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Lattice Paths and Rogers Identities

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Abstract

Recently we interpreted five *q*-series identities of Rogers combinatorially by using partitions with "*n* + *t* copies of *n*" of Agarwal and Andrews [1]. In this paper we use lattice paths of Agarwal and Bressoud [2] to provide new combinatorial interpretations of the same identities. This results in five new 3-way combinatorial identities.

Keywords: Lattice Paths, Colored Partitions, Generating Functions, Combinatorial Interpretations

1. Introduction Definitions and the Main Results

In the literature we find that several q-identities such as given in Slater's compendium [3] have been interpreted combinatorially using ordinary partitions by several authors (for example, see Connor [4], Subbarao [5], Subbarao and Agarwal [6] and Agarwal and Andrews [7]). In the early nineteen eighties Agarwal and Andrews introduced a new class of partitions called "$(n+t)$-color partitions" or partitions with "$(n+t)$ copies of n". Using these new partitions many more q-identities have been interpreted combinatorially in [8-12].

Recently in [13] we interpreted combinatorially the following q-identities of Rogers [14] by using colored partitions:

$$\sum_{n=0}^{\infty}\frac{q^{3n^2}}{\left(q;q^2\right)_n\left(q^4;q^4\right)_n}=\frac{\left(-q^3,-q^5,-q^7;q^{10}\right)_\infty}{\left(q^4,q^6;q^{10}\right)_\infty}, \quad (1.1)$$

$$\sum_{n=0}^{\infty}\frac{q^{3n^2-2n}}{\left(q;q^2\right)_n\left(q^4;q^4\right)_n}=\frac{\left(-q,-q^5,-q^9;q^{10}\right)_\infty}{\left(q^2,q^8;q^{10}\right)_\infty}, \quad (1.2)$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{\left(q;q^2\right)_n\left(q^4;q^4\right)_n}=\frac{\left(-q^3,-q^7,-q^{11};q^{14}\right)_\infty}{\left(q^2,q^6,q^8,q^{12};q^{14}\right)_\infty}, \quad (1.3)$$

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\left(q;q^2\right)_n\left(q^4;q^4\right)_n}=\frac{\left(-q^5,-q^7,-q^9;q^{14}\right)_\infty}{\left(q^4,q^6,q^8,q^{10};q^{14}\right)_\infty}, \quad (1.4)$$

and

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\left(q;q^2\right)_{n+1}\left(q^4;q^4\right)_n}=\frac{\left(-q,-q^7,-q^{13};q^{14}\right)_\infty}{\left(q^2,q^4,q^{10},q^{12};q^{14}\right)_\infty}. \quad (1.5)$$

In Equations (1.1)-(1.5), $(q;q)_n$ is a rising q-factorial which in general is defined as follows :

$$(a;q)_n=\prod_{i=0}^{\infty}\frac{\left(1-aq^i\right)}{\left(1-aq^{n+i}\right)}.$$

If n is a positive integer, then obviously

$$(a;q)_n=(1-a)(1-aq)\cdots\left(1-aq^{n-1}\right),$$

and

$$(a;q)_\infty=(1-a)(1-aq)\left(1-aq^2\right)\cdots,$$

and $\left(a_1,a_2,\cdots,a_t;z\right)_\infty$ is defined by

$$\left(a_1,a_2,\cdots,a_t;z\right)_\infty=\prod_{j=1}^{t}\left(a_j;z\right)_\infty.$$

We remark that Identities (1.1) and (1.2) were also derived by Bailey [15] and appear in [1], Identities (1.3)-(1.5) are also referred as Rogers-Selberg identities (see [3,14,16]).

In this paper we interpret the left-hand sides of (1.1)-(1.5) as generating functions for certain weighted lattice path functions defined by Agarwal and Bressoud in [17]. First we recall the definitions of the partitions with "$n+t$ copies of n" (also called $(n+t)$-color partitions) and their weighted difference from [12]:

Definition 1. A partition with "$n+t$ copies of n", $t\ge 0$, is a partition in which a part of size n, $n\ge 0$, can come in $n+t$ different colors denoted by subscripts:

$n_1, n_2, \cdots, n_{n+t}$.

