Open Journal of Applied Sciences
Supplement: 2012 world Congress on Engineering and Technology

Branches of solutions for an asymptotically linear elliptic problem on $\mathbb{R}^N$

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***Abstract*—We consider the following nonlinear schrödinger equation**

$$-\Delta u+\lambda V(x)u=f(x,u) \text{ with } u\in H^1(\mathbb{R}^N) \text{ and } u\not\equiv 0,(*)$$

where $\lambda>0$ and $f(x,s)$ is asymptotically linear with respect to s at origin and infinity. The potential $V(x)$ satisfies $V(x)\ge V_0>0$ for all $x\in\mathbb{R}^N$ and $\lim_{|x|\to+\infty}V(x)=V(\infty)\in(0,+\infty)$. We provethat problem $(*)$ has two connected sets of positive and negative solutions in $\mathbb{R}\times W^{2,p}(\mathbb{R}^N)$ for some $p\in[2,+\infty)\cap(\frac{N}{2},+\infty)$.

Keywords-Bifurcation, asymptotically linear, Fredholm operator of index zero.

I. INTRODUCTION

In this paper, we consider the following nonlinear Schrödingerequation

$$\begin{cases}-\Delta u+\lambda V(x)u=f(x,u),\\ u\in H^1(\mathbb{R}^N),\ N\ge 3,\end{cases}\qquad(1.1)$$

where $\lambda>0$ and the functions V and f satisfy the following assumptions:

(V_1) $V(x)\in C(R^N,R)$ and there exists $V_0>0$ such that $V(x)\ge V_0>0$ for all $x\in\mathbb{R}^N$;

(V_2) $\lim_{|x|\to+\infty}V(x)=V(\infty)\in(0,+\infty)$ and $eas\{x\in\mathbb{R}^N:V(x)<V(\infty)\}>0$;

(F_1) $f(x,s)\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ and $(x,\cdot)\in C^1(\mathbb{R},\mathbb{R})$;

(F_2) there exist two functions $h,g\in L^\infty(\mathbb{R}^N)$ such that $\lim_{s\to0}\frac{f(x,s)}{s}=h(x)$ and $\lim_{|s|\to+\infty}\frac{f(x,s)}{s}=g(x)$ uniformly in $x\in\mathbb{R}^N$, where h and g satisfy

(G) Setting $\Gamma=\inf\{\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx: u\in H^1(\mathbb{R}^N)\ and\ \int_{\mathbb{R}^N}u^2dx=1\}$, there exists $\alpha\in(\Gamma,V(\infty))$ such that $\lim_{|x|\to+\infty}g(x)=\inf_{x\in\mathbb{R}^N}g(x)=\alpha$;

(H) $|h|_\infty<\frac{\alpha V_0}{V(\infty)}$.

(F_3) $h(x)\le\frac{f(x,s)}{s}\le g(x)$ for all $(x,s)\in\mathbb{R}^N\times\mathbb{R}\backslash\{0\}$.

The existence of solutions of problem (1.1) has beeninvestigated extensively. For problem (1.1) with potentialwell and various conditions on $f(x,u)\not\equiv f(u)$, several authorshave obtained the existence of solutions for large λ byvariational methods,for example, [1], [2], [3], [5]. And other authors have got the existence of solutions for λ is not necessarily large by concentration compactnessargument and mountain pass geometry, for instance, [7], [8]. Stuart and Zhou [10] havestudied how the positive and negative solutions of problem(1.1)depend on λ by topological methods.

Inspired by the results we mentioned above, the main object of thisarticle is to investigate the relation between the positive andnegative solutions of problem (1.1)and the parameter λ, where the potential need not be well potential and $f(x,u)$ is asymptotically linear with respect to u at origin andinfinity.

For this purpose, we use the following global branch theorem established in [10].

Theorem 1.1: Let X and Y be real Banach spaces, $\mathcal{B}(X,Y)$ be thespace of bounded linear operators from X into Y with its usualnorm, $P:\mathbb{R}\times X\to\mathbb{R}$ denote theprojection $P(\lambda,u)=\lambda$, and

$\Phi_0(X,Y)=\{L\in\mathcal{B}(X,Y): L\ is\ a\ Fredholm\ operator\ of\ index\ zero\}$.

Let $L\in C^1(J,\mathcal{B}(X,Y))$ where J is an open interval and $L(\lambda)\in\Phi_0(X,Y)$ for all $\lambda\in J$. Suppose that there exists $\lambda_0\in J$ suchthat $\dim kerL(\lambda_0)$ is odd and

$$L'(\lambda_0)kerL(\lambda_0)\oplus regL(\lambda_0)=Y\qquad(1.2)$$

Let $K\in C(X,Y)$ be such that $K:X\to Y$ is compact and

$$\lim_{||u||_X\to0}\frac{||K(u)||_Y}{||u||_X}=0.\qquad(1.3)$$

Let $\tilde{Z}=Z\cup\{(\lambda_0,0)\}$ where $Z=\{(\lambda,u)\in J\times X: u\ne 0\ and\ L(\lambda)u+K(u)=0\}$ be considered with the metric inherited from $\mathbb{R}\times X$, and let $\mathcal{C}$ denote the connected component of $\tilde{Z}$ containing $(\lambda_0,0)$. Then $\mathcal{C}$ possesses at least one of the followingproperties:

(i) $\mathcal{C}$ is an unbounded subset of $\mathbb{R}\times X$;

(ii) $\bar{\mathcal{C}}\cap[J\times\{0\}]\ne\{(\lambda_0,0)\}$, where $\bar{\mathcal{C}}$ is the closure of $\mathcal{C}$ in $J\times X$;

(iii) either $supP\mathcal{C}=supJ$ or $infP\mathcal{C}=infJ$.

REMARK 1.1: For $K\in C(X,Y)$, the condition (1.3) is equivalent to theproperties $K(0)=0$ and $K:X\to Y$ is FrÉchetdifferentiable at zero with $K'(0)=0$.

By (F_1) and (F_2) we may define a function k having the following properties

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$$k(x,s) = g(x) - \frac{f(x,s)}{s} \qquad (1.4)$$

with $\lim_{s\to 0} k(x,s) = g(x) - h(x)$, $\lim_{|s|\to+\infty} k(x,s) = 0$ uniformly in $x \in \mathbb{R}^N$ and $0 \le k(x,s) \le g(x) - h(x)$. From the above notation (1.4), problem (1.1) is equivalent to

$-\Delta u - g(x)u + \lambda V(x)u + k(x,u)u = 0, u \in H^1(\mathbb{R}^N)$ (1.5) To prove the asymptotic bifurcation result, first we study the following formal asymptotic linearization of (1.5):

$$\begin{cases} -\Delta u - g(x)u + \lambda V(x)u = 0, \\ u \in H^1(\mathbb{R}^N),\ \lambda > 0. \end{cases} \qquad (1.6).$$

A number $\lambda > 0$ is said to be an eigenvalue of (1.6) if there exists $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$\int_{\mathbb{R}^N} [\nabla u \nabla v - g(x)uv + \lambda V uv]dx = 0 \; for\ all\ v \in H^1(\mathbb{R}^N)$.

For the discussion of equation (1.5), we take advantage of the additional regularity of solutions that follows from our assumptions (see Proposition 2.1 in [10]).

