

# Integral $\Phi_0$ -Stability of Impulsive Differential Equations

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## Abstract

In this paper, the notions of integral  $\phi$ -stability of ordinary impulsive differential equations are introduced. The definition of integral  $\phi$ -stability depends significantly on the fixed time impulses. Sufficient conditions for integral  $\phi$ -stability are obtained by using comparison principle and piecewise continuous cone valued Lyapunov functions. A new comparison lemma, connecting the solutions of given impulsive differential system to the solution of a vector valued impulsive differential system is also established.

## Keywords

Integral  $\phi$ -Stability, Cone Valued Lyapunov Functions, Impulsive Differential Equations, Fixed Time Impulses

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## 1. Introduction

Impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biological systems, industrial robotics, optimal control, bio-technology and so forth. In view of the vast applications, the fundamental and qualitative properties *i.e.* stability, boundedness etc. of such equations are studied extensively in past decades. Several types of stability have been defined and established in literature by academicians for impulsive ordinary differential equations. Various techniques such as scalar valued piecewise continuous Lyapunov functions, vector valued piecewise continuous Lyapunov functions, Rajumikhin method, comparison principle etc. have been employed to establish stability results.

To the best of our knowledge, the concept of integral stability and  $\phi_0$ -stability were introduced for ordinary differential equations by Lakshmikantham in 1969 [1] and by Akpan in 1992 [2] respectively. Later, these sta-

bilities were developed in [3] and [4] by Akpan, Soliman and Abdalla but for ordinary differential equations. In 2010, Integral stability was established for impulsive functional differential equations by Hristova. Motivated by these works, in this paper, we introduce and establish integral  $\phi_0$ -stability for impulsive ordinary differential equations:

$$\left. \begin{aligned} x' &= f(t, x), & t &\neq t_i \\ \Delta x &= I_i(x), & t &= t_i \\ x(t_0) &= x_0 \end{aligned} \right\} \tag{1}$$

where,  $i \in N$ ,  $t \in R^+$ ,  $I_i(x) = x(t_i^+) - x(t_i)$   $x \in R^n$ ,  $f: R^+ \times R^n \rightarrow R^n$ ,  $0 = t_0 < t_1 < t_2 < t_3 < \dots < \infty$  and  $I_i: R^n \rightarrow R^n$  are a sequence of instantaneous impulse operators and have been used to depict abrupt changes such as shocks, harvesting, natural disasters etc. and  $K$  is a cone defined in Section 2.

The paper is organized as follows:

In Section 2, some preliminaries notes and definitions are given. In Section 3, a new comparison lemma, connecting the solutions of given impulsive ordinary differential system to the solution of a vector valued impulsive differential system is worked out. This lemma plays an important role in establishing the main results of the paper. Sufficient conditions for integral  $\phi_0$ -stability are obtained by employing comparison principle and piecewise continuous cone valued Lyapunov functions.

## 2. Preliminaries

Let  $R^n$  denote the  $n$ -dimensional Euclidean space with any convenient norm  $\| \cdot \|$  and the scalar product  $(x, y) \leq \|x\| \|y\|$ ,  $R^+ = [0, \infty)$ ,  $J = [t_0, \infty)$ ,  $R = (-\infty, \infty)$ .

For any  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n) \in R^n$ , we will write  $x \leq y$  iff  $x_i \leq y_i$  for all  $i = 1, 2, 3, \dots$

Let  $x(t) = x(t; t_0, x_0)$  be the solution of system (1), having discontinuities of the first type (left continuous) at the moments when they meet the hyper planes  $t = t_i$ .

Together with system (1), let us consider, its perturbed IDS:

$$\left. \begin{aligned} x' &= f(t, x) + f^*(t, x), & t &\neq t_i \\ \Delta x &= I_i(x) + I_i^*(x), & t &= t_i \\ x(t_0) &= x_0 \end{aligned} \right\} \tag{2}$$

where,  $f^*(t, x): R^+ \times R^n \rightarrow R^n$ ,  $I_i^*(x): R^n \rightarrow R^n$ .

Let  $f(t, 0) = f^*(t, 0) = 0$ ,  $I_i(0) = I_i^*(0) = 0$  ( $i \in N$ ) so that the trivial solution of (1) and (2) exists.

Let us define the following:

**Definition 1.** A proper subset  $K$  of  $R^n$  is called a cone if (i)  $\lambda K \subset K$ ,  $\lambda \geq 0$  (ii)  $K + K \subset K$  (iii)  $K = \bar{K}$  (iv)  $K^0 \neq \emptyset$  (v)  $K \cap \{-K\} = \{0\}$ , where  $K^0$  and  $\bar{K}$  are interior and closure of  $K$  respectively.  $\partial K$  denotes the boundary of  $K$ .

**Definition 2.** The set  $K^* = \{ \phi \in R^n : (\phi, x) \geq 0 \forall x \in K \}$  is called the adjoint cone if it satisfies the properties (i)-(v) of definition 1.

The set  $x \in \partial K$  iff  $(\phi, x) = 0$  for some  $\phi \in K_0^*$ ,  $K_0 = K - \{0\}$ .

