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Dynamic properties of structures with dampers modelled using fractional order derivatives

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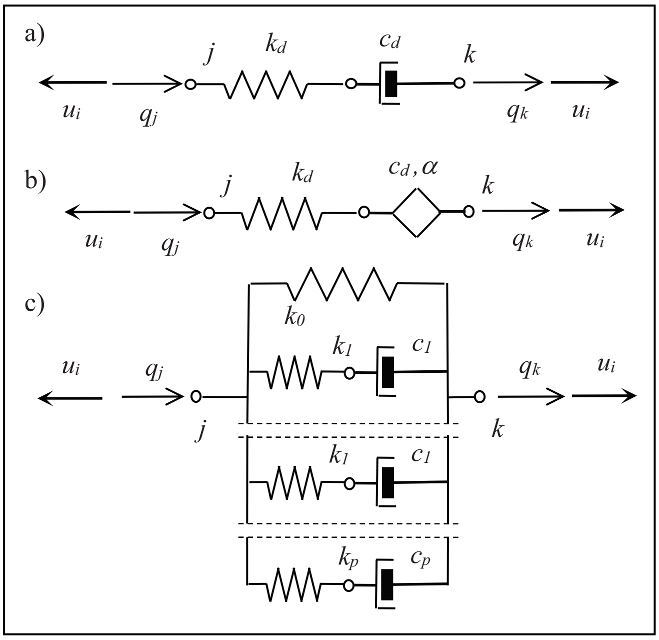
Abstract—The focus of this paper is on determination of the dynamic parameters of structural systems with viscoelastic (VE) dampers described by Maxwell rheological models. Such parameters could be obtained after solving the appropriately defined nonlinear eigenvalue problem for frames with VE dampers. The solution to the nonlinear eigenvalue problem is obtained by equating to zero the determinant of the considered system of equations. Apart from complex conjugate eigenvalues, the real ones occurred when dampers that are described by the classic Maxwell model, are also determined.

Keywords: viscoelastic damper; rheological model; fractional derivative; nonlinear eigenvalue problem; dynamic properties;

1. Introduction

In civil engineering passive damping systems are mounted on structures in order to reduce excessive vibrations caused by winds and earthquakes [1-3]. Different kinds of mechanical devices, such as viscous dampers, viscoelastic dampers, tuned mass dampers, or base isolation systems, are used in the passive systems. In the past, several rheological models were proposed for describing the dynamic behaviour of VE dampers and materials [1-3]. In recent years, the fractional calculus has received considerable attention and has been used in modelling the rheological behaviour of VE materials and dampers [4, 5]. The fractional models have an ability to correctly describe the behaviour of VE materials and dampers using a small number of model parameters. However, in this case, the VE damper equation of motion is a fractional differential equation [6]. It is the aim of this paper to find the dynamic properties (i.e., natural frequencies and non-dimensional damping factors) for structures with VE dampers. The above-mentioned properties are defined on the basis of eigenvalues, obtained from the nonlinear eigenproblem. The approach, as presented in this work, differs from the standard one which mostly uses the state-space variables and the dynamic parameters are derived from the linear eigenproblem [7] or the non-linear eigenproblem [8], depending on the assumed model of damper. One of the most important achievements of the proposed formulation is the dimension of the problem, which is much smaller, compared with the standard approach. The solution to the nonlinear eigenvalue problem is obtained by equating to zero the determinant of the considered system of equations. The results of sample numerical calculations are presented and discussed. It is shown that the results of nonlinear eigenproblem which correspond to the classic models differ qualitatively from the results obtained for the fractional model.

2. Rheological Model of Damper



The rheological properties of VE dampers were described using three different Maxwell models, i.e., classic model (Fig. 1a), fractional model (Fig. 1b) and generalized model (Fig. 1c). The classic Maxwell model consists of a dashpot with the constant c_d, connected in series with a spring of the stiffness k_d.

Figure 1. Rheological models of damper.

In the case of the fractional Maxwell model of damper, instead of the dashpot we have a fractional dashpot (see Fig. 1b) with the constants: c_d and α ($0 < \alpha \leq 1$), which denotes the order of fractional derivative [6].

