution of *X*, given $\Theta = \theta$.

First assume that Θ has a continuous distribution function *U* with density *u* over $(0,\infty)$. Then the p.f. of *X*, given by $p(x) = \int_0^\infty p(x|\theta)\mathrm{d}U(\theta)$, is a *U*-mixture of distributions. The high order equilibrium distributions of *p* are given in the following theorem.

Theorem 7 The *n*-th order equilibrium distribution of the mixed p.f.

$p(x) = \int_0^\infty p(x|\theta)\mathrm{d}U(\theta)$, is given by

$$p_n(x) = \int_0^\infty p_n(x|\theta)\mathrm{d}U_n(\theta), \tag{21}$$

where $\mathrm{d}U_n(\theta) = \dfrac{E\left[X^{(n)}|\theta\right] dU(\theta)}{E\left[X^{(n)}\right]}$.

Proof

$$p_n(x) = \frac{n}{\mu_{(n)}} \sum_{y \ge x+1} p(y)\left[y-(x+1)\right]^{(n-1)}$$

$$= n\frac{E\left(X-(x+1)\right)_+^{(n-1)}}{E\left[X^{(n)}\right]}$$

$$= \frac{n}{E\left[X^{(n)}\right]} \int_0^\infty E\left\{\left[X-(x+1)\right]_+^{(n-1)} |\theta\right\} \mathrm{d}U(\theta)$$

$$= \int_0^\infty n \frac{E\left\{\left[X-(x+1)\right]_+^{(n-1)} |\theta\right\}}{E\left[X^{(n)}|\theta\right]} \frac{E\left[X^{(n)}|\theta\right]\mathrm{d}U(\theta)}{E\left[X^{(n)}\right]}$$

$$= \int_0^\infty p_n(x|\theta)\mathrm{d}U_n(\theta).$$

□

This theorem shows that the *n*-th order equilibrium distribution of a mixed p.f. is the mixture of the *n*-th order equilibrium distribution of the conditional p.f., mixed by a new distribution U_n.

An important special case is the mixture of geometric p.f.'s, where $p(x) = \int_0^1 (1-\theta)\theta^x \mathrm{d}U(\theta)$. Geometric mixtures benefit from an important property, is that they are completely monotone distributions, in the sense that $(-1)^n \Delta^n p(x) \geq 0$, for all $x, n \in \mathbb{N}$. Then $p_n(x|\theta) = (1-\theta)\theta^x$ and

$$\mathrm{d}U_n(\theta) = \frac{E\left[X^{(n)}|\theta\right]\mathrm{d}U(\theta)}{E\left[X^{(n)}\right]}$$

$$= \frac{\sum_{x\geq 0} x^{(n)}(1-\theta)\theta^x \mathrm{d}U(\theta)}{E\left[X^{(n)}\right]}$$

$$= \frac{(1-\theta)\mathrm{d}U(\theta)}{E\left[X^{(n)}\right]}\sum_{x\geq 0} x^{(n)}\theta^x$$

$$= \frac{(1-\theta)\mathrm{d}U(\theta)}{E\left[X^{(n)}\right]}\frac{n!\theta^n}{(1-\theta)^{n+1}}, \quad \text{by()}$$

$$\propto \left(\frac{\theta}{1-\theta}\right)^n \mathrm{d}U(\theta),$$

showing that the *n*-th order equilibrium distribution of a geometric *U*-mixture is still a geometric, with same parameter. Here the new mixing density is proportional to the original one, $u_n(\theta) \propto \left(\frac{\theta}{1-\theta}\right)^n u(\theta)$.

Example 4 (Waring Distribution)

If $p(x|\theta) = (1-\theta)\theta^x$, for $x \in \mathbb{N}$, and

$$u(\theta) = \frac{1}{\beta(a,b)}\theta^{a-1}(1-\theta)^{b-1},$$

for $\theta \in (0,1)$, then

$$p(x) = \frac{\int_0^1 (1-\theta)^b \theta^{x+a-1} d\theta}{\beta(a,b)}$$

$$= \frac{\beta(x+a, b+1)}{\beta(a,b)}$$

$$= \frac{b\Gamma(a+b)}{\Gamma(a)}\frac{\Gamma(x+a)}{\Gamma(x+a+b+1)},$$

which is called a Waring distribution.