Proposition 1.1: (1) Assume that the conditions (F_1) (F_2) (V_1) (V_2) hold and $u \in H^1(\mathbb{R}^N)$ satisfies (1.5), then $u \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [2,+\infty)$ and hence $u \in C^1(\mathbb{R}^N)$ with $\lim_{|x|\to+\infty} u(x) = 0$ and $\lim_{|x|\to+\infty} \nabla u(x) = 0$.

(2) If V satisfies (V_1) (V_2) and $v \in H^1(\mathbb{R}^N)$ is an eigenfunction of (1.6), then $v \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [2,+\infty)$.

Our first result concerning the linearized equation (1.6) is the following :

Theorem 1.2: Assume that V and g satisfies (V_1) (V_2) and (G), then

(i) there exists an unique eigenvalue $\lambda = \Lambda(\alpha)$ of (1.6) having a positive eigenfunction. Furthermore $\Lambda(\alpha) > 1$, and it is simple in the sense that $\ker(A_{\Lambda(\alpha)}) = span\{u_{\Lambda(\alpha)}\}$ where A_λ denotes the Schrödinger operator $A_\lambda u = -\Delta u - g(x)u + \lambda V u$ and $u_{\Lambda(\alpha)} > 0$ on $\mathbb{R}^N$. All other eigenvalues of (1.6) are less than $\Lambda(\alpha)$ and their eigenfunctions change sign.

(ii) $\Lambda(\alpha)$ is the unique value of λ in the interval $[\frac{\alpha}{V(\infty)}, +\infty)$ for which 0 is the infimum of the spectrum of the Schrödinger operator A_λ.

Now we can state our main result concerning the nonlinear problem (1.1).

Theorem 1.3: Let the conditions (F_1) (F_2) (F_3) (V_1) (V_2) hold and fix $P \in [2,+\infty) \cap (\frac{N}{2},+\infty)$. Then there exist two connected subsets $\sum^+$ and $\sum^-$ of $\mathbb{R} \times W^{2,p}(\mathbb{R}^N)$, whose elements (λ, u) are, respectively, positive and negative solutions of problem (1.1), such that $inf\{\lambda : (\lambda,u) \in \sum^\pm\} = \frac{\alpha}{V(\infty)}$ and $up\{\lambda : (\lambda,u) \in \sum^\pm\} = \Lambda(\alpha)$, where $\Lambda(\alpha)$ is given by Theorem 1.2. Furthermore, $\sum^\pm$ is bounded away from the line of trivial solutions of $\mathbb{R} \times \{0\}$ and if $\{(\lambda_n, u_n)\} \subset \sum^\pm$ with $\lambda_n \xrightarrow{n} \lambda > \frac{\alpha}{V(\infty)}$, then $\max_{x\in\mathbb{R}^N} |u_n(x)| \xrightarrow{n} \infty$ if and only if $\lambda = \Lambda(\alpha)$.

II. EIGENVALUE PROBLEM

In this section, we prove Theorem 1.2. It follows from Proposition 1.1 that any eigenfunction u of equation (1.6) belongs to $C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$, and this leads us to introduce a Schrödinger operator having u as an eigenfunction. Define

$$A_\lambda : D(A_\lambda) = H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

by $A_\lambda = -\Delta u - g(x)u + \lambda V u$

Then A_λ is a self-adjoint operator in $L^2(\mathbb{R}^N)$ with spectrum $\sigma(A_\lambda)$ and essential spectrum $\sigma_e(A_\lambda) = [\lambda V(\infty) - \alpha, +\infty)$. Furthermore, setting $\sum(\lambda) = inf\,\sigma(A_\lambda)$, we have $\sum(\lambda) = \inf\{a_\lambda(u) : u \in H^1(\mathbb{R}^N) and\ \int_{\mathbb{R}^N} u^2 dx = 1\} > -\infty$, where $a_\lambda(u) = \int_{\mathbb{R}^N} [|\nabla u|^2 - g(x)u^2 + \lambda V u^2]dx$.

Lemma 2.1: Suppose that V satisfies (V_1) (V_2) and $\Gamma < \alpha$, then $\sum(1) < 0$. Moreover, there exists $\lambda_1 > 1$ such that $\sum(\lambda) < 0$ for all $\lambda \in (-\infty, \lambda_1]$.

proof : Since $\Gamma < \alpha$, there exists $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that $\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \alpha \int_{\mathbb{R}^N} u^2 dx \le \int_{\mathbb{R}^N} g(x)u^2 dx$. This means that $a_1(u) < 0$ and $\sum(1) < 0$. Hence there exists $u_1 \in H^1(\mathbb{R}^N)\ with\ \int_{\mathbb{R}^N} u_1^2 dx = 1$ such that $a_1(u_1) < 0$. By the definition of a_λ, we have

$a_\lambda(u_1) - a_1(u_1) = (\lambda - 1)\int_{\mathbb{R}^N} V(x)u_1^2 dx$ for all $\lambda \in \mathbb{R}$. (2.1)

By (V_1) (V_2), we have there exists $C > 0$ such that $V(x) \le C$ for all $x \in \mathbb{R}^N$. From (2.1), we have $a_\lambda(u_1) \le a_1(u_1) + C(\lambda - 1)$. Therefore choosing $\lambda_1 = 1 + \frac{-a_1(u_1)}{2C} > 1$, we get $a_{\lambda_1}(u_1) \le \frac{a_1(u_1)}{2} < 0$ and $\sum(\lambda_1) < 0$. Since $\sum(\lambda)$ is increasing with respect to $\lambda \in \mathbb{R}$, we have $\sum(\lambda) < 0$ for all $\lambda \in (-\infty, \lambda_1]$.

Lemma 2.2: Let V satisfies (V_1) (V_2). For $\Gamma < \alpha < V(\infty)$, if we set $S := \{\lambda \ge \frac{\alpha}{V(\infty)} : \sum(\lambda) < 0\}$ and $\Lambda(\alpha) = \sup\{\lambda : \lambda \in S\}$, then $\Lambda(\alpha) \in (1, +\infty)$.

Proof : From Lemma 2.1, we have $\Lambda(\alpha) > 1$. It is clear that S is an interval since $\sum(\lambda)$ is increasing in λ. Therefore, if $\Lambda(\alpha) = +\infty$, we have $S = [\frac{\alpha}{V(\infty)}, +\infty)$ and for any integer $n \ge \frac{\alpha}{V(\infty)}$, there exists $u_n \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_n^2 dx = 1$ such that

$$a_n(u_n) = \int_{\mathbb{R}^N} [|\nabla u_n|^2 - g(x)u_n^2 + nV(x)u_n^2]dx < 0. \qquad (2.2)$$

By condition (V_1) we see that (2.2) is impossible when $n \ge \frac{\alpha}{V_0}$.

Lemma 2.3: Assume that (V_1) (V_2) hold and $\Gamma < \alpha < V(\infty)$. Then $\lambda \in [\frac{\alpha}{V(\infty)}, +\infty)$ and $\sum(\lambda) = 0$ if and only if $\lambda = \Lambda(\alpha)$, where $\Lambda(\alpha)$ is given by Lemma 2.2.