Thus, for example, the partitions of 2 with "$n+1$ copies of n" are

$$2_1,\ 2_1+0_1,\ 1_1+1_1,\ 1_1+1_1+0_1,$$
$$2_2,\ 2_2+0_1,\ 1_2+1_1,\ 1_2+1_1+0_1,$$
$$2_3,\ 2_3+0_1,\ 1_2+1_2,\ 1_2+1_2+0_1.$$

Note that zeros are permitted if and only if t is greater than or equal to one.

Definition 2. The weighted difference of two elements m_i and n_j, $m \ge n$, is defined by $m-n-i-j$ and is denoted by $\left(m_i - n_j\right)$.

Next, we recall the following description of lattice paths from [17] which we shall be considering in this paper:

All lattice paths will be of finite length lying in the first quadrant. All paths will begin on the *y*-axis and terminate on the *x*-axis. Only three moves are allowed at each step:

northeast: from (i, j) to $(i+1, j+1)$,

southeast: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$,

horizontal: from $(i, 0)$ to $(i+1, 0)$, only allowed along *x*-axis.

All lattice paths are either empty or terminate with a southeast step: from $(i,1)$ to $(i+1,0)$.

In describing lattice paths, we shall use the following terminology:

PEAK: Either a vertex on the *y*-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.

VALLEY: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

MOUNTAIN: A section of the path which starts on either the *x*- or *y*-axis, which ends on the *x*-axis, and which does not touch the *x*-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

PLAIN: A section of path consisting of only horizontal steps which starts either on the *y*-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

Example: The following path has five peaks, three valleys, three mountains and one plain.

The **HEIGHT** of a vertex is its *y*-coordinate. The **Weight** of a vertex is its *x*-coordinate. The **WEIGHT OF A PATH** is the sum of the weights of its peaks.

In the example given above, there are two peaks of height three and three of height two, two valleys of height one and one of height zero.

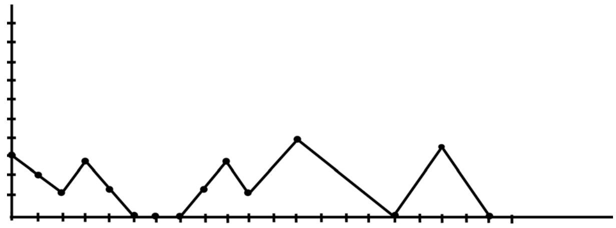

Figure 1. Contains five peaks, three valleys, three mountains and one plane.

The weight of this path is $0+3+9+12+17 = 41$.

Recently in [13] we showed that the identities (1.1)-(1.5) have their colored partition theoretic interpretations in the following theorems, respectively:

Theorem 1. Let $A_1(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd, all subscripts are greater than 2, if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Let

$$B_1(\nu) = \sum_{k=0}^{\nu} C_1(\nu-k) D_1(k),$$

where $C_1(\nu)$ is the number of partitions of ν into parts $\equiv \pm 4 \pmod{10}$ and $D_1(\nu)$ denotes the number of partitions of ν into distinct parts $\equiv \pm 3, 5 \pmod{10}$. Then

$$A_1(\nu) = B_1(\nu), \quad \text{for all } \nu.$$

Example. $A_1(15) = 6$, since the relevant partitions are 15_{15}, 15_{11}, 15_7, 15_3, 12_6+3_3, 11_3+4_4.

Also,

$$\begin{aligned} B_1(15) &= \sum_{k=0}^{15} C_1(\nu-k) D_1(k) \\ &= C_1(15)D_1(0) + C_1(14)D_1(1) + \cdots + C_1(0)D_1(15) \\ &= 0(1)+2(0)+0(0)+2(1)+0(0)+1(1)+0(0) \\ &\quad +1(1)+0(1)+1(0)+0(1)+1(0)+0(1) \\ &\quad +0(1)+0(0)+1(2) = 6. \end{aligned}$$