**Definition 3.** A function  $g: D \rightarrow R^n$ ,  $D \subset R^+ \times R^n$  is said to be quasi monotone relative to the cone  $K$  if for each  $t \in R^+$ ,  $u, v \in D$  and  $v - u \in \partial K$  imply that there exists  $\phi \in K_0^*$  such that  $(\phi, v - u) = 0$  and  $(\phi, g(t, v) - g(t, u)) \geq 0$ .

Consider the following sets:

$$\mathcal{K} = \{ a \in C[R^+, R^+] : a(0) = 0, a(r) \text{ is strictly increasing in } r \}$$

$$\begin{aligned}
 \mathcal{CK} &= \{b \in C[R^+ \times R^+, R^+] : b(t, \cdot) \in \mathcal{K} \text{ and for any fixed } t \in [0, \infty)\} \\
 \mathcal{PC}[R^+ \times R^n, R^n] &= \{f : R^+ \times R^n \rightarrow R^n : f \in C[(t_i, t_{i+1}) \times R^n, R^n], i = 1, 2, 3, \dots\} \\
 S(\rho, \phi_0) &= \{(t, x) \in R^+ \times K : (\phi_0, x) < \rho, \rho > 0\} \\
 S^c(\rho, \phi_0) &= \{(t, x) \in R^+ \times K : (\phi_0, x) \geq \rho, \rho > 0\} \\
 \Omega(t, T, \rho) &= \{x \in R^n : (t, x) \in S(\rho, \phi_0), s \in [t, t+T]\}.
 \end{aligned}$$

**Definition 4.** A function  $V : R^+ \times R^n \rightarrow K$  is said to belong to class  $\mathcal{L}$  if:

1.  $V(t, x)$  is a continuous function in  $G_i = (t_{i-1}, t_i), \forall u \in S(\rho, \phi_0)$ ;
2.  $V(t, x)$  is Lipschitz continuous relative to cone  $K$ , in its second argument;
3. For each  $i \in N, \lim_{t \rightarrow t_i^-} V(t, x) = V(t_i, x) = V(t_i, x)$  and  $\lim_{t \rightarrow t_i^+} V(t, x)$  exist.

And for  $t \neq t_k : k = 1, 2, 3, \dots$  we define derivative of the function  $V(t, x)$  along the trajectory of the system

$$(1) \text{ by } D^+ V_{(1)}(t, x) = \limsup_{h \rightarrow 0} \frac{1}{h} [V(t+h, x+h f(t, x)) - V(t, x)].$$

Now referring [5], let us define the following:

**Definition 5.** Let  $\phi_0 \in K_0^*$ . The function  $V(t, x) \in \mathcal{L}$  is said to be  $\phi_0$ -weakly decrescent, if there exists a  $\delta > 0$  and a function  $a \in \mathcal{CK}$  such that the inequality  $(\phi_0, x) < \delta$  implies that  $(\phi_0, V(t, x)) < a(t, (\phi_0, x))$ .

**Definition 6.** Let  $\phi_0 \in K_0^*$ . The function  $V(t, x) \in \mathcal{L}$  is said to be  $\phi_0$ -strongly decrescent, if there exists a  $\delta > 0$  and a function  $a \in \mathcal{K}$  such that the inequality  $(\phi_0, x) < \delta$  implies that  $(\phi_0, V(t, x)) < a((\phi_0, x))$ .

Throughout in the paper it was assumed that  $\|\phi_0\| \neq 0$ .

Let us consider the following comparison impulsive differential systems (referring [3] for Ordinary differential systems)

$$\left. \begin{aligned}
 u' &= g_1(t, u) & t \neq t_i \\
 \Delta u &= \xi_i(u(t_i)) & t = t_i \\
 u(t_0) &= u_0
 \end{aligned} \right\} \tag{3}$$

and

$$\left. \begin{aligned}
 w' &= g_2(t, w) & t \neq t_i \\
 \Delta w &= \eta_i(w(t_i)) & t = t_i \\
 w(t_0) &= w_0
 \end{aligned} \right\} \tag{4}$$

along with its perturbed system

$$\left. \begin{aligned}
 w' &= g_2(t, w) + p(t) & t \neq t_i \\
 \Delta w &= \eta_i(w(t_i)) + \gamma_i(t_i) & t = t_i \\
 w(t_0) &= w_0
 \end{aligned} \right\} \tag{5}$$

where  $g_1 \in PC[R^+ \times K \rightarrow R^n]$  is quasi monotone non decreasing in its second argument and  $\xi_i : K \rightarrow K$  is quasi monotone non decreasing satisfying  $g_1(t, 0) = 0, \xi_i(0) = 0, g_2 \in PC[R^+ \times K \rightarrow R^n], \eta_i : K \rightarrow K, p : R^+ \rightarrow R^n, \gamma_i : R^+ \rightarrow K, g_2(t, 0) = \eta_i(0) = 0, p(t)$  and  $\gamma_i$  are to be chosen later such that  $q(0) = \gamma_i(0) = 0$ .

**Definition 7.** The zero solution of (1) is said to be  $\phi_0$ -stable, if for every  $\alpha > 0$  and for any  $t_0 \in J$  there exists a positive function  $\beta = \beta(t_0, \alpha) \in \mathcal{K}$ , which is continuous in  $t_0$  for each  $\alpha$  such that the inequality  $(\phi_0, x_0) < \beta$  implies that  $(\phi_0, r(t)) < \alpha, t \geq t_0$  where  $\phi_0 \in K_0^*$  and  $r(t)$  is the maximal solution of (1) relative to the cone  $K$ .