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Proof : Since$\Gamma < \alpha < V(\infty)$and $\sum(\lambda)$ is increasing in λ, it follows from Lemma2.1 that $\sum\left(\frac{\alpha}{V(\infty)}\right) \leq \sum(1) < 0$. If $\lambda \in \left[\frac{\alpha}{V(\infty)}, +\infty\right)$and $\sum(\lambda) = 0$, then $\lambda > \frac{\alpha}{V(\infty)}$. Nowwe have $\sum(\lambda) = inf\sigma(A_\lambda) = 0$and $inf\sigma_e(A_\lambda) = \lambda V(\infty) - \alpha > \frac{\alpha}{V(\infty)} V(\infty) - \alpha = 0$. Hence 0 is aneigenvalue of A_λ and there exists $u_\lambda \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$such that $kerA_\lambda = span\{u_\lambda\}$ and$u_\lambda > 0$ on$\mathbb{R}^N$ (see [9],Theorem 3.20] for example). We mayassume that $\int_{\mathbb{R}^N} u_\lambda^2 dx = 1$ such that$a_\lambda(u_\lambda) = 0$. Then from the definition of $a_\lambda$we have for any $\varepsilon > 0$

$$a_{\lambda-\varepsilon}(u_\lambda) = -\varepsilon \int_{\mathbb{R}^N} V u_\lambda^2 dx \leq -\varepsilon V_0 \int_{\mathbb{R}^N} u_\lambda^2 dx < 0$$

and thismeans that $\lambda - \varepsilon \in S$ for any $\varepsilon > 0$. Thereforeλ =supS=$\Lambda(\alpha)$.

Conversely, if$\lambda = \Lambda(\alpha)$, by Lemma 2.2 wehave$\Lambda(\alpha) > 1 > \frac{\alpha}{V(\infty)}$. Hence it issufficient to prove$\Lambda(\alpha) \notin S \cup T$,where$T := \{\lambda \geq \frac{\alpha}{V(\infty)}: \sum(\lambda) > 0\}$. Indeed, if $\Lambda(\alpha) \in S$, then $\sum(\Lambda(\alpha)) < 0$. By the proof of Lemma 2.1 we see that there exists$\lambda_2 > \Lambda(\alpha)$such that $\sum(\lambda) < 0$ for all $\lambda \in (-\infty, \lambda_2]$. This contradicts the definition of $\Lambda(\alpha)$. On the other hand, if$\Lambda(\alpha) \in T$, then $\sum(\Lambda(\alpha)) > 0$. By the definition of a_λ and $V(x) \leq C$ for all$x \in \mathbb{R}^N$, we see that for any$\varepsilon > 0$ and $u \in H^1(\mathbb{R}^N)$ with$\int_{\mathbb{R}^N} u^2 dx = 1$,

$$a_{\Lambda(\alpha)-\varepsilon}(u) = a_{\Lambda(\alpha)}(u) - \varepsilon \int_{\mathbb{R}^N} V u^2\, dx \geq \sum(\Lambda(\alpha)) - \varepsilon C$$

Therefore we can choose $\varepsilon = \frac{\sum(\Lambda(\alpha))}{2C}$ such that $a_{\Lambda(\alpha)-\varepsilon}(u_\lambda) \geq \frac{\sum(\Lambda(\alpha))}{2} > 0$ for all $u \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u^2 dx = 1$. This means that $\sum(\Lambda(\alpha) - \varepsilon) > 0$ and also contradicts thedefinition of$\Lambda(\alpha)$.

Proof of Theorem 1.2 (i) From Lemma 2.2 and 2.3 we know that$\Lambda(\alpha) > 1$and $\sum(\Lambda(\alpha)) = inf\sigma(A_{\Lambda(\alpha)}) = 0$. Since $\alpha < V(\infty)$, we have $inf\sigma_e(A_{\Lambda(\alpha)}) = \Lambda(\alpha)V(\infty) - \alpha > 0$. Hence 0 is aneigenvalue of $A_{\Lambda(\alpha)}$ and there exists$u_{\Lambda(\alpha)} \in C(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$such that $kerA_{\Lambda(\alpha)} = span\{u_{\Lambda(\alpha)}\}$ and $u_{\Lambda(\alpha)} > 0$on $\mathbb{R}^N$. Suppose now that$\Lambda_1 \neq \Lambda(\alpha)$ is another eigenvalue of(1.6) with eigenfunction$u_1 \in H^1(\mathbb{R}^N)$. Then 0is an eigenvalue of $A_{\Lambda_1}$and $\sum(\Lambda_1) = inf\sigma(A_{\Lambda_1}) \leq 0$. It follows that$\Lambda_1 \leq \Lambda(\alpha)$. Otherwise, if $\Lambda_1 > \Lambda(\alpha)and\sum(\Lambda_1) \leq 0$, we divide two cases to deduce thecontradiction. One hand, if $\Lambda_1 > \Lambda(\alpha)$and $\sum(\Lambda_1) = 0$, it contradicts Lemma 2.3. On theother hand, if $\Lambda_1 > \Lambda(\alpha)$and $\sum(\Lambda_1) < 0$, by the proof of Lemma 2.1 we see thatthere exists $\lambda_3 > \Lambda_1$ such that $\sum(\lambda) < 0$ for all $\lambda \in (-\infty, \lambda_3]$. This contradicts thedefinition of$\Lambda(\alpha)$. Therefore$\Lambda(\alpha)$ isthe largest eigenvalue of (1.6). Furthermore, integratingby parts, we have

$$(\Lambda(\alpha) - \Lambda_1) \int_{\mathbb{R}^N} V u_1 u_{\Lambda(\alpha)} dx = 0.$$

For $\Lambda_1 < \Lambda(\alpha)$ and $V(x)u_{\Lambda(\alpha)} > 0$on $\mathbb{R}^N$, it follows that u_1 changes sign.

(ii) This follows from Lemma 2.3.

III. THE TRUNCATED PROBLEM

Let $p \in [2, +\infty) \cap (\frac{N}{2}, +\infty)$ be fixed and we set

$X = W^{2,p}(\mathbb{R}^N)$with $||\cdot|| = ||\cdot||_{W^{2,p}(\mathbb{R}^N)}$, and

$Y = L^p(\mathbb{R}^N)$with$|\cdot|_p = ||\cdot||_{L^p(\mathbb{R}^N)}$.

