

# A Study of Periodic Solution of a Duffing's Equation Using Implicit Function Theorem

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## Abstract

In this paper, the well known implicit function theorem was applied to study existence and uniqueness of periodic solution of Duffing-type equation. Under appropriate conditions around the origin, a unique periodic solution was obtained.

## Keywords

Implicit Function Theorem, Periodic Solution, Duffing Equation, Banach Spaces

## 1. Introduction

The well-known implicit function theorem has been employed by many authors to study existence of solution to non-linear differential equations of various types. [1] [2] [3] investigated the existence of solution to ordinary differential equations using implicit function theorem. Other researchers [4]-[10] used implicit function theorem to show the existence of periodic solution for non-linear partial differential equations. The Duffing equation (oscillator):

$$\ddot{x} + c\dot{x} + ax + bx^3 = h(t) \quad (1.1)$$

where  $a$ ,  $b$ ,  $c$  are real constants and  $h(t)$  is continuous, has been widely used in physics, economics, engineering, and many other physical phenomena. Given its characteristic of oscillation and chaotic nature, many scientists are inspired by this nonlinear differential equation given its nature to replicate similar dynamics in our natural world. This equation together with Van der Pol's equation has become one of the most common examples of nonlinear oscillation in textbooks and research articles. See for instance [11] [12] [13] [14] and the references

therein. Due to the importance of the Duffing equation in real world problems, the study of existence of solution of the equation has continued to attract the attention of many researchers. [15] [16] [17] [18] have proposed independently, the existence of periodic solution of Duffing equation of the general form:

$$\ddot{x} + c\dot{x} + g(t, x) = h(t) \tag{1.2}$$

where  $h(t)$  is continuous and  $2\pi$ -periodic in  $t \in R$  and  $g(t, x) = ax + bx^2 + \beta x^3$ .

Motivated by the above results, the purpose of this paper is to study the existence and uniqueness of periodic solution of Duffing equation of the form:

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = h(t) \tag{1.3}$$

using implicit function theorem where  $a, b, c$  are real constants and

$h: [0, 2\pi] \rightarrow R^n$  is continuous with boundary conditions

$$x(0) = x(2\pi)$$

$$\dot{x}(0) = \dot{x}(2\pi)$$

## 2. Preliminaries

**Definition 2.1.** Consider the general non-linear differential equation of the form

$$\dot{x} = f(t, x(t)) \tag{2.1}$$

where  $f: R \times R^n \rightarrow R^n$  is continuous. The function  $f$  in Equation (2.1) is said to be  $T$  periodic if for every  $(t, x) \in R \times R^n$  and some  $T > 0$ .  $f(t, x) = f(t + T, x)$  and  $f(t, x) \neq f(t + T^*, x)$  for all  $T^* < T$ .

**Definition 2.2.** A solution  $x$  of Equation (2.1) defined on  $R$  such that  $x(t + T) = x(t)$  for all  $t \in R$  is called  $T$  periodic solution or  $T$  periodic solution.

**Definition 2.3.** Let  $E, F$  be Banach spaces,  $U$  an open subset of  $E$  and let  $x_0 \in U$ . Let  $f: U \rightarrow F$  be a mapping of  $U$  into  $F$ .  $f$  is said to be Fréchet differentiable at  $x_0$  if there exists a continuous linear mapping;  $L: E \rightarrow F$  such that

$$\lim_{x \rightarrow x_0} \frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = 0$$

**Definition 2.4.** Let  $E, F, G$  be Banach spaces, let  $U = u_1 \times u_2 \in E \times F$  be open set and  $f: U \rightarrow G$  be a mapping of  $U$  into  $G$ , with  $(x_0, y_0) \in U$ .  $f$  is said to be Fréchet differentiable with respect to the first variable  $x$  at  $(x_0, y_0)$  if the following conditions hold.

- 1) There exists a continuous linear mapping  $L_1: E \rightarrow G$  such that  $\forall \varepsilon > 0 \exists \delta > 0$  such that  $\forall x, \bar{x} \in u_1$  then  $\|x - x_0\| \leq \delta$ ,  $\|\bar{x} - x_0\| \leq \delta$  and  $\forall y \in u_2$  such that  $\|y - y_0\| \leq \delta$ , it follows that.
- 2)  $\|f(x, y) - f(\bar{x}, y) - L_1(x - \bar{x})\| \leq \varepsilon \|x - \bar{x}\|$ .
- 3) The mapping  $y \rightarrow f(x_0, y)$  is continuous at  $y_0$ .