**Definition 8.** The zero solution of (1) is said to be integrally stable, if for every  $\alpha \geq 0$  and for any  $t_0 \in J$  there exists a positive function  $\beta = \beta(t_0, \alpha) \in \mathcal{K}$ , which is continuous in  $t_0$  for each  $\alpha$  such that for any solution  $x^*(t; t_0, x_0)$  of perturbed system (2), the inequality  $\|x^*\| < \beta$  holds provided that  $\|x_0\| \leq \alpha$  and for every  $T > 0$ , the perturbations  $f^*(t, x)$  and  $I_i^*(x)$ ,  $i = 1, 2, 3, \dots$  of RHS of (2) satisfy

$$\int_{t_0}^{t_0+T} \sup_{x: \|x^*\| < \beta} \|f^*(s, x^*)\| ds + \sum_{k: t_0 < t_k \leq t_0+T} \sup_{x: \|x^*(t_k)\| < \beta} \|I_i^*(x^*)\| \leq \alpha.$$

**Definition 9.** The trivial solution of (1) is said to be integrally  $\phi_0$ -stable, if for every  $\alpha \geq 0$  and for any  $t_0 \in J$  there exists a positive function  $\beta = \beta(t_0, \alpha) \in \mathcal{K}$ , which is continuous in  $t_0$  for each  $\alpha$  such that for any solution  $x^*(t; t_0, x_0)$  of perturbed system (2) and for  $\phi_0 \in K_0^*$ , the inequality  $(\phi_0, x^*) < \beta$  holds provided that

$$(\phi_0, x_0) \leq \alpha \tag{6}$$

and, for every  $T > 0$ , the perturbations  $f^*(t, x)$  and  $I_i^*(x)$ ,  $i = 1, 2, 3, \dots$  of RHS of (2) satisfy

$$\int_{t_0}^{t_0+T} \sup_{x^* \in \Omega(t, T, \beta)} \|f^*(s, x^*)\| ds + \sum_{k: t_0 < t_k \leq t_0+T} \sup_{x: (\phi_0, x^*(t_k)) < \beta} \|I_i^*(x^*)\| \leq \alpha. \tag{7}$$

### 3. Main Results

**Lemma 1:** Consider the comparison system (3) and assume that

- (i)  $g_1 \in PC[R^+ \times K \rightarrow R^n]$  where  $g_1$  is quasi monotone non decreasing in its second argument;
- (ii)  $V \in \mathcal{L}$  such that  $V \in PC[S(\rho, \phi_0) \times K]$  and satisfies

$$(\phi_0, D^+ V(t, x)) \leq (\phi_0, g_1(t, V(t, x))) : t \neq t_k : k = 1, 2, \dots$$

- (iii)  $\xi_k \in \mathcal{K}$  such that  $(\phi_0, V(t+0, x + I_k(x))) \leq \xi_k((\phi_0, V(t, x)))$  for  $t = t_k : k = 1, 2, 3, \dots$

Let  $r(t; t_0, u_0)$  be the maximal solution of (3) existing on  $J$ . Then for any solution  $x(t; t_0, x_0)$  of (1) existing on  $J$ , we have  $(\phi_0, V(t, x(t))) \leq (\phi_0, r(t; t_0, u_0))$  provided that  $(\phi_0, V(t_0+0, x_0)) \leq (\phi_0, u_0)$ .

**Proof:** Let  $x(t; t_0, x_0)$  be the solution of (1) existing for  $t \geq t_0$  such that  $(\phi_0, V(t_0+0, x_0)) \leq (\phi_0, u_0)$ .

Define  $m(t) = (\phi_0, V(t, x(t)))$  for  $t \neq t_k$  such that  $m(t_0^+) = (\phi_0, V(t_0+0, x_0)) \leq u_0$ . Then for small  $h > 0$ , we have

$$\begin{aligned} & m(t+h) - m(t) \\ &= (\phi_0, V(t+h, x(t+h))) - (\phi_0, V(t, x(t))) = (\phi_0, V(t+h, x(t+h)) - V(t, x(t))) \\ &= (\phi_0, V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t))) + V(t+h, x(t) + hf(t, x(t))) - V(t, x(t))) \\ &= (\phi_0, V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t)))) \\ & \quad + (\phi_0, V(t+h, x(t) + hf(t, x(t))) - V(t, x(t))) \\ &\leq \|\phi_0\| \|V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t)))\| + h \left\{ \phi_0, \left\{ \frac{V(t+h, x(t) + hf(t, x(t))) - V(t, x(t))}{h} \right\} \right\} \\ &\leq M \|\phi_0\| \|x(t+h) - x(t) - hf(t, x(t))\| + h \left\{ \phi_0, \left\{ \frac{V(t+h, x(t) + hf(t, x(t))) - V(t, x(t))}{h} \right\} \right\}, \end{aligned}$$

where  $M$  is the Lipschitz constant in  $(t_{k-1}, t_k]$ .