For k defined in (1.4), it can be shown that (see [10], Lemma B.1])

$$\frac{|k(x,u)u|_p}{||u||} \nrightarrow 0 \text{as} ||u|| \to +\infty$$

uniformly in $x \in \mathbb{R}^N$. Hence in order to make use ofTheorem 1.1 we need to introduce the following truncatedproblem

$$-\Delta u - g(x)u + \lambda V(x)u + \psi_n(x)k(x,u)u = 0 \quad (3.1)$$

where$u \in H^1(\mathbb{R}^N)$ and

$$\psi_n(x) = \begin{cases} 1, & if \quad |x| \leq n, \\ 0, & if \quad |x| > n. \end{cases}$$

Define $L(\lambda): X \to Y$ by

$$L(\lambda)u = -\Delta u - g(x)u + \lambda V(x)u \quad (3.2)$$

Using the inversion, $u \mapsto v = \frac{u}{||u||^2}$, problem (3.1) is equivalent to

$$L(\lambda)v + K_n(v) = 0 \quad (3.3)$$

where

$$K_n(v) = \begin{cases} \psi_n k\left(x, \dfrac{v}{||v||^2}\right) v, & for\ v \in X\backslash\{0\} \\ 0, & for\ v \equiv 0 \end{cases}$$

In the sequel we show that Theorem 1.1 is applicable tothe inverted truncated problem (3.3). First, it follows fromTheorem 4.3 of [6] that$L(\lambda) \in \Phi_0(X, Y)$ for all $> \frac{\alpha}{V(\infty)}$, where $\Phi_0(X, Y)$ is defined in Theorem 1.1 and

$$\text{Y}= kerL(\lambda) \oplus rgeL(\lambda) \text{for all} \lambda > \frac{\alpha}{V(\infty)} \quad (3.4)$$

From (3.4) and Theorem1.2 we can prove (1.2) holds with $\lambda_0 = \Lambda(\alpha)$ (see the proof of Theorem 4.2in [10]). By Remark 1.1 the following lemma(Lemma 3.2in [10]) can verify condition (1.3) for $K_n$defined in(3.3) .

Lemma 3.1: For all$n \in N$, $K_n \in C(X, Y) \cap C^1(X\backslash\{0\}, Y)$, $K_n: X \to Y$ is compact and it is FrÉchetdifferentiable at 0 with$K_n'(0) = 0$.

Now we can apply Theorem 1.1 for $L(\lambda)$ defined by (3.2) on the interval$J = (\frac{\alpha}{V(\infty)}, +\infty)$and at the point $\lambda_0 = \Lambda(\alpha)$. Set $\mathcal{Z}_n = \{(\lambda, v) \in (\frac{\alpha}{V(\infty)}, +\infty) \times X$: $v$is a nontrivial solution of

(3.3)}. Let $\mathcal{C}_n$ denote the connected component of $\mathcal{Z}_n \cup \{(\Lambda(\alpha),0)\}$containing$(\Lambda(\alpha),0)$. In order to get some initial informationabout $\mathcal{C}_n$ from Theorem 1.1, we establish someestimates about the solutions of (3.2) that will help us toobtain more precise information about $\mathcal{C}_n$.

Lemma 3.2: Let the conditions (V_1) (V_2) hold and $\alpha = l + \mu$Suppose that$\Gamma < \alpha < V(\infty)$and $0 \le \mu < \frac{lV_0}{V(\infty)-V_0}$. Then, there exists $T > 0$, such that for any $\beta > \frac{\alpha}{V(\infty)}$ there is $N_\beta \in \mathbb{N}$ satisfying $||v|| \le T$ for all $(\lambda, v) \in \mathcal{Z}_n$ with $\lambda \ge \beta$ and $n \ge N_\beta$.

Proof :Firstwe define the bound T. Since$0 \le \mu < \frac{lV_0}{V(\infty)-V_0}$and $\alpha = l + \mu$, we candeduce that $\mu < \frac{\alpha V_0}{V(\infty)}$ and choose $\eta = \frac{1}{2}\left(\frac{\alpha V_0}{V(\infty)} - \mu\right) > 0$. By (1.4) and (V_1) we know $k(x,0) = l$ and there exists $\delta_1$such that $k(x,s) \ge l - \eta = \frac{1}{2}l + \frac{1}{2}\left(\alpha - \frac{\alpha V_0}{V(\infty)}\right) \ge \frac{1}{2}l > 0$ for all $|s| \le \delta_1$. By Sobolevembeding there is C_1 suchthat $|u|_\infty \le C_1||u||$. Set$T = \frac{C_1}{\delta_1}$. Suppose that $v \in X$and $||v|| \ge T$. Then, for all $x \in \mathbb{R}^N$, we have$\frac{|v(x)|}{||v||^2} \le \frac{|v|_\infty}{||v||^2} \le \frac{C_1}{||v||} \le \frac{C_1}{\mathrm{T}} = \delta_1$. Consequently, $k\left(x, \frac{|v(x)|}{||v||^2}\right) \ge l - \eta$. For any$\beta > \frac{\alpha}{V(\infty)}$, we can choose $\delta_\beta = V(\infty) - \frac{\alpha}{\beta} > 0$. By (V_2), there exists$N_\beta \in \mathbb{N}$, such that $V(x) = V(\infty) - \delta_\beta$for all $|x| \ge N_\beta$. Hence

$$\lambda \mathrm{V(x)} \ge \beta(V(\infty) - \delta_\beta) = \alpha \quad (3.5)$$

for all $\lambda \ge \beta$ and $|x| \ge N_\beta$. On the otherhand, for $\lambda \ge \beta$, $n \ge N_\beta$, $|x| \le N_\beta$, and $||v|| \ge T$, we have

$$K_n(v) = \psi_n k\left(x, \frac{|v|}{||v||^2}\right) = k\left(x, \frac{|v|}{||v||^2}\right) \ge l - \eta,$$

and$-\alpha + \lambda V(x) + K_n(v) \ge -\alpha + \frac{\alpha}{V(\infty)}V_0 + l - \eta$

$$= -\mu - \eta + \frac{\alpha}{V(\infty)}V_0 = \frac{1}{2}\left(\frac{\alpha V_0}{V(\infty)} - \mu\right) > 0.$$

Combining the above inequality and (3.5), we get that for$(\lambda, v) \in \mathcal{Z}_n$ with $\lambda \ge \beta$and $n \ge N_\beta$, if$||v|| \ge T$, then

$$v(x)\Delta v(x) = v^2(x)[-\alpha + \lambda V(x) + \psi_n k\left(x, \frac{|v|}{||v||^2}\right)] \ge 0$$

for all $x \in \mathbb{R}^N$. The maximum principle now leads toa contradiction.

Lemma 3.3: Fix $\beta > \frac{\alpha}{V(\infty)}$. Then, for any $\varepsilon \in (0, V(\infty) - \frac{\alpha}{\beta})$, there exists $C_\varepsilon > 0$such that$|u(x)| \le |u|_\infty e^{-\sqrt{\xi}(|x|-C_\varepsilon)}$ for all $x \in \mathbb{R}^N$,where $\xi = \beta(V(\infty) - \varepsilon) - \alpha > 0$ for all $(\lambda, u) \in [\beta, +\infty)$with $uL(\lambda)u \le 0$ on$\mathbb{R}^N$.