Theorem 2. Let $A_2(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd, if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is ≥ 4 and is $\equiv 0 \pmod 4$. Let

$$B_2(\nu) = \sum_{k=0}^{\nu} C_2(\nu-k) D_2(k),$$

where $C_2(\nu)$ is the number of partitions of ν into parts $\equiv \pm 2 \pmod{10}$ and $D_2(\nu)$ denotes the number of partitions of ν into distinct parts $\equiv \pm 1, 5 \pmod{10}$. Then

$$A_2(\nu) = B_2(\nu), \quad \text{for all } \nu.$$

Theorem 3. Let $A_3(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd >1, if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Let

$$B_3(\nu) = \sum_{k=0}^{\nu} C_3(\nu-k) D_3(k),$$

where $C_3(\nu)$ is the number of partitions of ν into parts $\equiv \pm 2, \pm 6 \pmod{14}$ and $D_3(\nu)$ denotes the number of partitions of ν into distinct parts $\equiv \pm 3, 7 \pmod{14}$. Then

$$A_3(\nu) = B_3(\nu), \quad \text{for all } \nu.$$

Theorem 4. Let $A_4(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd with odd, all subscripts are >3, if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$ and the weighted difference of any two consecutive parts is ≥ -4 and is $\equiv 0 \pmod 4$. Let

$$B_4(\nu) = \sum_{k=0}^{\nu} C_4(\nu-k) D_4(k),$$

where $C_4(\nu)$ is the number of partitions of ν into parts $\equiv \pm 4, \pm 6 \pmod{14}$ and $D_4(\nu)$ denotes the number of partitions of ν into distinct parts $\equiv \pm 5, 7 \pmod{14}$. Then

$$A_4(\nu) = B_4(\nu), \text{ for all } \nu.$$

Theorem 5. Let $A_5(\nu)$ denote the number of partitions of ν with "$n+2$ copies of n" such that the even parts appear with even subscripts and odd with odd, all subscripts are >1, if m_i is the smallest or the only part in the partition, then $m \equiv i \pmod 4$, for some i, i_{i+2} is a part and the weighted difference of any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Let

$$B_5(\nu) = \sum_{k=0}^{\nu} C_5(\nu-k) D_5(k),$$

where $C_5(\nu)$ is the number of partitions of ν into parts $\equiv \pm 2, \pm 4 \pmod{14}$ and $D_5(\nu)$ denotes the number of partitions of ν into distinct parts $\equiv \pm 1, 7 \pmod{14}$. Then

$$A_5(\nu) = B_5(\nu), \quad \text{for all } \nu.$$

In this paper we prove the following combinatorial interpretations of the identities (1.1)-(1.5) in terms of lattice paths:

Theorem 6. Let $E_1(\nu)$ denote the number of lattice paths of weight ν which start from $(0,0)$, have no valley above height 0, the lengths of the plains, if any, are $\equiv 0 \pmod 4$ and the height of each peak is greater than 2. Then

$$E_1(\nu) = B_1(\nu), \quad \text{for all } \nu.$$

Example. $E_1(15) = 6$, since the relevant lattice paths are:

Theorem 7. Let $E_2(\nu)$ denote the number of lattice paths of weight ν which start from $(0,0)$, have no valley above height 0, the lengths of the plains are $\equiv 0 \pmod 4$ and there is a plain of length ≥ 4 between any two peaks. Then

$$E_2(\nu) = B_2(\nu), \quad \text{for all } \nu.$$

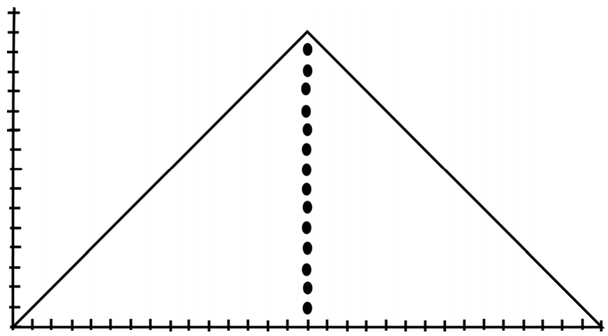

Figure 2. Contains one peak of height fifteen.

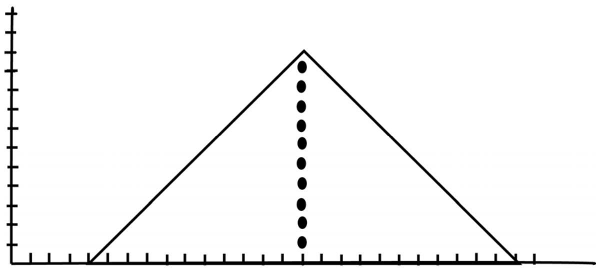

Figure 3. Contains one peak of height fifteen.