**Proposition 2.5.** Then condition (1) of the definition 2.4 is satisfied if the

partial Frechet derivative  $f_1^l(x_0, y_0)$  exists for  $(x, y)$  in a neighbourhood of  $(x_0, y_0)$  and if the mapping  $(x, y) \rightarrow f_1^l(x_0, y_0)$  is continuous at  $(x_0, y_0)$ .

**Proposition 2.6.** If  $f$  is Fréchet differentiable with respect to the first variable at  $(x_0, y_0)$ , it is Fréchet differential with respect to this variable at  $(x_0, y_0)$  with the same  $L_1$ . Moreover, this is unique.  $L_1$  is called the strong partial Frechet derivative with respect to the first variable at  $(x_0, y_0)$  and denoted by  $f_1^l(x_0, y_0)$ .

**Lemma 2.7.** (The Banach fixed point theorem) Let  $E$  be a Banach space and  $f: E \rightarrow E$  be a contraction mapping, then  $f$  has a unique fixed point in  $E$ , i.e. there exists a unique  $x \in E$  such that  $f(x) = x$ .

**Lemma 2.8.** (The implicit function theorem) Let  $E, F, G$  be Banach spaces and let  $U_1 \subset E, U_2 \subset F$ . Set  $U = U_1 \times U_2$ . For arbitrary  $(x_0, y_0) \in U$ , let  $f: U \rightarrow G$  be a mapping satisfying the following conditions.

- 1)  $f(x_0, y_0) = 0$ .
- 2)  $f$  is Fréchet differentiable with respect to the first variable at  $(x_0, y_0)$ .
- 3)  $f_1^l(x_0, y_0): E \rightarrow G$  is a linear homeomorphism.

Then there exists a neighborhood  $V_1 \times V_2 \subset U_1 \times U_2$  of  $(x_0, y_0)$  and a unique mapping  $\zeta: V_1 \rightarrow V_2$  such that for each  $y \in V_2$  the equation  $f(x, y) = 0$  has in  $V_1$ , the unique solution  $f(\zeta(y), y) = 0$ : Moreover,  $\zeta$  is continuous at  $y_0$ .

**Lemma 2.9.** If  $X$  and  $Y$  are Banach spaces and  $A \in B(X, Y)$  with  $R(A) = Y$  and  $N(A) = \{0\}$ , then  $A^{-1} \in B(Y, X)$  where  $N(A)$  is the Null spaces of  $A$  and  $R(A)$  is the range space of  $A$ .  $B(X, Y)$  is the space of bounded linear transformations from  $X$  to  $Y$ .

### 3. Main Result

We present in this section, the main result of this paper.

**Theorem 3.1.** Let  $C_{2\pi}^2 = \{x: [0, T] \rightarrow \mathbb{R}^n : x \text{ is a class of } C^2\}$  and equipped with the usual uniform norm  $\|x\|_\infty = \max \{|x(t)|, |\dot{x}(t)|, |\ddot{x}(t)|\}$   $C = \{x: [0, T] \rightarrow \mathbb{R} : x \text{ is continuous}\}$  with the usual norm,  $J = [0, T]$ ,  $J_{2\pi} = [0, 2\pi]$ .

Then, Equation (1.2) is equivalent to

$$f(x, h) = 0 \text{ in } C_{2\pi}^2 \times C \quad (3.1)$$

where  $f: C_{2\pi}^2 \times C \rightarrow C$  is defined by

$$f(x, h) = \ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 - h \quad (3.2)$$

**Proof:** We first remark that with the norm defined above,  $C_{2\pi}^2$  is a Banach space. The strategy for the proof involves application of the implicit function theorem to the function  $f$  defined in Equation (3.1). We split the proof into steps.