Proof : Since $\lim_{|x|\to+\infty} V(x) = V(\infty)$, for any $\varepsilon \in (0, V(\infty) - \frac{\alpha}{\beta})$, there exists $C_\varepsilon > 0$ suchthat $V(x) \ge V(\infty) - \varepsilon$ for all $|x| \ge C_\varepsilon$. Set $q(x) = |u|_\infty e^{-\sqrt{\xi}(|x|-C_\varepsilon)} - u(x)$ and $\Omega_\varepsilon = \{x \in \mathbb{R}^N: |x| > C_\varepsilon \; and \; q(x) < 0\}$. For all $x \in \Omega_\varepsilon$, we have $u(x) > 0$ and

$$0 \ge L(\lambda)u = -\Delta u - \alpha u + \lambda V(x)u$$

$$\ge -\Delta u - \alpha u + \beta(V(\infty) - \varepsilon)u = -\Delta u + \xi u,$$

since $\ge \beta$. By direct calculations, we have for$x \in \Omega_\varepsilon$

$$\Delta q(x) = |u|_\infty e^{\sqrt{\xi}C_\varepsilon}\left(\xi - \frac{N-1}{|x|}\sqrt{\xi}\right)e^{-\sqrt{\xi}|x|} - \Delta u$$

$$\le |u|_\infty e^{\sqrt{\xi}C_\varepsilon}\xi e^{-\sqrt{\xi}|x|} - \xi u = \xi q(x) < 0.$$

Since $q(x) \to 0$ as $|x| \to +\infty$ and $q(x) \ge 0$ for $|x| = C_\varepsilon$, we have $q(x) \ge 0$ for $x \in \Omega_\varepsilon$. If $\Omega_\varepsilon \ne \emptyset$, the weakmaximum principle (Theorem 8.1 in [4]) implies that $q(x) \ge 0$ in Ω_ε, a contradiction. Thus we see that$|u(x)| \le |u|_\infty e^{-\sqrt{\xi}(|x|-C_\varepsilon)}$for all$|x| \ge C_\varepsilon$.Replacing $u$by $-u$, we get the above inequality for $|u(x)|$.Hence, we complete the proof.

Lemma 3.4: For each $n \in \mathbb{N}$, there exists an open neighborhood $U$of$(\Lambda(\alpha), 0) \in \mathbb{R} \times X$, such that $u^2 > 0$on $\mathbb{R}^N$ for all $(\lambda, v) \in U \cap \mathcal{Z}_n$.

Proof : By contradiction, suppose that there exists asequence $\{(\lambda_i, u_i)\} \subset \mathcal{Z}_n$ such that $\lambda_i \xrightarrow{i} \Lambda(\alpha)$ and $||u||_i \xrightarrow{i} 0$ andeach continuous functionu_i has at least one zero in $\mathbb{R}^N$. We prove this leadsto a contradiction.

Setting $z_i = \frac{u_i}{||u||_i}$, by the definition of $\mathcal{Z}_n$, we have $L(\lambda_i)z_i + \frac{K_n(u_i)}{||u||_i} = 0$ on $\mathbb{R}^N$ .Since $V(x) \in L^\infty(\mathbb{R}^N)$, by Lemma 3.1, wefind that

$$L(\Lambda(\alpha))z_i = (\Lambda(\alpha) - \lambda_i)V(x)z_i - \frac{K_n(u_i)}{||u||_i} \xrightarrow{i} 0 \text{ in } Y.$$

On the other hand, by passing to a subsequence, we may supposethat $z_i \rightharpoonup z$ weakly in X. Since$\Lambda(\alpha) > 1 > \frac{\alpha}{V(\infty)}$, we have $L(\Lambda(\alpha)) \in \Phi_0(X, Y)$. By Lemma 3.5 of [10], we know that $z_i \xrightarrow{i} z$ strongly in X . This means that $||z|| = 1$ and $L(\Lambda(\alpha))z_i = 0$. By Lemma3.3, wehave $z \in H^1(\mathbb{R}^N)$. According to Theorem 1.2(i), we may as well suppose that $\mathrm{z} > 0$ on $\mathbb{R}^N$. Since $\lambda_i \xrightarrow{i} \Lambda(\alpha)$ and $\Lambda(\alpha) > 1$, choosing$\varepsilon = \frac{\Lambda(\alpha)-1}{2} > 0$, there exists $i_0 \in \mathbb{N}$ such that $\lambda_i \ge 1 + \varepsilon$for all $i \ge i_0$. By $(V_2)$and $\alpha < V(\infty)$, thereexists $R \ge n$ such that $V(x) \ge \alpha$ for all$|x| \ge R$. Hence, for all $i \ge i_0$ and $|x| \ge R$, we have$-\alpha + \lambda_i V(x) \ge -\alpha + (1+\varepsilon)\alpha > 0$.Since$R \ge n$, we have for $|x| > R$, $\psi_n(x) = 0$ and

$$0 = L(\lambda_i)z_i = -\Delta z_i + \left(-\alpha + \lambda_i V(x)\right)z_i \quad (3.6)$$

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Recalling that $X \subset C(\mathbb{R}^N)$, we have $\delta = \inf_{|x|\le R} z(x) > 0$. Since $z_i \xrightarrow{i} z$ strongly in X, we have that $\max_{|x|\le R} |z_i(x) - z(x)| \xrightarrow{i} 0$ and there exists $i_1 \ge i_0$ such that $z_i(x) \ge \frac{1}{2}\delta$ for all $|x| \le R$ and $i \ge i_1$. Since $z_i(x) \ge \frac{1}{2}\delta$ for $|x| = R$ and $\lim_{|x|\to+\infty} z_i(x) = 0$, we get $z_i(x) \ge 0$ on ∂B_R^c. Now we can apply the strong maximum principle (Theorem 8.19 in [4]) to equation (3.6) on the region B_R^c and deduce that $z_i > 0$ in B_R^c for all $i \ge i_1$. This contradicts that each u_i has at least one zero in $\mathbb{R}^N$.

Now we can give more information about $\mathcal{C}_n$.

Theorem 3.1: Let $\mathcal{C}_n$ denote the connected component of $\mathcal{Z}_n \cup \{(\Lambda(\alpha), 0)\}$ containing the point $(\Lambda(\alpha), 0)$. Then:

(i) $u^2 > 0$ on $\mathbb{R}^N$ and $\lambda \le \Lambda(\alpha)$ for all $(\lambda, u) \in \mathcal{C}_n \backslash \{(\Lambda(\alpha), 0)\}$;

(ii) for any $\beta > \frac{\alpha}{V(\infty)}$, there exist $T > 0$ and $N_\beta \in \mathbb{N}$ such that, for all $n \ge N_\beta$,

$inf P\mathcal{C}_n := \quad inf\{\lambda: (\lambda, u) \in \mathcal{C}_n\} < \beta$ and $||u|| \le T$

for all $(\lambda, v) \in \mathcal{C}_n$ with $\lambda \ge \beta$.

Proof : The first step is to show that if $(\lambda, u) \in \mathcal{C}_n \backslash \{(\Lambda(\alpha), 0)\}$, we have $\lambda \le \Lambda(\alpha)$. Since (λ, u) solves equation (3.3), we have

$$-\Delta u - \alpha u + \lambda V(x) u + \psi_n(x) k\left(x, \frac{u}{||u||^2}\right) u = 0 \qquad (3.7)$$

By Lemma 3.3, we know $u \in H^1(\mathbb{R}^N)$. Hence by Theorem 1.2 (ii) and (3.7),

$$0 = inf\left\{\int_{\mathbb{R}^N} [|\nabla v|^2 - \alpha v^2 + \Lambda(\alpha) V v^2] dx : v \in H^1(\mathbb{R}^N)\right\}$$

$$\le \int_{\mathbb{R}^N} [|\nabla u|^2 - \alpha u^2 + \Lambda(\alpha) V u^2 + \psi_n(x) k\left(x, \frac{|u|}{||u||^2}\right) u^2] dx$$

$$= (\Lambda(\alpha) - \lambda) \int_{\mathbb{R}^N} V(x) u^2 dx.$$

Since $\int_{\mathbb{R}^N} V(x) u^2 dx \ge V_0 \int_{\mathbb{R}^N} u^2 dx > 0$, we see that $\lambda \le \Lambda(\alpha)$.

