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Algorithm of Iterative Process for Some Mappings and Iterative Solution of Some Diffusion Equation

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Abstract—In Hilbert spaces , through improving some corresponding conditions in some literature and extending some recent relevent results, a strong convergence theorem of some implicit iteration process for pesudocon-traction mappings and explicit iteration process for nonexpansive mappings were established. And by using the result, some iterative solution for some equation of response diffusion were obtained.

Keywords—pesudocon-traction mappings; nonexpansive mappings; implit iteration process; explicit iteration process; diffusion equation.

1. Introduction

Let E be *Banach* space, and k be a nonempty closed convex subset of E. Suppose that T is a mapping from K to K ,and $F(T)$ is a set of fixed point of T with $F(T)\neq\phi$. Assume that $J:E\to 2^E$ is regular dual mapping on E, and $J(x)=\{f\in E^*|<x,f>=\|x\|\|f\|,\|f\|=\|x\|\},x\in E.$

As $E=H$ is *Hilbert* space, the internal product of H is donate by the symbol $<\bullet,\bullet>$, and the norm of H is designated by symbol $\|\bullet\|$.

Definition 1 Mapping $T:K\to K$ is said to be pseudo contraction if for arbitrary $x,y\in K$, there exits $j(x-y)\in J(x-y)$ such that $<Tx-Ty,j(x-y)>\leq\|x-y\|^2$.

T is said to be strong pseudo contraction if there is $k\in(0,1)$ such that $<Tx-Ty,j(x-y)>\leq k\|x-y\|^2$ for arbitrary $x,y\in K$.

Definition 2 Mapping $T:K\to K$ is said to be nonexpanxive if for abitrary $x,y\in K$, there is $\|Tx-Ty\|\leq\|x-y\|$.

As we all know, that T is pseudo contraction is equivalent to that for every $s>0$ and every $x,y\in K$,there is

$$\|x-y\|\leq\|x-y+s[(I-T)x-(I-T)y]\| \quad (1)$$

When $E=H$ is *Hilbert* space, $J:E\to 2^E$ is single value , and for abitrary $x,y\in K$,there Is

$$<x-y,j(x-y)>=<x-y,x-y>.$$

Obviously, nonexpansive mapping is pseudo contraction.

2. Lemmas and Methods

Lemma 1 [1,2] Let E be a real *Banach* space , and K be nonempty closed convex subset of E. Assume that $T:K\to K$ is continuous strong pseudo constraction mapping . Then T is unique fixed point in K.

Lemma2 Let E be a real reflexive *Banach* space satisfying *opial* condition, and K be a nonempty closed convex subset of E. Supposethat $T:K\to K$ is continuous strong pseudo constraction mapping . Then for abitrary $\{x_n\}\subset E$, x_n weakly converge to y, and $\|x_n-Tx_n\|\to 0$. So there is $(I-T)x=0$.

Lemma 3 [4] Let $p>1,r>0$ be two certain real number, then *Banach* space is $(I-T)x=0$ if and only if there is a strictly increasing continuous function $g:[0,+\infty)\to[0,+\infty)$, $g(0)=0$, such that

$$\|\lambda x+(1-\lambda)y\|^p\leq\lambda\|x\|^p+(1-\lambda)\|y\|^p-W_p(\lambda)g(\|x-y\|),$$

for all $x,y\in B_r$, where $\lambda\in[0,1]$, and B_r is a closed spheroid which center is origin and radius is r, and

$$W_p(\lambda)=\lambda^p(1-\lambda)+\lambda(1-\lambda)^p.$$

Lemma 4 Let nonnegative real sequence $\{a_n\}$ satisfy the inequaltiy : $a_{n+1}\leq(1-\gamma_n)a_n+\delta_n$, $n\geq 0$,where $\gamma_n\in[0,1)$, $\sum_{n=1}^{\infty}\gamma_n=\infty$, $\lim_{n\to\infty}\frac{\delta_n}{\gamma_n}=0$ or $\sum_{n=1}^{\infty}|\delta_n|<+\infty$, then $\lim_{n\to\infty}a_n=0$.

In *Hilbert* space, *Moudaf*[6] has get strong convergence theorem of implicit iteration process of nonexpansive mapping, and *Xu*[7] has improved and extended some relative results in Reference [7].

