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Stechkin-Marchaud Type Inequalities in L_p for Linear Combination of Bernstein-Durrmeyer Operators

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Abstract—In this paper, we use the equivalence relation between K-functional and modulus of smoothness, and give the Stechkin-Marchaud-type inequalities for linear combination of Bernstein-Durrmeyer operators . Moreover, we obtain the inverse result of approximation for linear combination of Bernstein-Durrmeyer operators with $\omega_{\varphi^\lambda}^{2r}(f;x)$. Meanwhile we unify and extend some previous results.

Keywords- Bernstein-Durrmeyer operators; linear combination; K-functional;Stechkin-Marchaud-type inequalities; modulus of smoothness

1. Introduction and Main Results

Let $f \in L_p[0,1],(1\le p\le\infty)$. The Bernstein-Durrmeyer operator $D_n(f;x)$ ($n\in\mathbb{N}:=$ set of naturals) is defined as follows

$$D_n(f;x)=\sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_0^1 p_{n,k}(t)f(t)dt, \tag{1.1}$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$.

which was first introduced and investigated by Derrieinnic[1] in 1985. The Linear combination of Bernstein-Durrmeyer operators given by

$$O_{n,r}(f;x)=\sum_{i=0}^{2r-1} c_i(n)D_{n_i}(f;x), \tag{1.2}$$

where n_i and $c_i(n)$ satisfy:

$$i)\ n\le n_0\le n_1\le\cdots\le n_{2r-1}\le c_n,\qquad ii)\sum_{i=0}^{2r-1}c_i(n)=1,$$

$$iii)\sum_{i=0}^{2r-1}|c_i(n)|\le M,$$

$$iv)\sum_{i=0}^{2r-1}c_i(n)D_{n_i}((t-x)^m;x)=0,\ m=1,2,\cdots,2r-1. \tag{1.3}$$

Ditzian and Ivanov [2], Zhou [3], Guo and Li [4] studied the Linear combination of Bernstein-Durrmeyer operators, and obtained the characterization of approximation, the relationship of differential and modulus of smoothness for $O_{n,r}(f;x)$.

In this paper, we first establish Bernstein-type inequality with parameter λ for $O_{n,r}(f;x)$. After that, we use the equivalence relation between K-functional and modulus of smoothness, and give the Stechkin-Marchaud type inequalities in $f\in L_p[0,1]$ for linear combination of Bernstein-Durrmeyer operators. Moreover, we obtain the inverse result of approximation for linear combination of Bernstein-Durrmeyer operators with $\omega_{\varphi^\lambda}^{2r}(f;x)$. Meanwhile we unify and extend [2-4] results.

First, we introduce some useful definitions and notations.

Definition1.1. Let $\varphi^2(x)=x(1+x),\ 0\le\lambda\le1, 1\le p\le\infty.$

The modulus of smoothness by

$$\omega_{\varphi^\lambda}^{2r}(f;t)_p=\sup_{0\le h<t}\left\|\Delta_{h\varphi^\lambda}^{2r}f\right\|_p,$$

where

$$\Delta_h^r f(x)=\sum_{k=0}^{r}\binom{r}{k}(-1)^k f(x+(\tfrac{r}{2}-k)h),\left[x-\tfrac{rh}{2},x+\tfrac{rh}{2}\right]\subseteq[0,1],$$

otherwise $\Delta_h^r f(x)=0.$

The K-functional by

$$K_{\varphi^\lambda}^{2r}(f;t^{2r})_p=\inf_{g\in G}\left\{\|f-g\|_p+t^{2r}\left\|\varphi^{2r\lambda}g^{(2r)}\right\|_p\right\},$$

where

$$G=\left\{g\,\middle|\,g\in L_p[0,1],g^{(2r-1)}\in A.C._{loc},\varphi^{2r\lambda}g^{(2r)}\in L_p[0,1]\right\}.$$

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By [5,pp.10-11] , there exists $M>0$, such that

$$M^{-1}K_{\varphi^{\lambda}}^{2r}(f;t^{2r})_p \le \omega_{\varphi^{\lambda}}^{2r}(f;t^{2r})_p \le MK_{\varphi^{\lambda}}^{2r}(f;t^{2r})_p.$$

We are now in a position to state our main results.