Therefore we have

$$\begin{aligned} \frac{m(t+h)-m(t)}{h} &\leq \|\phi_0\| M \left\| \frac{x(t+h)-x(t)-hf(t,x(t))}{h} \right\| + \left( \phi_0, \left\{ \frac{V(t+h,x(t)+hf(t,x(t)))-V(t,x(t))}{h} \right\} \right) \\ \frac{m(t+h)-m(t)}{h} &\leq \|\phi_0\| M \left\| \frac{x(t+h)-x(t)}{h} - f(t,x(t)) \right\| + \left( \phi_0, \left\{ \frac{V(t+h,x(t)+hf(t,x(t)))-V(t,x(t))}{h} \right\} \right) \\ &\Rightarrow D^+m(t) \leq (\phi_0, D_{(1)}^+V(t,x(t))) \leq (\phi_0, g_1(t,V(t,x))) \end{aligned}$$

Also  $m(t_0^+) \leq (\phi_0, u_0)$  and  $m(t_k^+) = (\phi_0, V(t_k^+, x(t_k^+))) = (\phi_0, V(t_k^+, x(t_k)) + I(x(t_k))) \leq (\phi_0, \xi_k(V(t_k, x)))$ .

Then by theorem (1.4.3) in [6], we observe the desired inequality

$$(\phi_0, V(t,x(t))) \leq (\phi_0, r(t:t_0, u_0)) \text{ for all } t \geq t_0.$$

**Theorem 1:** Let us assume the following:

1. Let  $f \in PC[R^+ \times R^n \rightarrow R^n]$  and  $I_k \in C[R^n \rightarrow R^n] : k = 1, 2, 3, \dots$
2. There exist  $V_1(t,x) \in \mathcal{L}, V_1(t,0) = 0$  such that
  - (i)  $V_1$  is  $\phi_0$ -weakly decrescent
  - (ii) For  $t \neq t_k : k = 1, 2, 3, \dots$  the inequality

$$(\phi_0, D_{(1)}^+V_1(t,x)) \leq (\phi_0, g_1(t, V_1(t,x))) \text{ holds for all } (t,x) \in S(\rho, \phi_0), t \neq t_k$$

where  $g_1$  monotone non decreasing in its second argument

- (iii)  $(\phi_0, V_1(t, x + I_k(x))) \leq (\phi_0, \xi_k(V_1(t,x)))$  for all  $(t,x) \in S(\rho, \phi_0), t = t_k : k = 1, 2, 3, \dots$  where  $\xi_k$

is monotone non decreasing, satisfying  $\xi_k(x) \geq x$

3. For any number  $\mu > 0$  there exists  $V_2^{(\mu)}(t,x) \in \mathcal{L}, V_2(t,0) = 0$  such that

- (iv)  $b((\phi_0, x)) \leq (\phi_0, V_2^{(\mu)}(t,x)) \leq a((\phi_0, x))$  for  $(t,x) \in S(\rho, \phi_0) \cap S^C(\mu, \phi_0)$  where  $a, b \in \mathcal{K}$

- (v) For  $t \neq t_k : k = 1, 2, 3, \dots$  the inequality

$$(\phi_0, D_{(1)}^+V_1(t,x) + D_{(1)}^+V_2^{(\mu)}(t,x)) \leq (\phi_0, g_2(t, V_1(t,x) + V_2^{(\mu)}(t,x)))$$

holds for any

$$(t,x) \in S(\rho, \phi_0) \cap S^C(\mu, \phi_0)$$

where  $g_2 \in PC[R^+ \times K \rightarrow R^n]$  is monotone non decreasing in its second argument.

- (vi)  $(\phi_0, V_1(t_k^+, x + I_k(x)) + V_2^{(\mu)}(t_k^+, x + I_k(x))) \leq (\phi_0, \eta_k(V_1(t_k, x(t_k)) + V_2^{(\mu)}(t_k, x(t_k))))$  for

$$(t_k, x) \in S(\rho, \phi_0) \cap S^C(\mu, \phi_0), k = 1, 2, 3, \dots$$

where  $\eta_k \in K, \eta_k(x) \geq x$

4. The system (3) and (4) have solutions, for any initial point  $t_0 \geq 0$ .

5. For any initial point  $(t_0, x_0) \in R^+ \times R^n$ , the system (1) has solution.

Let the zero solution of (3) be  $\phi_0$ -stable, and scalar IDE (4) is integrally  $\phi_0$ -stable, then the system (1) will be integrally  $\phi_0$ -stable.

**Proof:** Since  $V_1(t,x) \in \mathcal{L}$  is  $\phi_0$ -weakly decrescent, therefore there exists a  $\rho_1 > 0$  ( $\rho_1 < \rho$ ) and a function  $\psi_1 \in C\mathcal{K}$  such that the inequality  $(\phi_0, x) < \rho_1$  implies that

$$(\phi_0, V_1(t,x)) < \psi_1(t, (\phi_0, x)) \tag{8}$$

where  $\phi_0 \in K_0^*$ .

Let  $t_0 \geq 0$  be a fixed time. Choose a number  $\alpha > 0$  such that  $\alpha < \rho_1$ .

As  $V_1(t, x), V_2^{(\mu)}(t, x) \in \mathcal{L}$ , there exist Lipschitz constants  $M_1$  and  $M_2$  of  $V_1(t, x)$  and  $V_2^{(\mu)}(t, x)$  respectively. Let  $(M_1 + M_2)\|\phi_0\|\alpha = \alpha_1$ .