In the generalized Maxwell model (Fig. 1c), there is an additional element of the stiffness k_0, which is connected in parallel with the other elements of the system, described respectively by stiffness k_l and damping c_l , ($l = 1,2,..., p$). The equations of motion for the classic or fractional and generalized Maxwell models could be written as follows:

$$\begin{aligned} &D_t^{\alpha} u_i(t) + \nu\, u_i(t) = k_d D_t^{\alpha} \Delta q_i(t) \\ &\nu_l u_{li}(t) + D_t^1 u_{li}(t) = k_l D_t^1 \Delta q_i(t) \qquad , \\ &u_{0i}(t) = k_0 \Delta q_i(t) \qquad l = 1,2,..., p \end{aligned} \tag{1}$$

where, $\Delta q_i(t) = q_j - q_k$, $\nu = k_d / c_d$, $\nu_l = k_l / c_l$. Moreover, u_i , u_{li} and q_j , q_k denote the dampers force, the force in the *j*-th Maxwell unit and nodal displacements, respectively. The symbol $D_t^{\alpha}(\bullet)$ denotes the Riemann-Liouville fractional derivative of the order α with respect to time, *t*. More information concerning the fractional derivative can be found in [6]. For consistent notation, we introduce $D_t^1(q) = \dot{q}(t)$. The equation of motion for the classic Maxwell model could be obtained after introducing into (1) $\alpha = 1$.

3. Structural system with dampers

A. The equation of motion

In this paper, the structure with VE dampers is treated as an elastic linear system modelled as a shear frame. The mass of the system is lumped at the level of storeys. Viscoelastic dampers are installed between two successive storeys. The equation of motion of the structure with dampers can be written as follows:

$$\mathbf{M}_s \ddot{\mathbf{q}}_s(t) + \mathbf{C}_s \dot{\mathbf{q}}_s(t) + \mathbf{K}_s \mathbf{q}_s(t) = \mathbf{f}(t) + \mathbf{p}(t) \tag{2}$$

where the symbols $\mathbf{M}_s$, $\mathbf{C}_s$ and $\mathbf{K}_s$ denote the mass, the damping, and the stiffness matrices, respectively. Moreover, $\mathbf{q}_s(t) = [q_{s,1},..., q_{s,j},..., q_{s,n}]^T$ denotes the vector of displacements of the structure and $\mathbf{p}(t) = [p_1,..., p_j,..., p_n]^T$ the vector of excitation forces. The components of vector $\mathbf{f}(t) = [f_1, f_2,..., f_n]^T$ are the interaction forces between the frame and the dampers (Fig.. 2).

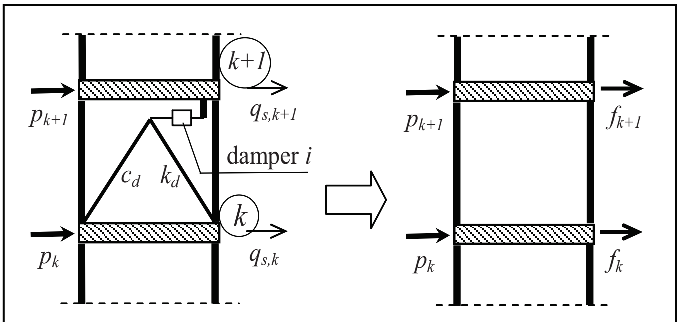


Figure 2. Diagram of frame and the interaction forces.

If a structure with only one damper denoted as the damper number *i*, mounted between two successive storeys, *k* and *k+1*, is considered, then the vector of damper forces could be written in the following form:

$$\mathbf{f}_i(t) = [0,..., f_k = u_i, f_{k+1} = -u_i,.....,0]^T = \mathbf{e}_i u_i(t) \tag{3}$$

where, $\mathbf{e}_i = [0,..., e_k = 1, e_{k+1} = -1,...,0]^T$. For a structure with *m* dampers the vector of interaction forces is the sum of vectors $\mathbf{f}_i(t)$, i.e.:

$$\mathbf{f}(t) = \sum_{i=1}^{m} \mathbf{f}_i(t) \,. \tag{4}$$

B. The Laplace transform

After applying the Laplace transform and taking into account that:

$$\mathcal{L}[\mathbf{q}(t)] = \bar{\mathbf{q}} \,,\ \mathcal{L}\left[D_t^{\alpha}\, \mathbf{q}(t)\right] = s^{\alpha}\, \bar{\mathbf{q}} \,,\ \mathcal{L}\left[D_t^1 \mathbf{q}(t)\right] = s\bar{\mathbf{q}} \,, \tag{5}$$

the equation of motion (2) can be written as:

$$\left(s^2 \mathbf{M}_s + s\ \mathbf{C}_s + \mathbf{K}_s\right) \bar{\mathbf{q}}_s(s) = \bar{\mathbf{f}}(s) + \bar{\mathbf{p}}(s) \,. \tag{6}$$

