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b-chromatic Number of $M[C_n]$, $M[P_n]$, $M[F_{1,n}]$ and $M[W_n]$

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Abstract

In this paper, we discuss about the b-colouring and b-chromatic number for middle graph of Cycle, Path, Fan graph and Wheel graph denoted as $M[C_n]$, $M[P_n]$, $M[F_{1,n}]$ and $M[W_n]$.

Keywords: Chromatic Number, b-chromatic, b-colouring, Middle Graph

1. Introduction

Let G be a finite undirected graph with no loops and multiple edges. A coloring (*i.e.*, proper coloring) of a graph $G = (V,E)$ is an assignment of colors to the vertices of G, such that any two adjacent vertices have different colors. A coloring is called a b-coloring [1], if for each color i there exists a vertex x_i of color i such that every color $j \neq i$, there exists a vertex y_j of color j adjacent to x_i, such a vertex x_i is called a dominating vertex for the colour class i or color dominating vertex which is known as b-chromatic vertex.

The b-chromatic number of a graph G, denoted by $\varphi(G)$ is the largest positive integer k such that G has a b-colouring by k colors. The b-chromatic number of a graph was introduced by R.W. Irwing and manlove [2] in the year 1999 by considering proper colorings that are minimal with respect to a partial order defined on the set of all partitions of $V(G)$. They proved that determining $\varphi(G)$[3] is NP-hard for general graphs, but polynomial for trees.

Let G be a graph with vertex set $V(G)$ and the edge set $E(G)$. The middle graph [4,5] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds;

1) x, y are in $E(G)$ and x, y are adjacent in G
2) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

2. b-chromatic Number of Middle Graph of Cycle

2.1. Definition of Cycle

A Cycle is a circuit in which no vertex except the first (which is also the last) appears more than once. A cycle with n vertices is denoted by Cn.

2.2. Theorem

For any $n \geq 3$, $\varphi[M(Cn)] = n$.

Proof

Let C_n be a cycle of length n with the vertices $V_1, V_2, \cdots, V_n$. By the definition of middle graph the edge V_{ij} for $1 \leq i \leq n$, $1 \leq j \leq n$ of the cycle C_n is subdivided by the vertex V'_m for $m = 1, 2, \cdots, n$. Here the vertices $V'_1, V'_2, \cdots, V'_n$, V_i induces a clique of order n.

Now assign a proper colouring to these vertices as follows. Consider a colour class $C = \{c_1, c_2, c_3 \cdots, c_n\}$. Assign the color c_i to the vertex V'_m for i = 1, 2, $\cdots$, n. Here $M(C_n)$ contains a clique of order n, so for proper colouring we require maximum n colours to colour the vertices of V'_m, which produces a b-chromatic coloring. Next we assign a colouring to the vertices V_i for $i = 1, 2, \cdots, n$. Suppose if we assign any new colour c_{n+1} to the vertex V_i $\forall\, i = 1, 2, \cdots, n$, it will not produce a b-chromatic colouring because none of the vertices v_i does not realizes its own colours. Therefore the only possibility is to assign an existing colors to the vertices v_i.

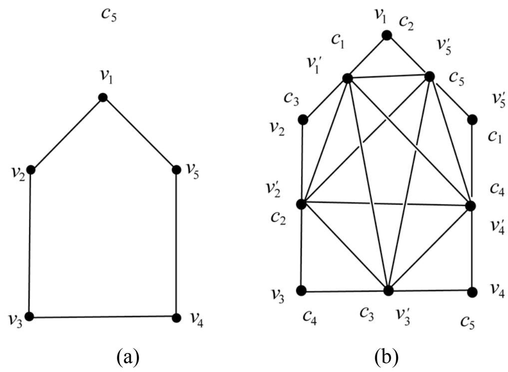

Figure 1. (a) [C_5]; (b) $\varphi\{M(C_5)\}$ = 5.

Hence by colouring procedure the above said colouring is maximal and b-chromatic.

$$\therefore \varphi\left[M\left(C_n\right)\right]=n \quad \text{for } n \geq 3$$

$$\text{Eg: } \varphi\left[M\left(C_5\right)\right]=5$$

3. b-chromatic Number of Middle Graph of Path

3.1. Definition of Path

A Path is a sequence of consecutive edges in a graph and the length of the path is the number of edges traversed. A path with n vertices is denoted as P_n.