**Step 1:**  $f(0, 0) = 0$ . This follows trivially from the definition of  $f$ : hence

$$f(0, h) = -h \quad (3.3)$$

**Step 2:**  $f$  is Fréchet differentiable with respect to  $x$  at  $(0; 0)$ . Observe that

$$\begin{aligned}
 & |f(x, h) - f(\bar{x}, h) - f_x(0, 0)(x, \bar{x})| \\
 &= \left| \dot{x} + c\dot{x} + ax + bx^2 + 2x^3 - h - (\ddot{x} + c\dot{\bar{x}} + a\bar{x} + b\bar{x}^2 + 2\bar{x}^3 - h) \right| \\
 &= \left| (\ddot{x} - \ddot{\bar{x}}) + c(\dot{x} - \dot{\bar{x}}) + a(x - \bar{x}) + b(x^2 - \bar{x}^2) + 2(x^3 - \bar{x}^3) \right| \\
 &\leq |x - \bar{x}| \cdot \left| b(x + \bar{x}) + 2(x^2 + x\bar{x} + \bar{x}^2) \right| \\
 &\leq |x - \bar{x}| b(|x| + |\bar{x}|) + 2|x^2 + x\bar{x} + \bar{x}^2| \\
 &\leq |x - \bar{x}| b(|x| + |\bar{x}|) + 2(|x|^2 + |x\bar{x}| + |\bar{x}|^2) \\
 &\leq |x - \bar{x}| (2bp + 6p^2), \text{ with } |x| \leq p, |\bar{x}| \leq p
 \end{aligned}$$

Consequently,

$$|f(x, h) - f(\bar{x}, h) - f_x(0, 0)(x, \bar{x})| \rightarrow 0, \text{ as } |x - \bar{x}| \rightarrow 0 \tag{3.4}$$

Combining (3.3) and (3.4), we obtain that  $f$  is Fréchet differentiable with respect to the first variable at  $(0, 0)$ .

**Step 3:**  $f_x(0, 0): C_{2\pi}^2 \rightarrow C$  defined by  $z \rightarrow \ddot{z} + c\dot{z} + az$  is a linear homeomorphism.

The mapping  $f_x(0, 0): z \rightarrow \ddot{z} + c\dot{z} + az$  is linear and continuous and hence bounded. It is also an onto mapping. Linear homeomorphism would have been established if the mapping is shown to be one to one. This is equivalent to requiring that

$$\ddot{z} + c\dot{z} + az = 0 \tag{3.5}$$

with

$$z(0) = z(2\pi) \text{ and } \dot{z}(0) = \dot{z}(2\pi) \tag{3.6}$$

be non-critical.

It suffices to place appropriate conditions on the constants  $a, c$  such that Equation (3.5) is solvable. The auxiliary equation of (3.5) is  $\lambda^2 + c\lambda + a = 0$ .

**Case I:**

If  $c = 0$ , and  $a = k^2$  where  $k$  is a natural number, then  $\lambda = \pm ik$  and

$$z(t) = c_1 \cos kt + c_2 \sin kt \tag{3.7}$$

for arbitrary constants  $c_1$  and  $c_2$ . Clearly

$$z(0) = z(2\pi) \text{ and } \dot{z}(0) = \dot{z}(2\pi) \tag{3.8}$$

and the solution is non-trivial.

**Case II:**

If  $c = 0$  and  $a \neq k^2$ , then condition

$$\dot{z}(0) = \dot{z}(2\pi) \tag{3.9}$$

is satisfied only by the trivial solution  $z = 0$ .

**Case III:**

If  $c \neq 0$  and  $a \neq 0$ , only the trivial solution exists. Most generally, put

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4a}}{2} = u + iv \tag{3.10}$$

for some real numbers  $u, v$ .

- 1) Choose  $c$  and  $a$  such  $u \neq 0$ .
- 2) Choose  $c$  and  $a$  such that  $u \neq 0, v = 0$  then  $z \rightarrow \ddot{z} + c\dot{z} + az$  is non-critical [19].
- 3) Choose  $c$  and  $a$  such that  $I - x(2\pi)$  is non-singular where  $x(t)$  is fundamental matrix of Equation (3.5) with  $x(0) = I$  the  $2 \times 2$  identity matrix [20].

Thus with any of these conditions imposed, one deduces the one to oneness of  $f_x^l(0, 0)$ . Hence by Lemma 2.8  $[f_x^l(0, 0)]^{-1}$  exists as a bounded linear operator. Linear homeomorphism of  $f_x^l(0, 0)$  follows. Existence of a unique solution is now assured by the implicit function theorem.

### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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