The vectors $\bar{\mathbf{q}}_s(s)$, $\bar{\mathbf{f}}(t)$ and $\bar{\mathbf{p}}(s)$ denote the Laplace transforms of displacements and forces, respectively. According to (3), for damper *i,* the force transform is $\bar{\mathbf{f}}_i(s) = \mathbf{e}_i \bar{u}_i(s)$. The Laplace transform converts (1) into one relationship which is valid for each considered model of damper:

$$\bar{u}_i(s) = \left[k_{vi} + G_i(s)\right]\ \Delta\bar{q}_i(s) \,. \tag{7}$$

The quantities k_{vi} and $G_i(s)$ are defined as:

$$\begin{aligned} &k_{vi} = 0 \qquad G_i(s) = \frac{s^{\alpha_i} \cdot k_{di}}{\nu_i + s^{\alpha_i}} \\ &k_{vi} = k_{0i} \qquad G_i(s) = \sum_{l=1}^{p} \frac{s \cdot k_{li}}{\nu_{li} + s} \end{aligned} \,. \tag{8}$$

for classic ($\alpha_i = 1$) or fractional models and for generalized Maxwell model, respectively. Now the second index in the symbols k_{0i} , k_{li} , ν_{li} refers to the damper's number. Moreover, $\Delta\bar{q}_i(s) = -\mathbf{e}_i^T \bar{\mathbf{q}}_s(s)$.

Finally, one may rewrite (6) in the following form:

$$\left[s^2 \mathbf{M}_s + s\ \mathbf{C}_s + \mathbf{K} + \mathbf{G}(s)\right] \bar{\mathbf{q}}_s(s) = \bar{\mathbf{p}}(s) \,, \tag{9}$$

where, $\mathbf{K} = \mathbf{K}_s + \mathbf{K}_v$, $\mathbf{K}_v = \sum_{i=1}^{m} k_{vi}\mathbf{e}_i\mathbf{e}_i^T$, $\mathbf{G}(s) = \sum_{i=1}^{m} G_i(s)\mathbf{e}_i\mathbf{e}_i^T$, m stands for the number of dampers.

C. Nonlinear eigenproblem

For $\overline{\mathbf{p}}(s) = \mathbf{0}$, the equation of motion (9) expresses a nonlinear eigenproblem from which the eigenvalues and eigenvectors can be determined [8]. In the case of the fractional Maxwell model it is possible to write the relationship:

$$\left[\left(s^2\mathbf{M}_s + s\ \mathbf{C}_s + \mathbf{K}\right) + \sum_{i=1}^{m} \frac{s^{\alpha_i}}{(\nu_i + s^{\alpha_i})}\ \mathbf{B}_i\right]\overline{\mathbf{q}}_s = \mathbf{0}, \quad (10)$$

where, $\mathbf{B}_i = k_i\mathbf{e}_i\mathbf{e}_i^T$. For the classic Maxwell model $\alpha_i = 1$.

The solution to the system (10) yields a set of complex and conjugate eigenvalues. The number of pairs of eigenvalues equals the number of degree of freedom for the considered system. Moreover, for a structural system with the classic or the generalized Maxwell models of dampers we obtain a set of real eigenvalues of which the number equals the number of all dashpots occurring in the damper models. The calculation carried out by the authors suggests that for the dampers described by the fractional Maxwell model, real solutions do not exist.