As the zero solution of (3) is  $\phi_0$ -stable, therefore for every  $\alpha_1 > 0$  and for any  $t_0 \in J$  there exists a positive function  $\delta_1 = \delta_1(t_0, \alpha_1)$  for each  $\alpha_1$  such that the inequality  $(\phi_0, u_0) < \delta_1$  implies that

$$(\phi_0, r(t; t_0, u_0)) < \frac{\alpha_1}{2}, \quad t \geq t_0 \tag{9}$$

where  $r(t; t_0, u_0)$  is the maximal solution of (3)

As  $\psi_1 \in C\mathcal{K}$ , there exists  $\delta_2 = \delta_2(\delta_1) > 0$  and hence  $\delta_2 = \delta_2(t_0, \alpha_1)$  such that

$$(\phi_0, u) < \delta_2 \Rightarrow \psi_1(t, (\phi_0, u)) < \delta_1. \tag{10}$$

Again in view of the fact that the perturbations in (5), depend only on  $t$  and system (4) is  $\phi_0$ -integrally stable, there exists a function  $\beta_1 = \beta_1(t_0, \alpha_1) \in \mathcal{K}$ , continuous in  $t_0$  for each  $\alpha_1$  (take in particular  $\alpha_1 \leq b(\alpha_1)$ ) such that for every solution  $w^*(t; t_0, w_0)$  of perturbed system (5), the inequality

$$(\phi_0, w^*(t; t_0, w_0)) < \beta_1 \tag{11}$$

holds provided that  $(\phi_0, w_0) \leq \alpha_1$  and for every  $T > 0$ , the perturbation terms  $p(t)$  and  $\gamma_k$  satisfy

$$\int_{t_0}^{t_0+T} |p(s)| ds + \sum_{k: t_0 < t_k \leq t_0+T} |\gamma_k(t_k)| \leq \alpha_1. \tag{12}$$

Since  $b \in \mathcal{K}$ ,  $\lim_{s \rightarrow \infty} b(s) = \infty$  let us choose  $\beta = \beta(\beta_1) > 0$  such that  $b(\beta) \geq \beta_1$  and  $\beta > \psi_2(\alpha)$  where  $\psi_2 \in \mathcal{K}$  is a function satisfying  $\psi_2(\alpha) < \rho_1$ .

Select  $\delta_3 = \delta_3(\alpha_1, \beta)$ ,  $\alpha < \delta_3 < \min\{\delta_2, \rho_1\}$  such that the inequalities

$$a(\delta_3) < \frac{\alpha_1}{2} \quad \text{and} \quad \psi_2(\delta_3) < \beta \quad \text{hold} \tag{13}$$

Let  $x^*(t; t_0, x_0)$  be the solution of (2). Now we will prove that if the inequalities (6) and (7) are satisfied then

$$(\phi_0, x^*) < \beta, \quad t \geq t_0 \tag{14}$$

If possible let this be false. Therefore there exists a point  $t^* > t_0$  such that

$$(\phi_0, x^*(t^*)) \geq \beta \quad \text{and} \quad (\phi_0, x^*(t)) < \beta, \quad t \in [t_0, t^*) \tag{15}$$

**Case 1:** Let  $t^* \neq t_k$  for any  $k = 1, 2, 3, \dots$ . Then the solution  $x^*(t; t_0, x_0)$  is continuous at  $t^*$ . Therefore  $(\phi_0, x^*(t^*)) = \beta$

In this case first we note that  $(\phi_0, x^*(t^*)) > \delta_3$ .

For if  $(\phi_0, x^*(t^*)) \leq \delta_3$ , then by the choice of  $\delta_3$  we get  $\psi_2(\phi_0, x^*(t^*)) < \beta$  which is a contradiction to (15).

Now let us consider the interval  $(t_0, t^*)$

**Subcase 1.1:** Let there exists  $t_0^* \in (t_0, t^*)$ ,  $t_0^* \neq t_k : k = 1, 2, 3, \dots$  such that  $\delta_3 = (\phi_0, x^*(t_0^*))$  and

$$(t, x^*(t)) \in S(\beta, \phi_0) \cap S^C(\rho, \phi_0) : t \in [t_0^*, t^*)$$

If  $r_1(t; t_0^*, u_0)$  is the maximal solution of (3) with  $u_0 = V_1(t_0^*, x^*(t_0^*))$ , then in view of the assumptions (ii) and (iii) of theorem, using lemma 1, we obtain

$$(\phi_0, V_1(t, x(t; t_0^*, x_0))) \leq (\phi_0, r_1(t; t_0^*, u_0)) : t \in [t_0^*, t^*) \tag{16}$$

where  $x(t:t_0^*, x_0)$  is a solution of (1), starting at  $t_0^*$ .