3.2. Theorem

For any $n \geq 2$, $\varphi\left[M\left[P_n\right]\right]=n$

Proof:

Let P_n be any path of length n – 1 with vertices v_1, v_2, $\cdots$, v_n. By the definition of middle graph each edge of v_{ij} for $1 \leq i \leq n$, $1 \leq j \leq n$ of the path graph P_n is subdivided by the vertex v'_m in $M[P_n]$ and the vertices $v'_1, v'_2, \cdots, v'_m$ along with $v_1, v_2, \ldots, v_n$ induces a clique of order n in $M[P_n]$.

$$i.e., \; V\left[M\left(P_n\right)\right]=\left\{v_i \middle| 1 \leq i \leq n\right\} \cup\left\{v_m \middle| 1 \leq m \leq n\right\}$$

Now consider a proper colouring to $M[P_n]$ as follows. Consider the colour class $C=\left\{c_1, c_2, \cdots, c_n\right\}$. Assign the color c_i to the vertices v'_m for i = 1, 2, $\cdots$, n. Here $M[P_n]$ contains a clique of order n. So for proper colouring it require n distinct colours which results in b-chromatic coloring. Next we assign the coloring to the vertices v_i for i = 1, 2, $\cdots$, n. Suppose if we assign the colour c_{n+1} to the vertex v_i $\forall\, i = \cdots n$ which does not produces b-coloring. Hence we should assign only an existing colours to the vertices $v_1, v_2, \ldots, v_n$. Hence by coloring procedure it is the maximal and b-chromatic coloring.

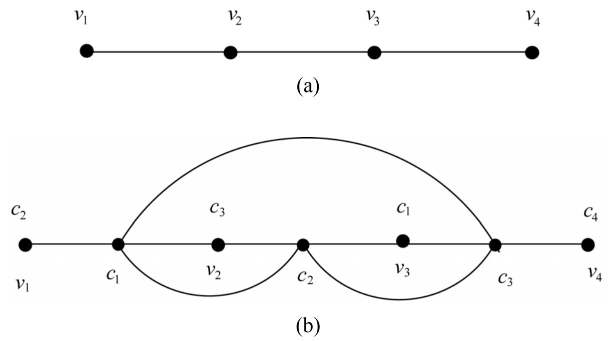

Figure 2. (a) P_3; (b) φ[M(P_3)] = 3.

$$\therefore \varphi\left[M\left(p_n\right)\right]=n \quad \text{for } n \geq 2 .$$

$$\text{Eg: } \varphi\left[M\left(p_3\right)\right]=3$$

4. b-chromatic Number of Midlle Graph of Fan Graph

4.1. Definition of Fan Graph

A fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m}+P_n$, where $\overline{K_m}$ is the empty graph on nodes and P_n is the path on n nodes.

4.2. Theorem

$$\varphi\left[M\left(F_{1,n}\right)\right]=n+1 \quad \text{for } n \geq 2$$

Proof

Let (x,y) be the bipartition of $F_{m,n}$ with $|x| = m$ and $|y| = n$. Let V be the only vertex of x and $y = \{v, v_2, \cdots, v_n\}$. By the definition of Middle graph each edge vv_i for i = 1, 2, 3, $\cdots$, n of $F_{1,n}$ is subdivided by the vertex v'_m in M[$F_{1,n}$] and the vertices $v'_1, v'_2, \cdots, v'_m$, v induces a clique of order n + 1 in $M[F_{1,n}]$.

$$i.e., \; \mathrm{V}\left[M\left(F_{1,n}\right)\right]=\left\{v_i / 1 \leq i \leq n\right\} \cup\left\{v'_m / 1 \leq m \leq n\right\} \cup V .$$

Now assign a proper colouring to these vertices as follows. Consider a colour class $C = \{c_1, c_2, \cdots, c_{n+1}\}$. First assign the colour c_1, c_2, $\cdots$, c_n to the vertices v'_m for m = 1, 2, $\cdots$, n. Here M[$F_{1,n}$] contains a clique of order n. So for proper colouring we require n distinct colours which results as b-chromatic colouring. Next assign the colour c_{n+1} to the vertex v and c_{n+2}, c_{n+2}, $\cdots$ to the vertices $v_1, v_2, \cdots, v_n$. Here the vertex v realizes its own colors but the vertices $v_1, v_2, \cdots, v_n$ does not realizes its own colors, so we cannot assign any new colours to the vertices v'_i for i = 1, 2, $\cdots$, n. Therefore by assigning only existing colors to the v_i produces a b-chromatic coloring. Hence by coloring procedure the above said col-