A nonlinear eigenproblem can be solved using the continuation method which is similar to the one described in the paper [8]. Another possibility to obtain the values s_i is a method of equating to zero the determinant of the system of equations [9]. It is to be noted that for $\alpha_i = 1$ the value existing in the denominator in (10) leads to the singularity when $s = -\nu_i$. In order to eliminate these singularities, we transform the system of equations (10) into the following form:

$$\left[\prod_{i=1}^{m}(\nu_i + s^{\alpha_i})\left(s^2\mathbf{M}_s + s\ \mathbf{C}_s + \mathbf{K}\right) + \sum_{i=1}^{m} s^{\alpha_i}\prod_{\substack{k=1\\k\neq i}}^{m}(\nu_k + s^{\alpha_k})\mathbf{B}_i\right]\overline{\mathbf{q}}_s = \mathbf{0} \quad (11)$$

In this work, real and complex eigenvalues are obtained by searching the value of determinant of the system (11) and by evaluating the roots of determinant function.

D. Dynamic properties of structure

The dynamic behaviour of a frame with viscoelastic dampers is characterized by the natural frequencies ω_i and the non-dimensional damping parameters γ_i. Similarly to viscous damping, the above-mentioned properties are defined as follows:

$$\omega_i^2 = \mu_i^2 + \eta_i^2, \qquad \gamma_i = -\mu_i / \omega_i, \quad (12)$$

where $\mu_i = \mathrm{Re}(s_i)$, $\eta_i = \mathrm{Im}(s_i)$. For the real eigenvalues s_i relationships (12) are not valid. The real eigenvalues correspond to the rheological properties of the considered dampers.

4. Results of Calculation

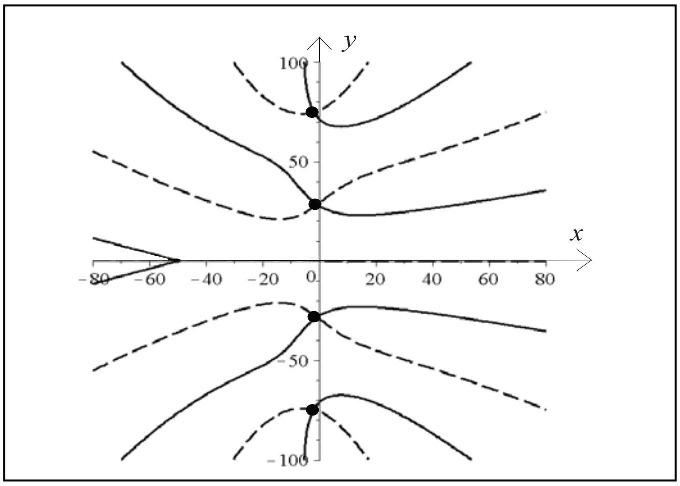
In the numerical example, a two-storey building structure modelled as a shear plane frame with the rigid beams is considered. The mass is lumped and same at every floor: $m_s = 2000kg$. The bending rigidity of each storey is $k_s = 4000\ kN/m$. The viscoelastic damper with the stiffness parameter $k_1 = 2500\ kN/m$ and damping $c_1 = 50\ kNs^{\alpha}/m$ is mounted on the first storey. Thus, the coefficient $\nu_1 = 50$. On the second floor there is a damper, characterized by the parameters: $k_2 = 600\ kN/m$, $c_2 = 30\ kNs^{\alpha}/m$, $\nu_2 = 20$.

Firstly, the calculations were carried out for a frame with dampers described by the fractional Maxwell model, for which the value of the fractional parameter was $\alpha = 0.6$. After equating to zero the determinant of the system of equations (11), we obtain the characteristic equation which enables four complex and conjugate eigenvalues s_i to be derived (see Table I).

TABLE I. THE EIGENVALUES – FRACTIONAL MAXWELL MODEL

Root number i	$\Re(s_i)$	$\Im(s_i)$
1, 2	-0.7254	±28.3351
3, 4	-1.8983	±74.8284

The value determinant $\det(A) = Z(s) = a + i \cdot b$ is a complex number which depends on the complex variable $s = x + i \cdot y$, where $i = \sqrt{-1}$. Thus, the value of the determinant equals zero only if its real and imaginary part



simultaneously is equal to zero, $a(x, y) = 0$ and $b(x, y) = 0$. The roots $s_i = x_i + i \cdot y_i$ of a characteristic equation are in a complex plane at the intersection of lines, along which the real part is equal to zero (solid line in Fig. 3) and the imaginary part is equal to zero (dashed line in Fig. 3).