As  $\delta_3 = (\phi_0, x^*(t_0^*))$  is chosen therefore we have  $(\phi_0, x^*(t_0^*)) = \delta_3 < \delta_2$  and  $(\phi_0, u_0) = (\phi_0, V_1(t_0^*, x^*(t_0^*)))$  by using (8) and then (10), we get

$$(\phi_0, u_0) = (\phi_0, V_1(t_0^*, x^*(t_0^*))) \leq \psi_1(t_0^*, (\phi_0, x^*(t_0^*))) < \delta_1$$

Now  $(\phi_0, V_1(t, x(t:t_0^*, x_0))) \leq (\phi_0, r_1(t:t_0^*, u_0)) : t \in [t_0^*, t^*]$  by virtue of (9) gives:

$$(\phi_0, V_1(t, x(t:t_0^*, x_0))) \leq (\phi_0, r_1(t:t_0^*, u_0)) < \frac{\alpha_1}{2} \text{ for } t \in [t_0^*, t^*] \tag{17}$$

Now from inequality (13) and condition (iv) of theorem, we get

$$(\phi_0, V_2^{(\alpha)}(t_0^*, x^*(t_0^*))) \leq a((\phi_0, x^*(t_0^*))) < a(\delta_1) < \frac{\alpha_1}{2} \tag{18}$$

Let us define the function  $V : R^+ \times R^n \rightarrow K$ ,  $V \in \mathcal{L}$  by  $V(t, x) = V_1(t, x) + V_2^{(\alpha)}(t, x)$

Now, for  $t \in [t_0^*, t^*]$  and  $t \neq t_k$ ,  $(t, x) \in S(\beta, \phi_0) \cap S^C(\alpha, \phi_0)$ , in view of (v) of theorem and lipschitz condition on  $V_1$  and  $V_2^{(\alpha)}$ , we have

$$\begin{aligned} & (\phi_0, D_{(2)}^+ V(t, x)) = (\phi_0, D_{(2)}^+ V_1(t, x) + D_{(2)}^+ V_2^{(\alpha)}(t, x)) \\ & = \left( \phi_0, \limsup_{h \rightarrow 0} \frac{1}{h} \{V_1(t+h, x+h[f(t, x) + f^*(t, x)]) - V_1(t, x)\} \right. \\ & \quad \left. + \limsup_{k \rightarrow 0} \frac{1}{k} \{V_2^{(\alpha)}(t+k, x+k[f(t, x) + f^*(t, x)]) - V_2^{(\alpha)}(t, x)\} \right) \\ & = \left( \phi_0, \limsup_{h \rightarrow 0} \frac{1}{h} \{V_1(t+h, x+h[f(t, x) + f^*(t, x)]) - V_1(t+h, x+hf(t, x))\} \right. \\ & \quad \left. + V_1(t+h, x+hf(t, x)) - V_1(t, x) \right) + \left( \phi_0, \limsup_{h \rightarrow 0} \frac{1}{k} \{V_2^{(\alpha)}(t+k, x+k[f(t, x) + f^*(t, x)]) \right. \\ & \quad \left. - V_2^{(\alpha)}(t+k, x+kf(t, x)) + V_2^{(\alpha)}(t+k, x+kf(t, x)) - V_2^{(\alpha)}(t, x) \right) \\ & = \left( \phi_0, \limsup_{h \rightarrow 0} \frac{1}{h} \{V_1(t+h, x+h[f(t, x) + f^*(t, x)]) - V_1(t+h, x+hf(t, x))\} \right) \\ & \quad + \left( \phi_0, \limsup_{h \rightarrow 0} \frac{1}{k} \{V_2^{(\alpha)}(t+k, x+k[f(t, x) + f^*(t, x)]) - V_2^{(\alpha)}(t+k, x+kf(t, x))\} \right) \\ & \quad + \left( \phi_0, \limsup_{h \rightarrow 0} \frac{1}{h} \{V_1(t+h, x+hf(t, x)) - V_1(t, x)\} \right) + \left( \phi_0, \limsup_{h \rightarrow 0} \frac{1}{k} \{V_2^{(\alpha)}(t+k, x+kf(t, x)) - V_2^{(\alpha)}(t, x)\} \right) \\ & \leq \|\phi_0\| M_1 \sup_{x \in \Omega(t_0^*, T^*, \beta)} \|f^*(t, x)\| + \|\phi_0\| M_2 \sup_{x \in \Omega(t_0^*, T^*, \beta)} \|f^*(t, x)\| + (\phi_0, D_{(1)}^+ V_1(t, x)) + (\phi_0, D_{(1)}^+ V_2^{(\alpha)}(t, x)) \\ & \leq (\phi_0, g_2(t, V_1(t, x) + V_2^{(\alpha)}(t, x))) + \|\phi_0\| (M_1 + M_2) \sup_{x \in \Omega(t_0^*, T^*, \beta)} \|f^*(t, x)\| \\ & = (\phi_0, g_2(t, V(t, x))) + \|\phi_0\| (M_1 + M_2) \sup_{x \in \Omega(t_0^*, T^*, \beta)} \|f^*(t, x)\|, \text{ where } T^* = t^* - t_0^* \end{aligned} \tag{19}$$