In this paper, by applying a new implicit iteration sequence $x_n=\alpha_n f(x_n)+\beta_n x_n+\gamma_n Tx_n$,and explicit iterative sequence $y_{n+1}=\alpha_n f(y_n)+\beta_n y_n+\gamma_n Ty_n$, we shall consider the problem involving the fixed point of strong pseudo constraction and nonexpansive mapping on closed convex set K . When exact conditions are satisfied, $\{x_n\}$ and $\{y_n\}$ all strongly converge to the fixed point of T . When the conditions for $\{\alpha_n\}$ and f in Reference[6],[7] are widened, and as $\beta_n=0$, we can obtain the iterative sequence in Reference [6],[7], and then we improve and extend some ralative results and obtain some equation of diffusion by applying the above results.

Let $T:K\to K$ be continuous pseudo constraction mapping , and $f:K\to K$ be continuous strong pseudo constraction mapping with constant α $(0<\alpha<1)$. Suppose that $\alpha_n+\beta_n+\gamma_n=1$ for $\alpha_n,\beta_n,\gamma_n\in(0,1)$, and we stucture mapping $S_n:K\to K$, $S_n x=\alpha_n f(x)+\beta_n x+\gamma_n Tx$ .Then S_n is continuous strong pseudo constraction mapping . By virtue of Lemma 1 , S_n hasunique fixed point x_n , then we have

$$x_n=S_n x_n=\alpha_n f(x_n)+\beta_n x_n+\gamma_n Tx_n$$

(2)

3. Main Results

Theorem 1 Let E be a *Hilbert* space , and K be a nonempty closed convex subset of E . Assume that $f:K\to K$ is continuous strong pseudo constraction mapping with constant α $(0<\alpha<1)$, and f is bounded on bounded set , and $T:K\to K$ is continuous pseudo constraction mapping .Then

(a) If $\sigma_n=\frac{\alpha_n}{1-\beta_n}\to 0$ or $\limsup_{n\to\infty}\sigma_n<1$, and there is $p\in F(T)$ such that $\|f(x_n)-p\|^2-\|x_n-p\|^2\to 0$, then implict iterative sequence（2）strongly converges to the point of $F(T)$.

(b) If T is nonexpansive mapping and f is constraction mapping with constant α ,as $\frac{|\sigma_n-\sigma_{n-1}|}{\alpha_n\sigma_n}\to 0$ and $\sum\alpha_n=\infty$, the explicit iterative sequence $y_{n+1}=\alpha_n f(y_n)+\beta_n y_n+\gamma_n Ty_n$ strongly converges to the point of $F(T)$.

Proof.

（a）Because $\forall p\in F(T)$,

$$\|x_n-p\|^2$$
$$=<\alpha_n(f(x_n)-p)+\beta_n(x_n-p)+\gamma_n(Tx_n-p),x_n-p>$$
$$\le\alpha\alpha_n\|x_n-p\|^2+\alpha_n\|f(p)-p\|\|x_n-p\|+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2,$$

we have $\|x_n-p\|\le\frac{\alpha_n}{1-\alpha\alpha_n-\beta_n-\gamma_n}\|f(p)-p\|=\frac{\alpha_n}{1-\alpha}\|f(p)-p\|$.

Hence $\{x_n\},\{f(x_n)\},\{Tx_n\}$ are bounded.

If $\sigma_n=\frac{\alpha_n}{1-\beta_n}\to 0$,then using formula (2),we can write $x_n=\sigma_n f(x_n)+(1-\sigma_n)Tx_n$,and then we obtain

$$\|x_n-Tx_n\|=\sigma_n\|f(x_n)-Tx_n\|\to 0. \quad (3)$$

If $\limsup_{n\to\infty}\sigma_n<1$ and there is $p\in F(T)$ such that $\|f(x_n)-p\|^2-\|x_n-p\|\to 0$, then by virtue of formula (1) and Lemma 3, we obtain

$$\|x_n-p\|^2$$
$$\le\left\|x_n-p+\frac{1-\sigma_n}{2\sigma_n}(x_n-Tx_n)\right\|^2$$
$$=\left\|x_n-p+\frac{1-\sigma_n}{2}(f(x_n)-Tx_n)\right\|^2$$
$$=\left\|\frac{1}{2}(f(x_n)-p)+\frac{1}{2}(x_n-p)\right\|^2$$
$$\le\frac{1}{2}\|f(x_n)-p\|^2+\frac{1}{2}\|x_n-p\|^2-\frac{1}{4}g(\|x_n-f(x_n)\|),$$

and then

$$\frac{1}{2}g(\|x_n-f(x_n)\|)\le\|f(x_n)-p\|^2-\|x_n-p\|^2\to 0.$$

So we have $\|x_n-f(x_n)\|\to 0$.