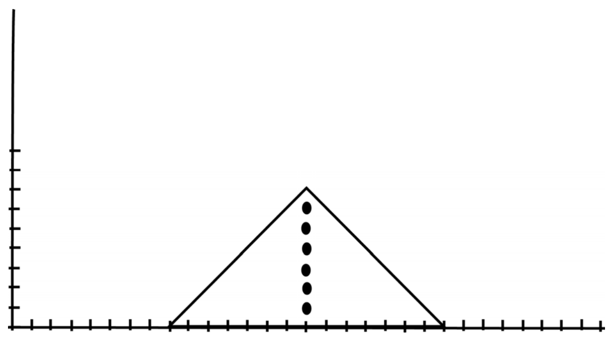

Figure 4. Contains one plain of length eight and one peak of height seven.

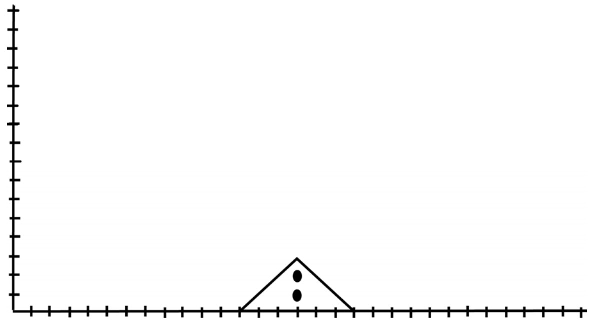

Figure 5. Contains one plain of length eight and one peak of height seven.

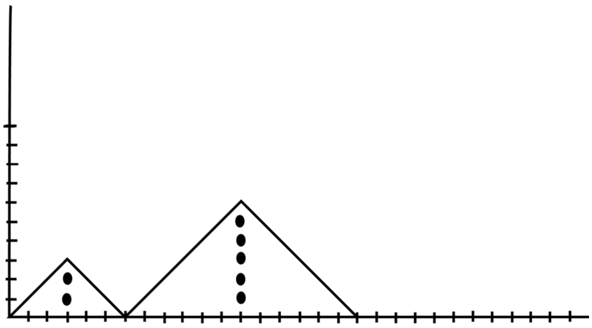

Figure 6. Contains one plain of length eight and one peak of height seven.

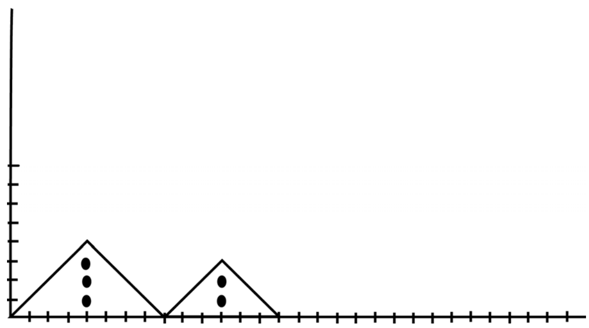

Figure 7. Contains two peaks of height four, three and one valley at height zero.

Theorem 8. Let $E_3(\nu)$ denote the number of lattice paths of weight ν which start from $(0,0)$, have no valley above height 0, the lengths of the plains, if any, is $\equiv 0 \pmod 4$ and the height of each peak is greater than 1. Then

$$E_3(\nu) = B_3(\nu), \quad \text{for all } \nu.$$

Theorem 9. Let $E_4(\nu)$ denote the number of lattice paths of weight ν which start from $(0,0)$, have no valley above height 0, the height of each peak is >1, there is a plain of length $\equiv 2 \pmod 4$ in the beginning of the path and the lengths of the other plains, if any, are $\equiv 0 \pmod 4$ Then

$$E_4(\nu) = B_4(\nu), \quad \text{for all } \nu.$$

Theorem 10. Let $E_5(\nu)$ denote the number of lattice paths of weight ν which start from $(0,2)$, have no valley above height 0, the height of each peak is >1, the lengths of the plains, if any, are $\equiv 0 \pmod 4$ Then

$$E_5(\nu) = B_5(\nu), \quad \text{for all } \nu.$$

Theorems 6-10 lead to the following 3-way extension of Theorems 1-5:

Theorem 11. For $1 \le k \le 5$, we have

$$A_k(\nu) = B_k(\nu) = E_k(\nu), \quad \text{for all } \nu.$$

In [13] we have shown that for $1 \le k \le 5$ the left-hand side of the Equation $(1.k)$ generates $A_k(\nu)$ and consequently $A_k(\nu) = B_k(\nu)$. Here we shall prove that the left-hand side of equation $(1.k)$ generates $E_k(\nu)$ also. We shall also show bijectively that $A_k(\nu) = E_k(\nu)$.

Furthermore, since each of these five cases is proved in a similar way, we provide the details for $k = 1$ in our next section and sketch the changes required to treat the remainder in Section 3.

2. Proof of Theorem 6

In $\dfrac{q^{3m^2}}{(q;q^2)_m (q^4;q^4)_m}$ the factor q^{3m^2} generates the lattice path of m peaks each of height 3 starting at (0,0) and terminating at $(6m,0)$.

If $m = 4$, the path begins as:

The factor $1/(q^4;q^4)_m$ generates m-nonnegative multiples of 4, say $a_1 \ge a_2 \ge \cdots a_m \ge 0$, which are encoded by inserting a_m horizontal steps in front of the first mountain and $a_i - a_{i+1}$ horizontal steps in front of the $(m-i+1)st$ mountain, $1 \le i \le m.$

If $a_1 = 8$, $a_2 = 4$, $a_3 = 4$, $a_4 = 0$, then our above graph becomes:

The factor $1/(q;q^2)_m$ generates nonnegative multiples of $(2i-1)$, $1 \le i \le m$, say, $b_1 \times 1,\ b_2 \times 3, \cdots, b_m \times (2m-1)$. This is encoded by having the ith peak grow to height

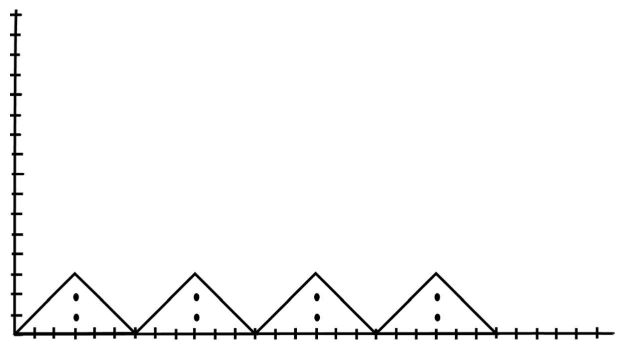

Figure 8. Contains four peaks each of height three and three valleys each at height zero.

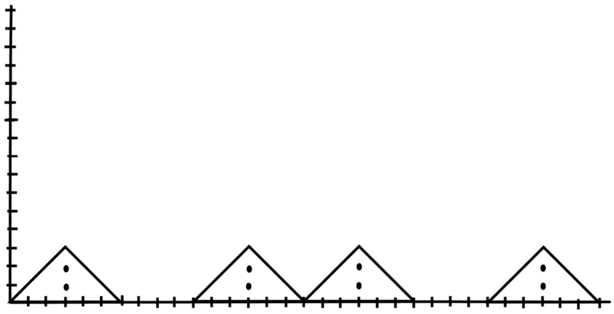

Figure 9. Contains two plains each of length four and four peaks each of height three and one valley at height zero.

$b_{m-i+1}+3$. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two.

If b_1 = 3, b_2 = 1, b_3 = 2, b_4 = 0, then our example becomes:

In the Graph-8, we consider two successive peaks, say i th and $(i+1)$ th and denote them by P_1 and P_2, respectively

Now, due to the impact of the factor $1/(q^4;q^4)_m$, the **Figure 11** changes to **Figure 12**

Again by taking into consideration, the impact of the factor $1/(q;q^2)_m$, the **Figure 12** changes to **Figure 13** or **Figure 14** depending on whether $b_{m-i} > b_{m-i+1}$ or $b_{m-i} < b_{m-i+1}$. In the case when $b_{m-i} = b_{m-i+1}$, the new graph will look like **Figure 12**.