It follows that

$$p_n(x|\theta) = p(x|\theta)$$

$$= (1-\theta)\theta^x, u_n(\theta) \propto \theta^{a+n-1}(1-\theta)^{b-n-1},$$

which is a $\beta(a+n, b-n)$ distribution when $b > n$. Then

$$p_n(x) = \int_0^1 (1-\theta)\theta^x \mathrm{d}U_n(\theta)$$

$$= \frac{\beta(x+a+n, b-n+1)}{\beta(a+n, b-n)}$$

$$= \frac{(b-n)\Gamma(a+b)}{\Gamma(a+n)}\frac{\Gamma(x+a+n)}{\Gamma(x+a+b+1)}$$

when $b \leq n$, then $p_n(x)$ does not exist.

If instead, the geometric p.f. is mixed over another discrete p.f., that is $p(x) = \sum_{j=1}^k f(j)(1-\theta_j)\theta_j^x$, for $x \in \mathbb{N}$, where $0 < \theta_j < 1$, $f(j) > 0$ and $\sum_{j=1}^k f(j) = 1$, then

$$p_n(x) = \sum_{j=1}^k f_n(j)(1-\theta_j)\theta_j^x, \quad x \in \mathbb{N},$$

where $f_n(j) = \frac{\left(\frac{\theta_j}{1-\theta_j}\right)^n f(j)}{\sum_{i=1}^k \left(\frac{\theta_i}{1-\theta_i}\right)^n f(i)}$, for $j = 1, 2, \cdots, k$.

The following theorem gives the aging properties of the higher order equilibrium distributions of geometric mixtures.

Theorem 8 The *n*-th order equilibrium distribution of a geometric mixture is DFR_d and $IMRL_d$. Also $\overline{P}_n(x) \leq \overline{P}_{n+1}(x)$, for $x \geq 0$, or equivalently, $p_n <_{st} p_{n+1}$.

Proof: Since p_n is also a geometric mixture, it is completely monotone. Then p_n is DFR_d, see [9]. Further, $r_n(x) = 1/(h_{n+1}(x))$, by (17), hence p_n is $IMRL_d$. Lastly,

$$\frac{\overline{P}_n(x)}{\overline{P}_{n+1}(x)} = \frac{r_n(0)-1}{r_n(x)-1} \leq 1,$$

by (16). □

6. Equilibrium Distribution of the (*a*,*b*) Family

We consider here the equilibrium distributions of the (a,b) class of discrete distributions, or more precisely, of the important subclass of the (a,b) family called the $(a,b,0)$ class, see [11,12]. This class of counting distributions has support on the non-negative integers on which the recurrence relation $p(x) = (a + (b/x))p(x-1)$ holds for $x = 1, 2, \cdots$. The members of the this class are binomial, Poisson and negative binomial distributions (with their corresponding special cases). It is easily seen that

$$\mu_{(1)} = \frac{a+b}{1-a} \text{ and } \mu_{(k)} = \frac{(ak+b)\mu_{(k-1)}}{1-a}, \text{ for } k \geq 2.$$

Then we have the following recursive formula for p_n.

Theorem 9 The *n*-th order equilibrium distribution p_n in the $(a,b,0)$ class of distributions satisfies the following recursion for $n \in \mathbb{N}^+$:

$$p_{n+1}(x+1) = ap_{n+1}(x) - \frac{(n+1)}{n}\frac{(1-a)(x+n+1)}{an+a+b}p_n(x+1) + \frac{(n+1)}{n}\frac{(1-a)(an+a+b+ax)}{an+a+b}p_n(x), \quad x \in \mathbb{N}. \tag{22}$$

The starting points of the recursion are

$$p_1(x) = \frac{1}{\mu_{(1)}}\overline{P}(x), \quad p_1(0) = \frac{1}{\mu_{(1)}}[1-p(0)]$$

and

$$p_n(0) = \frac{n!}{\mu_{(n)}}\left\{(-1)^{n-1}[1-p(0)] + \sum_{k=1}^{n-1}\frac{(-1)^{n-1-k}}{k!}\mu_{(k)}\right\},$$

for $n \geq 2$.