Whereas

$$\|x_n-Tx_n\|=\sigma_n\|f(x_n)-Tx_n\|=\frac{\sigma_n}{1-\sigma_n}\|x_n-f(x_n)\|\to 0$$

(4)

Because $\{x_n\}$ is bounded , and E is *Hilbert* space , we have that x_n weakly converge to $q\in K$ .By virtue of formula

(3)or (4) and Lemma 2, we have $q \in F(T)$.

Because

$$\|x_n - q\|^2 = < \sigma_n(f(x_n)-q)+(1-\sigma_n)(Tx_n - q), x_n - q >$$
$$\le \alpha\sigma_n\|x_n - q\|^2 + \sigma_n < f(q)-q, x_n - q > + (1-\sigma_n)\|x_n - q\|^2,$$

we obtain $\|x_n - q\|^2 \le \frac{1}{1-\alpha} < f(q)-q, x_n - q >$. Since x_n weakly converges to q, x_n strongly converges to $q \in F(T)$.

(b) Because

$$\|x_n - x_{n-1}\|$$
$$= \|\sigma_n f(x_n) + (1-\sigma_n)Tx_n - \sigma_{n-1} f(x_{n-1}) - (1-\sigma_n)Tx_{n-1}\|$$
$$\le \sigma_n\alpha\|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|f(x_{n-1})\| + (1-\sigma_n)\|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Tx_{n-1}\|,$$

we obtain

$$\|x_n - x_{n-1}\| \le \frac{|\sigma_n - \sigma_{n-1}|}{\sigma_n(1-\alpha)} M, \qquad (5)$$

where $\|f(x_{n-1})\| \le \frac{M}{2}$, $\|Tx_{n-1}\| \le \frac{M}{2}$.

$$\|y_{n+1} - y_n\|$$
$$= \|\alpha_n f(y_n) + \beta_n y_n + \gamma_n Ty_n - \alpha_n f(x_n) - \beta_n x_n - \gamma_n Tx_n\|$$
$$\le \alpha\alpha_n\|y_n - x_n\| + \beta_n\|y_n - x_n\| + \gamma_n\|y_n - x_n\| = [1-\alpha_n(1-\alpha)]\|y_n - x_n\|$$
$$\le [1-\alpha_n(1-\alpha)]\|y_n - x_{n-1}\| + [1-\alpha_n(1-\alpha)]\|x_n - x_{n-1}\|$$

Since $\sum \alpha_n = \infty$, $\frac{|\sigma_n - \sigma_{n-1}|}{\alpha_n\sigma_n(1-\alpha)} \to 0$, formula (5) and Lemma 4, we obtain $\|y_{n+1} - x_n\| \to 0$.

Hence we have $\|y_{n+1} - q\| \le \|y_{n+1} - x_n\| + \|x_n - q\| \to 0$, which means that $\{y_n\}$ strongly converges to $q \in F(T)$.

Note.

Theorem 1 improves and extends some relative results in Reference [6] and [7].

As follows, we will discuss iterative solution of some response diffusion equation .

Let $E = L^2(I) = \{ x(t,s) | (t,s) \in I$, $x(t,s)$ and $x^2(t,s)$ are Lebsgue intergrable on I},where $I = [a,b]\times[c,d]$, and $\forall x, y \in E$,we define $< x, y > = \iint_I x(t,s)y(t,s)dtds$ .Then E is *Hilbert* space, and $\|x\|^2 = < x, x > = \iint_I x^2(t,s)dtds$,

$$< y, j(x) > = < y, x >, \forall x, y \in E.$$

Consider the problem involving solution of some first order diffusion equation:

$$\begin{cases} \frac{\partial x}{\partial t} = -u_0\frac{\partial x}{\partial s} - xGx - hx, \\ x(s,0) = x_0(s), x(0,t) = x_1(t), \end{cases} \qquad (6)$$

where G is continuous mapping on E, and $u_0 \ge 0$ is constant , and $h = h(t,s) \ge 0$.