Theorem1.1. For $f\in G, r\in\mathbb{N}, 0\le\lambda\le 1, \delta_n(x)=\varphi(x)+\frac{1}{\sqrt{n}}$, one has the Steckin-Marchaud inequality

$$\omega_{\varphi^{\lambda}}^{2r}(f;n^{-\frac{r}{2}}\delta_n^{r(1-\lambda)}(x))_p \le Mn^{-1}\sum_{k=1}^{n}\left\|O_{k,r}(f)-f\right\|_p.$$

Theorem1.2. Let $f\in G, r\in\mathbb{N}, 0\le\alpha\le 2r$ .Then

$$\left\|O_{n,r}(f)-f\right\|_p = O(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)) \Rightarrow \omega_{\varphi^{\lambda}}^{2r}(f;t)_p = O(t^{\alpha}).$$

Remark 1.3. For the inverse result, it is obvious that the result of [2] is a special case of the Theorem 1.2 with $\lambda=1$, the result of [3] is a special case of the Theorem 1.2 with $\lambda=0, p=\infty$, and the result of [4] is a special case of the Theorem 1.2 with $p=\infty$.

Throughout this paper, M denotes a positive constant independent of and f which may be different in different places.

I. AUXILIARY LEMMAS

To prove the theorems, we need also the following Lemmas.

Lemma2.1. If $c<\frac{1}{2}, d<\frac{1}{2}$. Then

$$\int_0^1 p_{n,k}(t)t^{-c}(1-t)^{-d}dt \le Mn^{-1}\left(\tfrac{k+1}{n}\right)^{-c}\left(1-\tfrac{k-1}{n}\right)^{-d}. \quad (2.1)$$

Proof. We notice [5, pp.164]

$$\int_0^1 p_{n,k}(t)t^{\eta}dt \le Mn^{-1}\left(\tfrac{k+1}{n}\right)^{\eta}, \quad \eta>-1,$$

$$\int_0^1 p_{n,k}(t)(1-t)^{\xi}dt \le Mn^{-1}\left(1-\tfrac{k-1}{n}\right)^{\xi}, \quad \xi>-1.$$

Using Holder inequality，we have

$$\int_0^1 p_{n,k}(t)t^{-c}(1-t)^{-d}dt$$

$$\le\left(\int_0^1 p_{n,k}(t)t^{-2c}dt\right)^{\frac{1}{2}}\left(\int_0^1 p_{n,k}(t)(1-t)^{-2d}dt\right)^{\frac{1}{2}}$$

$$\le Mn^{-1}\left(\tfrac{k+1}{n}\right)^{-c}\left(1-\tfrac{k-1}{n}\right)^{-d}.$$

Lemma2.2. If $c\ge 0, d\ge 0, x>0$. Then

$$\sum_{k=0}^{n}p_{n,k}(x)\left(\tfrac{k+1}{n}\right)^{-c}\left(1-\tfrac{k-1}{n}\right)^{-d} \le Mx^{-c}(1-x)^{-d}. \quad (2.2)$$

Proof. We notice [5, pp.164]

$$\sum_{k=0}^{n}p_{n,k}(x)\left(\tfrac{n}{k+1}\right)^{l} \le Mx^{-l}, l\in\mathbb{N},$$

$$\sum_{k=0}^{n}p_{n,k}(x)\left(\tfrac{n}{n-k+1}\right)^{\zeta} \le M(1-x)^{-\zeta}, \zeta\in\mathbb{N}.$$

For $c=0, d=0$, the result of (2.2) is obvious.

For $c>0, d>0$, using Holder inequality, we have

$$\sum_{k=0}^{n}p_{n,k}(x)\left(\tfrac{k+1}{n}\right)^{-c}\left(1-\tfrac{k-1}{n}\right)^{-d}$$

$$\le\left(\sum_{k=0}^{n}p_{n,k}(x)\left(\tfrac{k+1}{n}\right)^{-2c}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}p_{n,k}(x)\left(1-\tfrac{k-1}{n}\right)^{-2d}\right)^{\frac{1}{2}}$$

$$\le\left(\sum_{k=0}^{n}p_{n,k}(x)\left(\tfrac{n}{k+1}\right)^{([2c]+1)}\right)^{\frac{c}{[2c]+1}}$$

$$\cdot\left(\sum_{k=0}^{n}p_{n,k}(x)\left(\tfrac{n}{n-k+1}\right)^{[2d]+1}\right)^{\frac{d}{[2d]+1}}$$

$$\le M\left(x^{-([2c]+1)}\right)^{\frac{c}{[2c]+1}}\left((1-x)^{-([2d]+1)}\right)^{\frac{d}{[2d]+1}} \le Mx^{-c}(1-x)^{-d}.$$

For $c>0, d=0$, or $c=0, d>0$, the proof is similar. Thus, this proof is complete.