The next step is to show that if $(\lambda, u) \in \mathcal{C}_n \backslash \{(\Lambda(\alpha), 0)\}$, then $u^2 > 0$ on $\mathbb{R}^N$. Set

$$Q = \{(\lambda, u) \in \mathcal{C}_n : u^2 > 0 \text{ on } \mathbb{R}^N\} \cap \{(\Lambda(\alpha), 0)\}.$$

We prove that Q is both an open and closed subset of $\mathcal{C}_n$, then by the connectedness of $\mathcal{C}_n$ we have $Q = \mathcal{C}_n$.

First we prove that Q is open in $\mathcal{C}_n$. Given $(\lambda, u) \in Q$, we show that there exists an open neighborhood U of (λ, u) in $\mathbb{R}^N \times X$ such that $U \cap \mathcal{C}_n \subset Q$. For $(\lambda, u) = (\Lambda(\alpha), 0)$ this is established in Lemma 3.4. For $(\lambda, u) \in \mathcal{C}_n$ with $u^2 > 0$ on $\mathbb{R}^N$, we have that u does not change sign since $X \subset C(\mathbb{R}^N)$. We suppose that $u > 0$ on $\mathbb{R}^N$, the case $u < 0$ being similar. By (V_2) and $\lambda > \frac{\alpha}{V(\infty)}$, there exist $r > 0$ and $R \ge n$ such that for all η with $|\lambda - \eta| \le r$, $-\alpha + \eta V(x) > 0$ for all $|x| \ge R$. Let $\delta = \inf_{|x|\le R} u(x)$. Then, $\delta > 0$ and there exists an open neighborhood U of (λ, u) in $\mathbb{R}^N \times X$ such that $|\lambda - \eta| \le r$ and $\inf_{|x|\le R} v(x) \ge \frac{\delta}{2}$ for all $(\eta, v) \in U$. Since $R \ge n$, for $|x| > R$, we have $\psi_n(x) = 0$ and for $(\eta, v) \in U \cap \mathcal{Z}_n$, $L(\eta) v = -\Delta v + (-\alpha + \eta V(x)) v = 0$. As the proof of Lemma 3.4, the maximum principle implies that $v(x) > 0$ for $|x| > R$ and then $v > 0$ on $\mathbb{R}^N$. Hence $U \cap \mathcal{C}_n \subset Q$ and Q is open.

Now we show that Q is closed in $\mathcal{C}_n$. Suppose that $(\lambda, u) \in \mathcal{C}_n$ and there exists a sequence $\{(\lambda_i, u_i)\} \subset Q$ such that $\lambda_i \xrightarrow{i} \lambda$ and $||u_i - u|| \xrightarrow{i} 0$. If $u = 0$, we must have $\lambda = \Lambda(\alpha)$ since $\mathcal{C}_n \cap [\mathbb{R} \times \{0\}] = \{(\Lambda(\alpha), 0)\}$ and so $(\lambda, u) \in Q$. If $u \ne 0$, passing to a subsequence, we may as well suppose that $u_i > 0$ on $\mathbb{R}^N$ for all $i \in \mathbb{N}$. It follows that $u \ge 0$ on $\mathbb{R}^N$ and $-\Delta u + c_+ u = c_- u \ge 0$ on $\mathbb{R}^N$, where $c(x) = -\alpha + \lambda V(x) + +\psi_n k\left(x, \frac{|u|}{||u||^2}\right)$. By the strong maximum principle we have $u > 0$ on $\mathbb{R}^N$. Hence $(\lambda, u) \in Q$ and Q is closed in $\mathcal{C}_n$.

Now we know that $Q = \mathcal{C}_n$. We claim that this means that case (ii) in Theorem 1.1 cannot occur. Indeed, if $\mathcal{C}_n$ has the property (ii), there exist $\lambda \in J \backslash \{\Lambda(\alpha)\}$ and a sequence $\{(\lambda_i, u_i)\} \subset \mathcal{C}_n$ such that $\lambda_i \xrightarrow{i} \lambda$ and $||u_i|| \xrightarrow{i} 0$. Setting $z_i = \frac{u_i}{||u_i||}$ and arguing as in the proof of Lemma 3.4, we may assume that $z_i \xrightarrow{i} z$ strongly in X and $L(\lambda) z = 0$ with $||z|| = 1$. By Lemma 3.3, we know $z \in H^1(\mathbb{R}^N)$. It follows from Theorem 1.2 (i) that $\lambda < \Lambda(\alpha)$ and z changes sign on $\mathbb{R}^N$. On the other hand, since $Q = \mathcal{C}_n$, the sequence z_i can be chosen so that $z_i > 0$ on $\mathbb{R}^N$. Then we have $z \ge 0$ on $\mathbb{R}^N$ and this contradicts the earlier conclusion.

For any $\beta > \frac{\alpha}{V(\infty)}$, by Lemma 3.2 there exists $N_\beta \in \mathbb{N}$ such that $||v|| \le T$ for all $(\lambda, v) \in \mathcal{C}_n$ with $\lambda \ge \beta$ and $n \ge N_\beta$. Thus, if $n \ge N_\beta$ and $inf P\mathcal{C}_n \ge \beta$, we deduce that $\mathcal{C}_n$ is bounded in $\mathbb{R} \times X$. Since we have shown that $\mathcal{C}_n$ has not the property (ii) in Theorem 1.1, we must have $\mathcal{C}_n$ satisfying $inf P\mathcal{C}_n = inf J = \frac{\alpha}{V(\infty)}$. This contradicts our earlier assumption $inf P\mathcal{C}_n \ge \beta > \frac{\alpha}{V(\infty)}$ and we complete the proof.

IV. PROOF OF THEOREM 1.3

In this section, by using the global bifurcation result for the inverted truncated problem (3.3), we first prove the bifurcation result for the following inverted problem

$$L(\lambda)v + K(v) = 0, \quad (4.1)$$

Where $K(v) = \begin{cases} k\left(x, \frac{|v|}{||v||^2}\right)v, & for\ v \in X\backslash\{0\} \\ 0, & for\ v \equiv 0. \end{cases}$

Set $\mathcal{Z} = \{(\lambda, v) \in (\frac{\alpha}{V(\infty)}, +\infty) \times X: v$ is a nontrivial solution of (4.1)$\}$ and $\tilde{\mathcal{Z}} = \{(\lambda, v) \in \mathcal{Z}: v^2 > 0\ on\ \mathbb{R}^N\}$.

Lemma 4.1: Let $\{(\lambda_n, u_n)\} \subset \tilde{\mathcal{Z}}$ be a sequence such that $\lambda_n \overset{n}{\to} \lambda > \frac{\alpha}{V(\infty)}$ and $||v_n|| \overset{n}{\to} 0$. Then $\lambda = \Lambda(\alpha)$.