This problem is equivalent to the integral equation as follows:

$$\int_0^x x(t,s)ds + u_0\int_0^t x(t,s)dt + \int_0^t\int_0^s h(t,s)x(t,s)dtds + \int_0^t\int_0^s xGxdtds$$
$$-\int_0^s x_0(s)ds - u_0\int_0^t x_1(t)dt = 0 \qquad (7)$$

Let $K = \{x \in E | x(t,s)$ is continuous function on I $\}$, then K is nonempty closed convex subset of E.

Let $H : K \to K$.

$$Hx = u_0\int_0^t x(t,s)dt + \int_0^s x(t,s)ds + \int_0^t\int_0^s h(t-x)x(t-s)dtds + \int_0^t\int_0^s xGxdtds$$
$$-\int_0^s x_0(s)ds - u_0\int_0^t x_1(t)dt.$$

If G satisfies (A): $\forall x, y \in K, xGx \le yGy$, then let

$$T : K \to K, Tx = -Hx + x.$$

If G satisfies (B) : there is $L_1 > 0$ such that $\|xGx - yGy\| \le L_1\|x - y\|$ for abitrary $x, y \in K$. Then H is *Lipschitz* mapping on K, and then we have $L > 0$ such that $\forall x, y \in K, \|Hx - Hy\| \le L\|x - y\|$.

Let $H_1 = \frac{2}{L}H$, $T_1 : K \to K, T_1x = -H_1x + x.$

Theorem 2 Let integral equation（7）has solution，then

(i) If G satisfies (A), when $\sigma_n = \frac{\alpha_n}{1-\beta_n} \to 0$ or $\limsup_{n\to\infty}\sigma_n < 1$,and there is $p \in F(T)$ such that $\|f(x_n) - p\|^2 - \|x_n - p\|^2 \to 0$, Implicit iterative sequence $x_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Tx_n$ strongly converges to the fixed point of T which is solution of equation (7).

(ii) If G satisfies（B）,when $\sum\alpha_n = \infty$ and $\frac{|\sigma_n - \sigma_{n-1}|}{\alpha_n\sigma_n} \to 0$, explicit iterative sequence $y_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n Ty_n$ strongly converges to the fixed point of T which is solution of equation (7).

Proof.

(i) Now, $\forall x, y \in K, (Hx - Hy)(x - y)$ is nonnegative on

$E_1 = \{(t,s) \in I \,|\, x - y \le 0\}$ and $E_2 = \{(t,s) \in I \,|\, x - y \ge 0\}$.

Then we have $< Hx - Hy, x - y >\ge 0$, that is said that T is pseudo constraction mapping on K. Using Theorem 1, we obtain the result .

(ii) Now , $\forall x, y \in K, \|H_1 x - H_1 y\| \le 2\|x - y\|$ （8）

$$(T_1 x - T_1 y)^2$$
$$= [-H_1 x + x - (-H_1 y + y)]^2$$
$$= (H_1 x - H_1 y)^2 - 2(y - x)(H_1 y - H_1 x) + (x - y)^2$$
$$= (H_1 y - H_1 x)^2 - 2|y - x||H_1 y - H_1 x| + (x - y)^2$$
$$= (H_1 y - H_1 x)(|H_1 y - H_1 x| - 2|y - x|) + (x - y)^2$$

If $|H_1 y - H_1 x| \le 2|y - x|$, then we obtain

$$(T_1 x - T_1 y)^2 \le (x - y)^2.$$

If $|H_1 y - H_1 x| \ge 2|y - x| \ge 0,$ then we obtain

$$(T_1 x - T_1 y)^2 \le (|H_1 y - H_1 x| + 2|y - x|)(|H_1 y - H_1 x| - 2|y - x|) + (x - y)^2$$
$$= (H_1 y - H_1 x)^2 - 4(x - y)^2 + (x - y)^2.$$

Hence, by virtue of formula (8), we have

$$\|T_1 x - T_1 y\|^2 \le \|x - y\|^2.$$

That is said that T_1 is nonexpansive mapping on K. Using Theorem 1, we obtain the result.

Therefore，through improving some corresponding conditions in literature [6],[7]，and extending some recent relevent results, Theorem 1 was established. Theorem 1 is a strong convergence theorem of some implicit iteration process for pesudocon-traction mappings and explicit iteration process for nonexpansive mappings . By applying Theorem 1, the iterative solution for some equation of response diffusion was obtained, Theorem 2 was established.

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