Lemma2.3. For $f\in L_p[0,1], r\in\mathbb{N}, 0\le\lambda\le 1, \delta_n(x)=\varphi(x)+\frac{1}{\sqrt{n}}, n\ge 2r$, one has the Bernstein-type inequality

$$\left\|\varphi^{2r\lambda}O_{n,r}^{(2r)}\right\|_p \le Mn^{r}\delta_n^{2r(1-\lambda)}(x)\left\|f\right\|_p. \quad (2.3)$$

Proof. For $p=1$, if $x\in E_n=\left[\frac{1}{n},1-\frac{1}{n}\right], \varphi^{-\xi}(x)\le n^{\frac{\xi}{2}}$, $\xi>0$, by simple computation, we have

$$D_n^{(2r)}(f;x)=\left(x(1-x)\right)^{-2r}\sum_{i=0}^{2r}Q_i(x,n)n^{i}\sum_{k=0}^{n}p_{n,k}(x)\cdot\left(\tfrac{k}{n}-x\right)^{i}(n+1)\int_0^1 p_{n,k}(u)f(u)du,$$

(2.4)

with $Q_i(x,n)$ is a polynomial in $nx(1-x)$ of degree $[(2r-i)/2]$ with non-constant bounded coefficients. Therefore,

$$\left|Q_i(x,n)n^i\right| \le M\left(x(1-x)\right)^{r-\frac{1}{2}} n^{r+\frac{1}{2}}, \quad x \in E_n.$$

Thus,

$$\left|\varphi^{2r\lambda} D_n^{(2r)}(x)(f;x)\right| \le Mn^{r(2-\lambda)}\left|\sum_{i=0}^{2r} n^{\frac{i}{2}}\varphi^{-i}(x)\sum_{k=0}^{n} p_{n,k}(x)\right.$$

$$\left.\cdot\left(\tfrac{k}{n}-x\right)^i (n+1)\int_0^1 p_{n,k}(u)f(u)du\right|. \quad (2.5)$$

Note that [5, pp.129]

$$\int_{E_n}\varphi^{-2m}(x)p_{n,k}(x)\left(\tfrac{k}{n}-x\right)^{2m}dx \le Mn^{-m-1},$$

We can write

$$\left\|\varphi^{2r\lambda}D_n^{(2r)}(f)\right\|_{1(E_n)} \le Mn^{r(2-\lambda)}\left|\sum_{i=0}^{2r} n^{\frac{i}{2}}\sum_{k=0}^{n}\varphi^{-i}(x)p_{n,k}(x)\right.$$

$$\left.\cdot\left(\tfrac{k}{n}-x\right)^i (n+1)\int_0^1 p_{n,k}(u)f(u)du\right|$$

$$\le Mn^{r(2-\lambda)}\sum_{k=0}^{n}\int_0^1 p_{n,k}(u)\left|f(u)\right|du$$

$$\le Mn^{r(2-\lambda)}\left\|f\right\|_1. \quad (2.6)$$

If $x \in E_n^c = \left[0,\frac{1}{n}\right)\cup\left(1-\frac{1}{n},1\right]$, then $\frac{n!}{(n-2r)!} \sim n^{2r}$, $\left\|\varphi^{2r\lambda}\right\|_\infty \sim n^{-r\lambda}, \int_0^1 p_{n,k}(x)dx = \frac{1}{n}$. By simple calculation, we have

$$D_n^{(2r)}(f;x) = \tfrac{n!}{(n-r)!}\sum_{k=0}^{n-2r} p_{n-2r,k}(x)(n+1)$$

$$\times\int_0^1\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}p_{n,k+j}(u)du, \quad (2.7)$$

$$\left\|\varphi^{2r\lambda}D_n^{(2r)}(f)\right\|_{1(E_n^c)} \le Mn^{2r(2-\lambda)}\sum_{k=0}^{n-2r}\int_0^1 p_{n-2r,k}(x)dx$$