Proof: $p(x) = \left(a + \frac{b}{x}\right)p(x-1)$, or equivalently, $(a+b)p(x) = (x+1)p(x+1) - axp(x)$, for $x = 0,1,2,\cdots$. Then

$$\begin{aligned}
&(a+b)p_n(x) \\
&= \frac{n}{\mu_{(n)}}(a+b)\sum_{y\geq x+1}p(y)[y-(x+1)]^{(n-1)}, \quad n \geq 1 \\
&= \frac{n}{\mu_{(n)}}\Bigg[\sum_{y\geq x+1}(y+1)p(y+1)[y-(x+1)]^{(n-1)} - a\sum_{y\geq x+1}yp(y)[y-(x+1)]^{(n-1)}\Bigg] \\
&= \frac{n}{\mu_{(n)}}\Bigg[\sum_{y\geq x+2}yp(y)[y-(x+2)]^{(n-1)} - a\sum_{y\geq x+1}[y-(x+n)]p(y)[y-(x+1)]^{(n-1)} \\
&\qquad - a(x+n)\sum_{y\geq x+1}p(y)[y-(x+1)]^{(n-1)}\Bigg] \\
&= \frac{n}{\mu_{(n)}}\sum_{y\geq x+2}p(y)[y-(x+2)]^{(n)} + \frac{n(x+n+1)}{\mu_{(n)}}\sum_{y\geq x+1}p(y)[y-(x+2)]^{(n-1)} \\
&\qquad - a\frac{n}{\mu_{(n)}}\sum_{y\geq x+1}p(y)[y-(x+1)]^{(n)} - a\frac{n(x+n)}{\mu_{(n)}}\sum_{y\geq x+1}p(y)[y-(x+1)]^{(n-1)} \\
&= \frac{n}{\mu_{(n)}}\frac{\mu_{(n+1)}}{n+1}p_{n+1}(x+1) - (x+n+1)p_n(x+1) \\
&\qquad - a\frac{n}{\mu_{(n)}}\frac{\mu_{(n+1)}}{n+1}p_{n+1}(x) - a(x+n)p_n(x),
\end{aligned}$$

which in turns implies

$$p_{n+1}(x+1) = ap_{n+1}(x) - \frac{\mu_{(n)}}{\mu_{(n+1)}}\frac{n+1}{n}p_n(x+1) + \frac{\mu_{(n)}}{\mu_{(n+1)}}\frac{n+1}{n}(an+a+b+ax)p_n(x), \quad n \geq 1$$

Since $\frac{\mu_{(n)}}{\mu_{(n+1)}} = \frac{1-a}{an+a+b}$, we get (22).

Finally, we have

$$\begin{aligned}
p_n(0) &= \frac{n}{\mu_{(n)}}\sum_{y\geq 1}p(y)[y-1]^{(n-1)} \\
&= \frac{n}{\mu_{(n)}}\sum_{y\geq 1}p(y)\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{(n-1-k)}y^{(k)} \\
&= \frac{n!}{\mu_{(n)}}\sum_{k=0}^{n-1}\frac{(-1)^{n-1-k}}{k!}\sum_{y\geq 1}p(y)y^{(k)} \\
&= \frac{n!}{\mu_{(n)}}\left\{(-1)^{n-1}[1-p(0)] + I(n\geq 2)\sum_{k=1}^{n-1}\frac{(-1)^{n-1-k}}{k!}\mu_{(k)}\right\}.
\end{aligned}$$

This completes the proof. □

For another subclass of the (a,b) family, the $(a,b,1)$ class of distributions, the relation $p(x) = (a+(b/x))p(x-1)$ holds for $x \geq 2$, where $p(0)$ is an arbitrarily selected value in $[0,1]$. In this case, it is easy to see that $\mu_{(k)} = \frac{ak+b}{1-a}\mu_{(k-1)}$, for $k \geq 2$. The above recursive formula (22) and those for the starting points still hold true here, the only change being that

$$\mu_{(1)} = \frac{p(1)+(a+b)(1-p(0))}{1-a}.$$

7. Conclusions

This paper investigates the higher order equilibrium distributions of counting random variables. The above results can be used in Risk Theory to derive bounds of ruin probabilities in the discrete time risk model. They also lead to the factorial moments of three related random variables: the surplus before ruin, the deficit at ruin and the time of ruin (see Li and Garrido [13] for details).

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