Again for  $t_k \in (t_0^*, t^*)$  such that  $(t_k, x) \in S(\beta, \phi_0) \cap S^C(\alpha, \phi_0)$ , by using condition (vi) of theorem and Lipschitz conditions on  $V_1$  and  $V_2^{(\alpha)}$ , we get

$$\begin{aligned}
 & \left( \phi_0, V(t_k^+, x + I_k(x) + I_k^*(x)) \right) \\
 &= \left( \phi_0, V(t_k^+, x + I_k(x)) + \left\{ V(t_k^+, x + I_k(x) + I_k^*(x)) - V(t_k^+, x + I_k(x)) \right\} \right) \\
 &= \left( \phi_0, V(t_k^+, x + I_k(x)) \right) + \left( \phi_0, V(t_k^+, x + I_k(x) + I_k^*(x)) - V(t_k^+, x + I_k(x)) \right) \\
 &= \left( \phi_0, V(t_k^+, x + I_k(x)) \right) \\
 &\quad + \left( \phi_0, V_1(t_k^+, x + I_k(x) + I_k^*(x)) + V_2^{(\alpha)}(t_k^+, x + I_k(x) + I_k^*(x)) - V_1(t_k^+, x + I_k(x)) - V_2^{(\alpha)}(t_k^+, x + I_k(x)) \right) \quad (20) \\
 &= \left( \phi_0, V(t_k^+, x + I_k(x)) \right) + \left( \phi_0, V_1(t_k^+, x + I_k(x) + I_k^*(x)) - V_1(t_k^+, x + I_k(x)) \right) \\
 &\quad + \left( \phi_0, V_2^{(\alpha)}(t_k^+, x + I_k(x) + I_k^*(x)) - V_2^{(\alpha)}(t_k^+, x + I_k(x)) \right) \\
 &\leq \left( \phi_0, \eta_k(V(t_k, x(t_k))) \right) + M_1 \|\phi_0\| \|I_k^*(x)\| + M_2 \|\phi_0\| \|I_k^*(x)\| = \left( \phi_0, \eta_k(V(t_k, x(t_k))) \right) \\
 &\quad + (M_1 + M_2) \|\phi_0\| \|I_k^*(x)\| \leq \left( \phi_0, \eta_k(V(t_k, x(t_k))) \right) + (M_1 + M_2) \|\phi_0\| \sum_{k:t_0 < t_k \leq t_0 + T} \sup_{x: (\phi_0, x(t_k)) < \beta} \|I_k^*(x)\|
 \end{aligned}$$

For the impulsive differential system (5) which is the perturbed system of (4), set the perturbations on RHS of (5) as

$$p(t) = (M_1 + M_2) \|\phi_0\| \sup_{x \in \Omega(t_0^+, T^+, \beta)} \|f^*(t, x)\| \quad \text{and} \quad \gamma_k(t_k) = (M_1 + M_2) \|\phi_0\| \sup_{x: (\phi_0, x(t_k)) < \beta} \|I_k^*(x)\|$$

Therefore (19) and (20) can be written as

$$\left( \phi_0, D_{(2)}^+ V(t, x) \right) \leq \left( \phi_0, g_2(t, V(t, x)) \right) + p(t)$$

and

$$\left( \phi_0, V(t_k + 0, I_k(x) + I_k^*(x)) \right) \leq \left( \phi_0, \eta_k(V(t_k, x(t_k))) \right) + \gamma_k(t_k).$$

If we consider the comparison system (5) with maximal solution  $r^*(t; t_0^*, w_0^*)$ , through the point  $(t_0^*, w_0^*)$  where  $w_0^* = V(t_0^*, x^*(t_0^*))$ , using (19), (20) and lemma 1, we get

$$\left( \phi_0, V(t, x^*(t; t_0, x_0)) \right) \leq \left( \phi_0, r^*(t; t_0^*, w_0^*) \right), \quad t \in H \cap [t_0^*, t^*]$$

where  $H$  is the interval of existence of maximal solution

$$r^*(t; t_0^*, w_0^*) \quad (21)$$

Now by using the inequality (7) for  $T = t^* - t_0^*$  in the interval  $t \in [t_0^*, t^*]$  and from the choice of  $\alpha_1$ ,

$$\begin{aligned}
 & \int_{t_0^*}^{t^*} p(s) ds + \sum_{k:t_0^* < t_k \leq t^*} \gamma_k \\
 &= \|\phi_0\| (M_1 + M_2) \int_{t_0^*}^{t^*} \sup_{x^* \in \Omega(t, T, \beta)} \|f^*(s, x)\| ds + \|\phi_0\| (M_1 + M_2) \sum_{k:t_0^* < t_k \leq t^*} \sup_{x: (\phi_0, x(t_k)) < \beta} \|I_k^*(x)\| \quad (22) \\
 &= \|\phi_0\| (M_1 + M_2) \left\{ \int_{t_0^*}^{t^*} \sup_{x^* \in \Omega(t, T, \beta)} \|f^*(s, x)\| ds + \sum_{k:t_0^* < t_k \leq t^*} \sup_{x: (\phi_0, x(t_k)) < \beta} \|I_k^*(x)\| \right\} < \alpha \|\phi_0\| (M_1 + M_2) < \alpha_1
 \end{aligned}$$

Let us choose a point  $T^* > t^*$  such that  $\int_{t_0^*}^{t^*} p(s) ds + \frac{1}{2}(T^* - t^*) p(t^*) < \alpha_1$ .