$$\times\sum_{j=0}^{2r}\binom{2r}{j}(n+1)\int_0^1 p_{n,k+j}(u)\left|f(u)\right|du$$

$$\le Mn^{2r(2-\lambda)}\sum_{j=0}^{2r}\binom{2r}{j}\sum_{k=0}^{n-2r}\int_0^1 p_{n,k+j}(u)\left|f(u)\right|du$$

$$\le Mn^{2r(2-\lambda)}\left\|f\right\|_1. \quad (2.8)$$

For $p=\infty$, if $x \in E_n$, by(2.5) we can now write

$$\left|\varphi^{2r\lambda}(x)D_n^{(2r)}(f;x)\right| \le Mn^{r(2-\lambda)}\left\|f\right\|_\infty\sum_{i=1}^{2r} n^{\frac{i}{2}}\varphi^{-i}(x)$$

$$\cdot\sum_{k=0}^{n} p_{n,k}(x)\left(\tfrac{k}{n}-x\right)^i (n+1)\int_0^1 p_{n,k}(u)du$$

$$\le Mn^{r(2-\lambda)}\left\|f\right\|_\infty \quad (2.9)$$

If $x \in E_n^c$, by (2.7), the proof is similar to that (2.9), it is enough to show

$$\left|\varphi^{2r\lambda}(x)D_n^{(2r)}(f;x)\right| \le Mn^{r(2-\lambda)}\left\|f\right\|_\infty. \quad (2.10)$$

By (2.6), (2.8), (2.9), (2.100 applying Riesz-Thorin theorem, we get

$$\left\|\varphi^{2r\lambda}D_n^{(2r)}(f)\right\|_p \le Mn^{r(2-\lambda)}\left\|f\right\|_p \le Mn^r\delta_n^{2r(1-\lambda)}(x)\left\|f\right\|_p.$$

Combining (iii) of (1.3), we obtain

$$\left\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\right\|_p \le Mn^r\delta_n^{2r(1-\lambda)}(x)\left\|f\right\|_p.$$

Lemma 2.4. If $f \in G, r \in \mathbb{N}, 0 \le \lambda \le 1, n > 2r$, Then

$$\left\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\right\|_p \le M\left\|\varphi^{2r\lambda}f^{(2r)}\right\|_p. \quad (2.11)$$

Lemma 2.5. If $f \in G, r \in \mathbb{N}, 0 \le \lambda \le 1,$ Then

$$\left\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\right\|_p$$

$$\le Mn^{r-1}\delta_n^{2r(\lambda-1)}(x)\sum_{k=1}^{n}\left\|O_{k,r}(f)-f\right\|_p.$$

Proof. By Lemma 2.3., Lemma2.4., note that $O_{1,r}^{(2r)} = 0$, we have

$$n^{-r}\left\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\right\|_p$$

$$\le n^{-r}\left\|\varphi^{2r}O_{n,r}^{(2r)}(O_{k,r}(f))\right\|_p$$

$$+n^{-r}\left\|\varphi^{2r}O_{n,r}^{(2r)}(O_{k,r}(f)-f)\right\|_p$$

$$\le M_2 n^{-r}\left\|\varphi^{2r}O_{k,r}^{(2r)}(f)\right\|_p$$

$$+M_1\delta_n^{2r(\lambda-1)}(x)\left\|O_{k,r}(f)-f\right\|_p. \quad (2.12)$$

We write $\left\|O_{q,r}(f)-f\right\|_p = \max_{1\le k\le n}\left\|O_{k,r}(f)-f\right\|$.

For $\left\|O_{q,r}(f)-f)\right\|_p$, there exists M_3, and $k: 1\le k\le n$, such that $\left\|O_{q,r}(f)-f)\right\|_p \le M\left\|O_{k,r}(f)-f)\right\|.$

Therefore,

$$M_2 n^{-r}\left\|\varphi^{2r}O_{k,r}^{(2r)}(f)\right\|_p$$
$$\le \frac{M_2}{n^r}\left\|\varphi^{2r\lambda}O_{k,r}^{(2r)}(O_{1,r}(f)-f)\right\|_p$$
$$+\frac{M_2}{n^r}\left\|\varphi^{2r\lambda}O_{k,r}^{(2r)}(O_{1,r}(f))\right\|_p$$
$$\le M_1M_2\delta_k^{2r(\lambda-1)}\left\|O_{1,r}(f)-f\right\|_p$$
$$+M_2^2\delta_k^{2r(\lambda-1)}\left\|O_{k,r}^{(2r)}(f)\right\|_p$$

$$\le M_1M_2\delta_k^{2r(\lambda-1)}\left\|O_{q,r}(f)-f\right\|_p$$
$$\le M_1M_2M_3\delta_k^{2r(\lambda-1)}\left\|O_{k.r}(f)-f\right\|_p. \qquad (2.13)$$