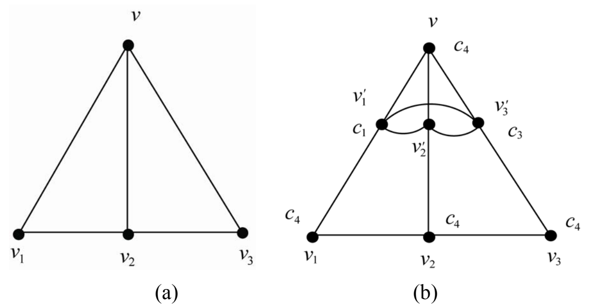

Figure 3. (a) [$F_{1,3}$]; (b) $\varphi[M(F_{1,3})] = 4$.

oring is maximal.

$$\therefore \varphi\left[M\left(F_{1,n}\right)\right] = n+1 \quad \text{for } n \geq 2.$$

$$\text{Eg: } \varphi\left[M\left(F_{1,3}\right)\right] = 4$$

5. b-chromatic Number of Middle Graph of Wheel Graph

5.1. Definition of Wheel Graph

A graph W_n of order n which contains a cycle of order $n - 1$, and for which every graph vertex in the cycle is connected to one other graph vertex (which is known as hub). The edges of a wheel which include the hub are spokes.

5.2. Theorem

For any $n > 4$, $\varphi[M(W_n)] = n$

Proof

Let $v_1, v_2, \cdots, v_n$ be the vertices taken in anticlock wise direction in the wheel graph w_n, where v_n is the hub. In $M(w_n)$, by the definition of middle graph the edge incident with v_i together with vertex v_i induces a clique of n vertices in $M(w_n)$. Let v'_m be the clique in $M(w_n)$ for $i = 1, 2, \cdots, n$.

Now consider a proper colouring to these vertices as follows. Consider the color class $C = \{c_1, c_2, \cdots, c_n\}$. First assign the color $c_1, c_2, \cdots, c_n$ to the vertex v'_m for $i = 1, 2, \cdots, n$. By the above statement that $M(w_n)$ contains a clique of order n, so we need only n colors to colour the vertices. Next we assign the color c_{n+1} to the hub. Here the vertices $v'_1, v'_2, \cdots, v'_n$ and the hub v_n realizes its colors, which produces a b-chromatic coloring. Next if we assign any new color to the vertices v_i for $i = 1, 2, \cdots, n-1$, it will not produce a b-chromatic coloring. So we should assign the existing colors c_{n+1} to the vertices v_i for $i = 1, 2, \cdots, n-2$ and c_1 to the verter v_{n-1}. Hence by coloring procedure it is the maximum and b-chromatic coloring.

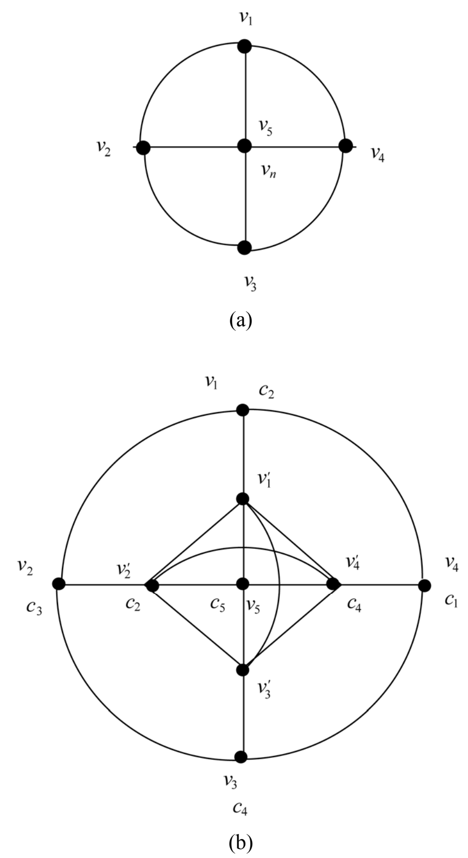

Figure 4. (a) [w_5]; (b) $\varphi[M(w_5)] = 5$.

$$\therefore \varphi\left[M\left(w_n\right)\right] = n \quad \text{for } n \geq 4.$$

$$\text{Eg: } \varphi\left[M\left(w_5\right)\right] = 5$$

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