Proof: First we show that

$$\frac{|K\left(\frac{|v_n|}{||v_n||^2}\right)|_p}{||v_n||} \overset{n}{\to} 0 \quad (4.2)$$

for all $p \in (1, +\infty)$. Since (λ_n, v_n) solves (4.1), we have $v_n L(\lambda_n) v_n \le 0$ on $\mathbb{R}^N$. From $\lambda_n \overset{n}{\to} \lambda > \frac{\alpha}{V(\infty)}$, we may choose $\beta = \frac{\alpha}{V(\infty)} + \frac{1}{2}\left(\lambda - \frac{\alpha}{V(\infty)}\right) > \frac{\alpha}{V(\infty)}$ such that $\lambda_n \ge \beta$ for n large. It follows from Lemma 3.3 that there exist $L > 0$ and $\gamma > 0$ which are independent of n such that

$$|v_n(x)| \le L|v_n|_\infty e^{-\gamma|x|} \text{for all } x \in \mathbb{R}^N. \quad (4.3)$$

By the Sobolev embedding, there is $C > 0$ which is independent of n such that $|v_n|_\infty \le C||v_n||$. Thus, we have

$$\frac{|K\left(\frac{|v_n|}{||v_n||^2}\right)|}{||v_n||} \le LCe^{-\gamma|x|} \text{for all } x \in \mathbb{R}^N. \quad (4.4)$$

Conseuqently, for every fixed $x \in \mathbb{R}^N$,

$$\frac{|K\left(\frac{|v_n|}{||v_n||^2}\right)|}{||v_n||} \overset{n}{\to} 0 \quad (4.5)$$

Combining (4.3) and (4.5), it follows from the dominated convergence that (4.2) holds. Let $\omega_n = \frac{v_n}{||v_n||}$. Since $v_n^2 > 0$ on $\mathbb{R}^N$, passing to a subsequence, we may assume that $\omega_n > 0$ on $\mathbb{R}^N$ for all $n \in \mathbb{N}$ and for some $\omega \in X$ with $\omega \ge 0$ on $\mathbb{R}^N$, $\omega_n \rightharpoonup \omega$ weakly in X. By (4.2) we deduce that

$$L(\lambda)\omega_n = L(\lambda_n)\omega_n - (\lambda_n - \lambda)V(x)\omega_n \overset{n}{\to} 0\ in\ Y$$

Since $\lambda > \frac{\alpha}{V(\infty)}$, we have $L(\lambda) \in \Phi_0(X, Y)$. By Lemma 3.5 of [1] we know that $\omega_n \overset{n}{\to} \omega$ strongly in X. So $L(\lambda)\omega = 0$ and $||\omega|| = 1$. By Lemma 3.3 we have $\omega \in H^1(\mathbb{R}^N)$. This means that λ is an eigenvalue of equation (1.6) and its corresponding eigenfunction ω does not change sign. Thus it follows from Theorem 1.2 (i) that $\lambda = \Lambda(\alpha)$.

Theorem 4.1: Let $\mathcal{C}$ denote the connected component of $\tilde{\mathcal{Z}} \cup \{(\Lambda(\alpha), 0)\}$ containing $\{(\Lambda(\alpha), 0)\}$. The following hold.

(i) $\mathcal{C}$ is bounded with $infP\mathcal{C} = \frac{\alpha}{V(\infty)}$ and $supP\mathcal{C} = \Lambda(\alpha)$.

(ii) If $\{(\lambda_n, v_n)\} \subset \mathcal{C}$ with $\lim_{n\to+\infty} \lambda_n = \Lambda(\alpha)$, then $\lim_{n\to+\infty} ||v_n|| = 0$.

Proof: (i) First we show that if $(\lambda, v) \in \tilde{\mathcal{Z}}$, then $\lambda < \Lambda(\alpha)$. Since $v \ne 0$ solves equation (4.1), by Lemma 3.3, we know $v \in H^1(\mathbb{R}^N)$ and then we deduce that $u = \frac{v}{||v||^2}$ solves equation (1.5). By Theorem 1.2 (ii) we have

$$0 = inf\left\{\int_{\mathbb{R}^N} [|\nabla\omega|^2 - \alpha\omega^2 + \lambda V\omega^2]dx: \omega \in H^1(\mathbb{R}^N)\ and \int_{\mathbb{R}^N} \omega^2 dx = 1\}\right\}$$

$$\le \frac{1}{\int_{\mathbb{R}^N} u^2 dx}[(\Lambda(\alpha) - \lambda)\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} k(x, u)u^2 dx]$$

Thus $(\Lambda(\alpha) - \lambda)\int_{\mathbb{R}^N} V(x)u^2 dx \ge \int_{\mathbb{R}^N} k(x, u)u^2 dx$. We claim that $\int_{\mathbb{R}^N} k(x, u)u^2 dx > 0$. Indeed, since $\lim_{s\to 0} k(x, s) = l > 0$ uniformly in $x \in \mathbb{R}^N$, there exists $\delta > 0$ such that $k(x, s) \ge \frac{l}{2}$ for all $|s| \le \delta$ and $x \in \mathbb{R}^N$. By Proposition 1.1 we have $\lim_{|x|\to+\infty} u(x) = 0$ and there exists $R > 0$ such that $|u(x)| \le \delta$ for $|x| \ge R$. Since $(\lambda, v) \in \tilde{\mathcal{Z}}$, we have $v^2 > 0$ on $\mathbb{R}^N$ and $u^2 > 0$ on $\mathbb{R}^N$. Therefore we deduce that

$$\int_{\mathbb{R}^N} k(x, u)u^2 dx \ge \int_{|x|\ge R} k(x, u)u^2 dx \ge \frac{l}{2}\int_{|x|\ge R} u^2 dx.$$

It follows that $\lambda < \Lambda(\alpha)$. Hence $\{\lambda: (\lambda, v) \in \tilde{\mathcal{C}}\backslash\{(\Lambda(\alpha), 0)\} \subset (\frac{\alpha}{V(\infty)}, \Lambda(\alpha))$ and $supP\mathcal{C} = \Lambda(\alpha)$.

Secondly we show that $||v|| \le T$ for all $(\lambda, v) \in \tilde{\mathcal{Z}}$, where $T > 0$ is defined in Lemma 3.2. By contradiction, if $(\lambda, v) \in \mathcal{Z}$ and $||v|| \ge T$, similar to the proof of Lemma 3.2, we have for all $x \in \mathbb{R}^N$, $K\left(\frac{v(x)}{||v||^2}\right) \ge l - \eta$, where $\eta = \frac{1}{2}\left(\frac{\alpha V_0}{V(\infty)} - \mu\right) > 0$. Consequently,

$$v\Delta v = [-\alpha + \lambda V(x) + k\left(x, \frac{v(x)}{||v||^2}\right)]v^2(x)$$

$$\ge \left[-\mu + \frac{\alpha V_0}{V(\infty)} - \eta\right]v^2(x) = \eta v^2(x) \ge 0.$$

The maximum principle leads to a contradiction. Therefore we see that $\tilde{\mathcal{Z}}$ is bounded in $\mathbb{R} \times X$ and so is $\mathcal{C}$.

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The next step is to prove that $infP\mathcal{C} = \frac{\alpha}{V(\infty)}$. By contradiction, supposethat $\rho := infP\mathcal{C} > \frac{\alpha}{V(\infty)}$. For any $\bar{\rho} \in (\frac{\alpha}{V(\infty)}, \rho)$, we set $A = \{(\lambda, v) \in Q: \lambda \geq \bar{\rho}\}$, $A_1 = \{(\Lambda(\alpha), 0)\}$, $A_2 = \{(\lambda, v) \in Q: \lambda = \bar{\rho}\}$, where $Q = \tilde{\mathcal{Z}} \cup \{(\Lambda(\alpha), 0)\}$. We claim that $A$is a compact subset of $\mathbb{R} \times X$.