Now let us define a continuous function  $p^*(t): [t_0^*, \infty) \rightarrow R$  given by

$$p^*(t) = \begin{cases} p(t) & : t \in [t_0^*, t^*] \\ \frac{p(t^*)}{t^* - T^*}(t - T^*) & : t \in (t^*, T^*] \\ 0 & : t \geq T^* \end{cases}$$

and the sequence of numbers.

$$\gamma_k^* = \begin{cases} \gamma_k & : t_k \in (t_0^*, t^*] \\ 0 & : t_k > t^* \end{cases}$$

We see that if (7) holds then from (22), for every  $T > 0$

$$\int_{t_0^*}^{t_0^*+T} |p^*(s)| ds + \sum_{k: t_0 < t_k \leq t_0+T} |\gamma_k^*| < \alpha_1 \tag{23}$$

let  $r^{**}(t; t_0^*, w_0^*)$  be the maximal solution of (5), through the point  $(t_0^*, w_0^*)$  where the perturbations terms are defined by  $p^*(s)$  and  $\gamma_k^*$ . Note that here we have  $r^{**}(t; t_0^*, w_0^*) = r^*(t; t_0^*, w_0^*) ; t \in [t_0^*, t^*]$ .

From inequalities (17) and (18) we see,  $(\phi_0, V(t_0^*, x^*(t_0^*))) = (\phi_0, V_1(t_0^*, x^*(t_0^*)) + V_2^{(\alpha)}(t_0^*, x^*(t_0^*))) < \alpha_1$  i.e.

$$(\phi_0, w_0^*) \leq \alpha_1 \tag{24}$$

and hence from (11), we get

$$(\phi_0, r^{**}(t; t_0^*, w_0^*)) < \beta_1 \text{ for } t \geq t_0^*. \tag{25}$$

Now from the choice of  $\beta_1$ , inequalities (21), (25) and condition (iv) of statement of theorem, we get

$$\begin{aligned} b(\beta) &\geq \beta_1 > (\phi_0, r^{**}(t; t_0^*, w_0^*)) = (\phi_0, r^*(t; t_0^*, w_0^*)) \geq (\phi_0, V(t^*, x^*(t^*; t_0^*, w_0^*))) \\ &= (\phi_0, V_1(t^*, x^*(t^*; t_0^*, w_0^*)) + V_2^{(\alpha)}(t^*, x^*(t^*; t_0^*, w_0^*))) \geq (\phi_0, V_2^{(\alpha)}(t^*, x^*(t^*; t_0^*, w_0^*))) \\ &\geq b((\phi_0, x^*(t^*; t_0^*, w_0^*))) = b(\beta) \end{aligned}$$

which yields  $b(\beta) > b(\beta)$ , a contradiction and therefore the inequality (14) is valid for  $t \geq t_0$ .

**Subcase 1.2:** Let there exist a point  $t_k \in (t_0, t^*)$  for some  $k = 1, 2, 3, \dots$  such that  $\delta_3 = (\phi_0, x^*(t_k))$  and  $(t, x^*(t)) \in S(\beta, \phi_0) \cap S^c(\rho, \phi_0) : t \in [t_k, t^*)$ .

Choose  $\bar{\delta}_3$  satisfying  $\delta_3 < \bar{\delta}_3 < \beta$  with  $\bar{\delta}_3 = (\phi_0, x^*(t_0^*; t_0, x_0)) : t_0^* (\neq t_k) \in (t_0, t^*)$  Now if we take  $\bar{\delta}_3$  in place of  $\delta_3$  and repeat the proof of subcase 1.1 we arrive at contradiction that assures the validity of (14).

**Case 2:** If  $t^* = t_k$  for some  $k \in N$  then from (15),

$$\begin{aligned} (\phi_0, x^*(t_k)) &\geq \beta \text{ and } (\phi_0, x^*(t)) < \beta, t \in [t_0, t_k) \\ (\phi_0, x^*(t_k^+; t_0, x_0)) &= (\phi_0, x^*(t_k)) + (I_k + I_k^+)(x^*(t_k)) \end{aligned}$$

Let us select  $\beta = \beta(\beta_1) > 0$  such that  $b(\beta) \geq \sup_k (\phi_0, \eta_k(r^{**}(t_k; t_0, w_0^*)))$

Now by adopting the procedure as in case 1, we get the inequalities (21) and (25). Then by using these inequalities along with the conditions (iv) and (vi) of the statement of theorem, we have

$$\begin{aligned}
b(\beta) &\geq \sup_k \left( \phi_0, \eta_k \left( r^{**} \left( t_k; t_0, w_0^* \right) \right) \right) > \left( \phi_0, \eta_k \left( r^* \left( t_k; t_0, w_0^* \right) \right) \right) \geq \left( \phi_0, \eta_k \left( V \left( t_k, x^* \left( t_k \right) \right) \right) \right) \\
&\geq \left( \phi_0, V \left( t_k, x^* \left( t_k \right) \right) \right) \geq \left( \phi_0, V_2^{(\alpha)} \left( t_k, x^* \left( t_k \right) \right) \right) \geq b \left( \phi_0, x^* \left( t_k \right) \right) \geq b(\beta)
\end{aligned}$$

and that again is a contradiction. Therefore inequality (14) is valid.

Thus in all the cases, validity of (14) proves that system (1) is integrally  $\phi_0$ -stable.

#### 4. Conclusion

Results in [1] [4] [7] have been exploited and extended to establish the new type of stability *i.e.* integral  $\phi_0$ -stability for the impulsive differential systems. Sufficient conditions are obtained by employing comparison principle and piecewise continuous cone valued Lyapunov functions.

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