Note that $\delta_k^{2r(\lambda-1)}(x)\le \delta_n^{2r(\lambda-1)}(x)$,

by (2.12), (2.13), we have

$$\left\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\right\|_p \le Mn^{r-1}\delta_n^{2r(\lambda-1)}(x)\sum_{k=1}^{n}\left\|O_{k,r}(f)-f\right\|_p.$$

where $M=M_1+M_1M_2M_3.$

2. Proofs of Theorems

Proof of Theorem 1.

Proof. For $n>2,$ there exists $m\in\mathbb{N},$ such that $\frac{n}{2}\le m\le n,$ and

$$\left\|O_{m,r}(f)-f\right\|_p=\min_{\frac{n}{2}\le k\le n}\left\|O_{k,r}(f)-f\right\|_p,$$

$$\left\|O_{m,r}(f)-f\right\|_p\le 2n^{-1}\sum_{\frac{n}{2}\le k\le n}\left\|O_{k,r}(f)-f\right\|_p.$$

Therefore, using the definition of $K_{\varphi^\lambda}^{2r}(f;t^{2r})_p$, and Lemma 2.5., note that $\delta_m^{2r(\lambda-1)}(x)\le\delta_n^{2r(\lambda-1)}(x)$, we have

$$K_{\varphi^\lambda}^{2r}(f;n^{-r}\delta_n^{2r(1-\lambda)}(x))_p$$
$$\le\left\|O_{m,r}(f)-f\right\|_p+n^{-r}\delta_n^{2r(1-\lambda)}\left\|\varphi^{2r\lambda}O_{m,r}^{(2r)}(f)\right\|_p$$
$$+Mn^{-r}\delta_n^{2r(1-\lambda)}\sum_{k=1}^{m}\left\|O_{k,r}(f)-f\right\|_p$$
$$\le Mn^{-1}\sum_{k=1}^{n}\left\|O_{k,r}(f)-f\right\|_p.$$

By relationship of K-functional and modulus of smoothness, we get

$$\omega_{\varphi^\lambda}^{2r}(f;n^{-\frac{r}{2}}\delta_n^{r(1-\lambda)})_p\le Mn^{-1}\sum_{k=1}^{n}\left\|O_{k,r}(f)-f\right\|_p.$$

This completes the proof of Theorem !.

Proof of Theorem2.

Proof. By $\left\|O_{n,r}(f)-f\right\|_p\le M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)$, Acording to the definition of $K_{\varphi^\lambda}^{2r}(f;t^{2r})$, we have

$$K_{\varphi^\lambda}^{2r}(f;t^{2r})_p\le\left\|f-O_{n,r}(f)\right\|_p+t^{2r}\left\|\varphi^{2r\lambda}O_{n,r}^{(2r)}(f)\right\|_p$$

$$\le M[(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha+t^{2r}\left(\left\|\varphi^{2r\lambda}O_n^{(2r)}(f-g)\right\|_p\right.$$
$$\left.+\left\|\varphi^{2r\lambda}O_n^{(2r)}(g)\right\|_p\right)]$$
$$\le M\left[(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha+t^{2r}\left(n^r\delta_n^{2r(\lambda-1)}(x)\left\|f-g\right\|_p\right.\right.$$
$$\left.+\left\|\varphi^{2r\lambda}g^{(2r)}\right\|_p)\right]$$
$$\le M\left((n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha+\frac{t^{2r}}{n^{-r}\delta_n^{2r(1-\lambda)}(x)}K_{\varphi^\lambda}^{2r}(f;n^{-r}\varphi^{2r(1-\lambda)})_p\right)$$

By Berens-Lorens theorem, and relationship of K-functional and modulus of smoothness, we have

$$\omega_{\varphi^\lambda}^{2r}(f;t)_p\le Mt^\alpha.$$

This completes the proof of Theorem2.

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