Indeed, let$\{(\lambda_n, v_n)\} \subset A$ be an infinitely sequence. Passing toa subsequence, we may suppose that $v_n > 0$ on $\mathbb{R}^N$ forall $n$and$\lambda_n \overset{n}{\to} \lambda \in [\bar{\rho}, \Lambda(\alpha)]$,$v_n \rightharpoonup v$ weakly in X,$||v_n|| \overset{n}{\to} \tau \geq 0$.To prove this claim, it is sufficient to prove $v_n \overset{n}{\to} v$ strongly in $X$with$(\lambda, v) \in A$.If $\tau \geq 0$, then $v_n \overset{n}{\to} v$ strongly in X. By Lemma 4.1, we must have $(\lambda, v) = (\Lambda(\alpha), 0) \in A$. If $\tau \neq 0$, similar to the proof ofLemma 4.1, we can deduce that$||v_n - v|| \overset{n}{\to} 0$ and hence $(\lambda, v) \in \mathcal{Z}$. The strong maximum implies that $(\lambda, v) \in A$. By a result of Whyburn [11] (see Lemma C.2 in [10]), we can prove that there exists a connected subset A_0 of A suchthat$A_0 \cap A_1 \neq \emptyset$ and $A_0 \cap A_2 \neq \emptyset$ .It follows that $infPA_0 = \bar{\rho} < \rho$. But since$A_0 \subset \mathcal{C}$, we have $infP\mathcal{C} \leq \bar{\rho} < \rho$, a contradiction.

(ii) By contradiction, suppose that there exists a sequence$\{(\lambda_n, v_n)\} \subset \mathcal{C} \backslash \{(\Lambda(\alpha), 0)\}$ such that $\lambda_n \overset{n}{\to} \Lambda(\alpha)$ and $||v_n|| \geq \delta > 0$ for all $n \in \mathbb{N}$. Bythe proof of part (i) we have$||v_n|| \leq T$. Passing to asubsequence, we may assume that $v_n \rightharpoonup v$ weaklyin X. Similar to the proof of Lemma 4.1, we candeduce that $||v_n - v|| \overset{n}{\to} 0$and $(\Lambda(\alpha), v) \in \mathcal{Z}$. Then by the strong maximum principle wehave$(\Lambda(\alpha), v) \in \tilde{\mathcal{Z}}$. But at thebeginning of the proof of part (i) we proved that $\lambda < \Lambda(\alpha)$ for all $(\lambda, v) \in \tilde{\mathcal{Z}}$. This is a contradiction.

We have established the global properties of a connected subset of$\tilde{\mathcal{Z}} \cup \{(\Lambda(\alpha), 0)\}$containing$\{(\Lambda(\alpha), 0)\}$. However, in order to maintainconnectedness under inversion, we need to find a connected subset of$\tilde{\mathcal{Z}}$ having similar properties.

Set $\mathcal{Z}^+ = \{(\lambda, v) \in \mathcal{Z}: v > 0 \; on \; \mathbb{R}^N\}$ and $\mathcal{Z}^- = \{(\lambda, v) \in \mathcal{Z}: v < 0 \; on \; \mathbb{R}^N\}$.

Corollary 4.1: Let the function f be odd. Then there exist two bounded connectedsubsets $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ of $\mathcal{Z}^+$ and $\mathcal{Z}^-$, respectively, satisfying the followingproperties:

(i) $infP\mathcal{C}_0^{\pm} = \frac{\alpha}{V(\infty)}and(\Lambda(\alpha), 0) \in \overline{\mathcal{C}_0^{\pm}}$.

(ii) $supP\mathcal{C}_0^{\pm} = \Lambda(\alpha)$ and $0 < ||v|| \leq T$ for all $(\lambda, v) \in \mathcal{C}_0^{\pm}$, where $T > 0$ isgiven by Lemma 3.2.

Proof : The proof is the same as the one of Corollary 5.3 in [10].

Proof of Theorem1.3

Let f_R and f_L be the odd functions defined by

$$f_R(s) = \begin{cases} f(x,s), & for \; s \geq 0, \\ -f(x,-s), & for \; s < 0 \end{cases}$$

and$f_L(s) = \begin{cases} -f(x,-s), & for \; s \geq 0, \\ f(x,s), & for \; s < 0. \end{cases}$

By corollary 4.1 there exist two bounded connected subsets$\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ of positive or negativesolutions for problem (1.1) with f_R or$f_L(s)$respectively. Setting

$$\sum^{\pm} = \{\left(\lambda, \frac{v}{||v||^2}\right): (\lambda, v) \in \mathcal{C}_0^{\pm}\},$$

it follows that $\sum^{\pm}$ areconnected sets of $(\frac{\alpha}{V(\infty)}, \Lambda(\alpha)) \times W^{2,p}(\mathbb{R}^N)$ consisting of, respectively, positive and negative solutions of (1.1) with $infP\sum^{\pm} = \frac{\alpha}{V(\infty)}$, $supP\sum^{\pm} = \Lambda(\alpha)$and $||u|| \geq \frac{1}{T}$for all $(\lambda, v) \in \sum^{\pm}$.

Suppose that $\{(\lambda_n, u_n)\} \subset \sum^{\pm}$ with $\lambda_n \overset{n}{\to} \lambda > \frac{\alpha}{V(\infty)}$ and $\max_{x \in \mathbb{R}^N} |u_n(x)| \overset{n}{\to} \infty$. Then $||u_n|| \overset{n}{\to} +\infty$ by the Sobolev embedding. Hence$(\lambda_n, v_n) \in \tilde{\mathcal{Z}}$ with $v_n = \frac{u_n}{||u_n||^2}$and $||v_n|| \overset{n}{\to} 0$. ByLemma 4.1 we have $\lambda = \Lambda(\alpha)$. On the otherhand, if$\{(\lambda_n, u_n)\} \subset \sum^{\pm}$ with $\lambda_n \overset{n}{\to} \Lambda(\alpha)$, by setting $v_n = \frac{u_n}{||u_n||^2}$ we know that$(\lambda_n, v_n) \in \mathcal{C}$. Then by Theorem 4.1 (ii) we have $||v_n|| \overset{n}{\to} 0$. This means that$||u_n|| \overset{n}{\to} +\infty$. We claim that $\max_{x \in \mathbb{R}^N} |u_n| \overset{n}{\to} +\infty$. Otherwise, passing to a subsequence, there is $C > 0$ such that$\max_{x \in \mathbb{R}^N} |u_n| \leq C$ for all$n \in \mathbb{N}$. Since(λ_n, u_n) is a solution of problem (1.1) we have

$$L(\lambda_n)u_n + k(x, u_n)u_n = 0 \quad (4.6)$$

By Lemma 3.3, we know that $u_n$isbounded in$Y$. Therefore, $\{(-\Delta + 1)u_n\}$is bounded in Y by (4.6). Since$-\Delta + 1: X \to Y$ isan isomorphism, this implies that $\{u_n\}$ is bounded in X, a contradiction.

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