Figure 3. Plot of functions $\Re(Z)$, $\Im(Z)$ - fractional Maxwell model.

In Fig. 3 one may observe four such intersection points of which the coordinates coincide with the values given in Table I.

Next, the dampers were modeled using the classic Maxwell model. The eigenproblem derived in the form of (11) was solved by equating the determinant of the system of equations to zero. This leads to a characteristic equation of which the solution yields four complex, conjugate eigenvalues s_i and four real eigenvalues (see Table II).

TABLE II. THE EIGENVALUES – CLASSIC MAXWELL MODEL

Root number i	$\Re(s_i)$	$\Im(s_i)$
1, 2	-2.9931	±29.7644
3, 4	-3.6167	±80.3574
5	-17.6578	0
6	-20	0
7	-39.1227	0
8	-50	0

The roots of the number 6 and 8 correspond to the solutions $s_i = -\nu_i$, that means a singular solution of (10), which should not be treated as the eigenvalues. For these points, the value of the determinant, as derived from eigenproblem (10), tends to infinity.

The discussed solutions are presented in Fig. 4, as the points of intersection of the zero level lines of the surface $\Re(Z(x,y))$ and surface $\Im(Z(x,y))$ derived from (11).

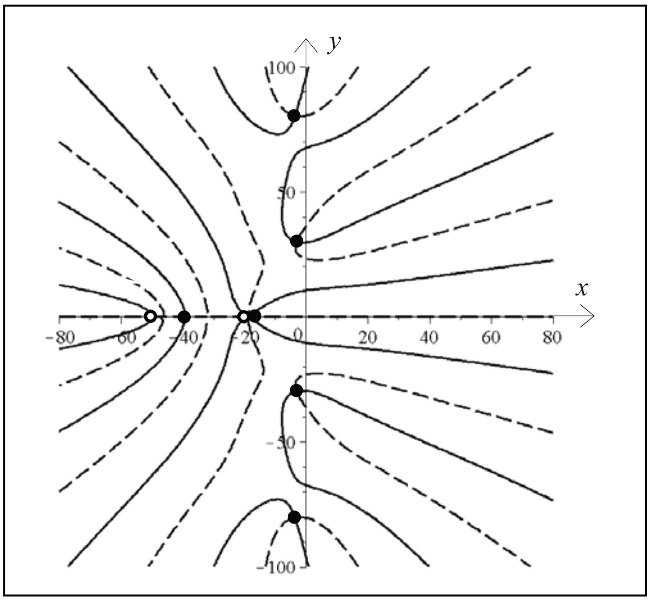


Figure 4. Plot of functions $\Re(Z)$ and $\Im(Z)$ - classic Maxwell model.

The real solutions s_5 and s_7 given in Table II coincide with the rheological properties of dampers mounted in structure.

5. Concluding Remarks

Comparing the results of calculations for a frame with dampers modeled using the classic Maxwell model and the results obtained for the fractional Maxwell model, we may observe qualitative differences. The solution to the nonlinear eigenproblem leads to a number of pairs of complex and conjugate eigenvalues $s_i = x_i \pm i \cdot y_i$. Moreover, in the case of the classic Maxwell model of damper we obtain some real eigenvalues for $x_i < 0$ and $y_i = 0$. For the fractional Maxwell model, real solutions do not exist because of discontinuity in the imaginary part of the determinant $\det(A) = Z(x,y)$ (see Fig. 5).

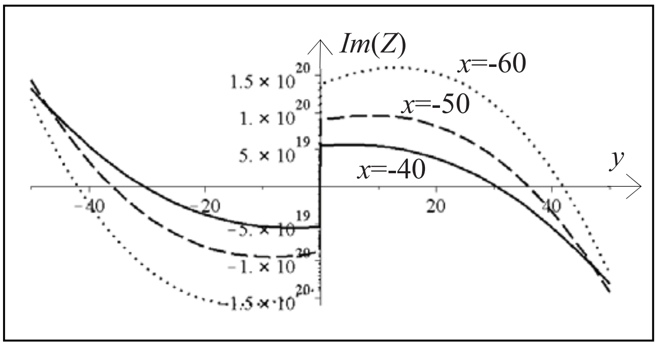


Figure 5. Diagram of imaginary part of determinant function.

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