Every lattice path enumerated by $E_1(\nu)$ is uniquely generated in this manner. This proves that the L.H.S. of (1.1) generates $E_1(\nu)$.

We now establish a $1-1$ correspondence between the lattice paths enumerated by $E_1(\nu)$ and the n-color partitions enumerated by $A_1(\nu)$.

We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak.

Thus, if we denote the two peaks in **Figure 13** (or **Figure 14**) by A_x and B_y, respectively, then

$$A=(6i-3)+a_{m-i+1}+2(b_m+b_{m-1}+\cdots+b_{m-i+2})+b_{m-i+1}$$
$$x=b_{m-i+1}+3$$
$$B=(6i+3)+a_{m-i}+2(b_m+b_{m-1}+\cdots+b_{m-i+1})+b_{m-i}$$
$$y=b_{m-i}+3.$$

If we look at the n-color part A_x, we find that the parity of both A and x is determined by b_{m-i+1}. If b_{m-i+1} is odd, then both A and x are even and if b_{m-i+1} is even, then both A and x are odd. This proves that even parts appear with even subscripts and odd with odd. Clearly, all subscripts x are >2.

The weighted difference of these two consecutive parts is

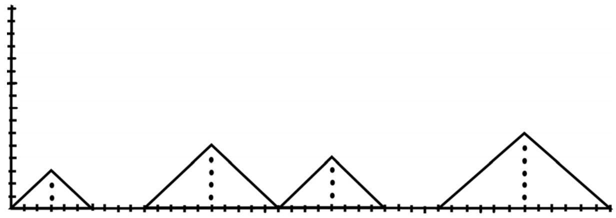

Figure 10. Contains two plains each of length four and four peaks of height three, five, four, six respectively and one valley at height zero.

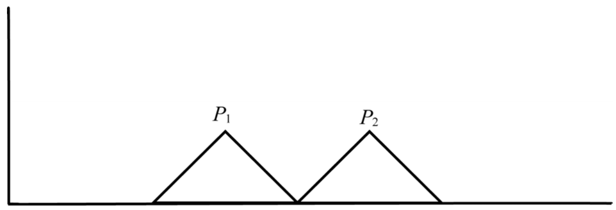

Figure 11. Contains two peaks of same height.

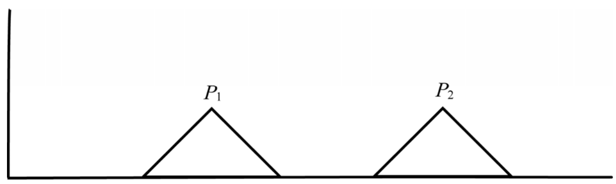

Figure 12. Contains two peaks separated by a plane and length of the plane is a multiple of four.

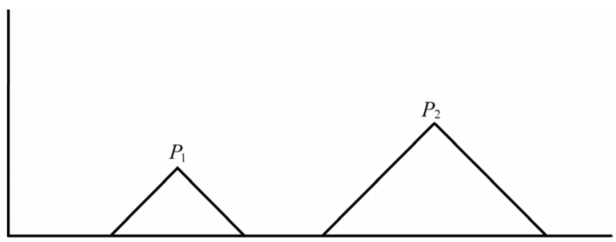

Figure 13. Contains two peaks of which height differs by an odd number and separated by a plane, *P*2 has more height than *P*1.

$$\begin{aligned}(B_y-A_x) &= B-A-x-y\\ &=(6i+3+a_{m-i}+2(b_m+b_{m-1}+\cdots+b_{m-i+1})+b_{m-i})\\ &\quad-(6i-3+a_{m-i+1}+2(b_m+b_{m-1}+\cdots+b_{m-i+2})+b_{m-i+1})\\ &\quad-b_{m-i+1}-3-b_{m-i}-3\\ &=a_{m-i}-a_{m-i+1}\equiv 0 \pmod 4.\end{aligned}$$

Obviously, if (A,x) is the first peak in the lattice path then it will correspond to the smallest part in the corresponding n-color partition or to the singleton part if the n-color partition has only one part and in both cases

$$A-x=a_m\equiv 0 \pmod 4.$$

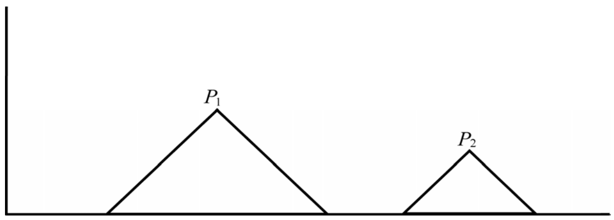

Figure 14. Contains two peaks of which height differs by an odd number and separated by a plane, *P*1 has more height than *P*2.

To see the reverse implication, we consider two n-color parts of a partition enumerated by $E_1(\nu)$, say, C_u and D_v.

Let $Q_1 \equiv (C,u)$ and $Q_2 \equiv (D,v)$ be the corresponding peaks in the associated lattice path.

The length of the plain between the two peaks is $D-C-u-v$ which is the weighted difference between the two parts C_u and D_v and is therefore nonnegative and $\equiv 0(\text{mod}\,4)$.

Also, there can not be a valley above height 0. This can be proved by contradiction.

Suppose, there is a valley V of height r $(r>0)$ between the peaks Q_1 and Q_2.

In this case there is a descent of $u-r$ from Q_1 to V and an ascent of $v-r$ from V to Q_2. This implies

$$D = C+(u-r)+(v-r)$$
$$\Rightarrow D-C-u-v=-2r.$$

But since the weighted difference is nonnegative, therefore $r=0$.

Also, $u,v>2$ imply that the height of each peak is atleast 3. This completes the proof of Theorem 6.

3. Sketch of the proofs of Theorems 7-10

Case $k = 2$ is treated in exactly the same manner as the first case except that now the path begins with m peaks each of height 1 and with a plain of length $4i$, $1 \le i \le m-1$ between ith and $(i+1)$th peak.

In the Case $k = 3$, the only point of departure from the first case is that the path begins with m peaks each of height 2.

Case $k = 4$ is treated in exactly the same manner as the previous case except that the extra factor q^{2m} puts a plain of length of 2 in front of the first peak. This increases the weight of each peak by 2 and so the weight of the lattice path is increased by $2m$.

Comparing the case $k = 5$ with the case $k = 3$, we see that in this case there are two extra factors, viz., q^{2m} and $\left(1-q^{2m+1}\right)^{-1}$. The extra factor q^{2m} puts two south east steps: (0,2) to (1,1) and (1,1) to (2,0). Thus there are now $m+1$ peaks starting from (0,2) and the extra factor

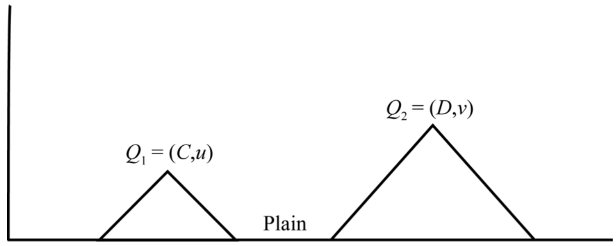

Figure 15. Contains two peaks separated by a plain.

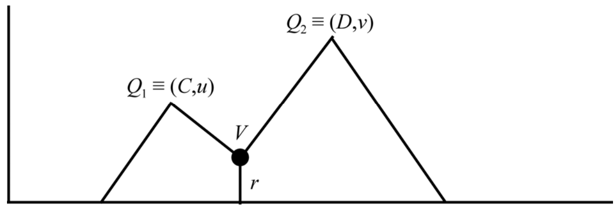

Figure 16. Contains two peaks and a valley at height r.

$\left(1-q^{2m+1}\right)^{-1}$ introduces a nonnegative multiple of $2m+1$, say $b_{m+1}\times(2m+1)$. This is encoded by having the first peak grow to height $b_{m+1}+2$. Clearly, $\left(b_{m+1}\right)_{b_{m+1}+2}$ which is of the form i_{i+2} will be the colored part corresponding to the first peak.

4. Conclusions

The sum-product identities like (1.1) to (1.5) are generally known as Rogers-Ramanujan type identities. They have applications in different areas such as Orthogonal polynomials, Lie-algebras, Combinatorics, Particle physics and Statistical mechanics.

The most obvious question arising from this work is:

Do Theorems 1.6-1.10 admit generalization analogous to the generalized results of [12,17]?

5. References

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