

Unification and Application of 3-point Approximating Subdivision Schemes of Varying Arity

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Abstract—In this paper, we propose and analyze a subdivision scheme which unifies 3-point approximating subdivision schemes of any arity in its compact form and has less support, computational cost and error bounds. The usefulness of the scheme is illustrated by considering different examples along with its comparison with the established subdivision schemes. Moreover, B-splines of degree 4 and well known 3-point schemes [1, 2, 3, 4, 6, 11, 12, 14, 15] are special cases of our proposed scheme.

Keywords—component; Approximating subdivision scheme; binary; ternary; α -ary; continuity and Laurent polynomial

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1. Introduction

In recent years, subdivision schemes has become one of the most popular methods of creating geometric objects in computer aided geometric design and animation industry. Their popularity is due to the facts that subdivision algorithms are easy to implement and suitable for computer applications. Subdivision schemes can be classified into approximating and interpolating ones.

The beginning of the subdivision story can be dated back to the papers of Chaikin [2] over thirty years ago, but the idea of families of subdivision schemes of higher arity is relatively new. Based on wavelet theory, Lian [8] introduced $2m - 1$ -point a -ary for any $a \geq 2$ and $(2m + 1)$ -point a -ary for any odd $a \geq 3$ interpolatory subdivision schemes for curve design. These schemes include the extended family of the classical 4- and 6-point interpolatory α -ary schemes [9] and the family of the 3- and 5-point α -ary interpolatory schemes [10]. Mustafa and Khan [7] offered a new 4-point quaternary approximating subdivision scheme. Siddiqi and Rehan [15] introduced a modified form of binary and ternary 3-point subdivision schemes which are C^1 and C^2 in the intervals $(\frac{-1}{8}, \frac{1}{5})$ and $(\frac{-1}{72}, \frac{7}{72})$ respectively. Most work in the area of subdivision schemes has considered binary and ternary schemes. But the research communities are gaining interest in introducing higher arity schemes (i.e. ternary, quaternary, ..., n -ary) that give better results and less computational cost. This motivates us to present the family of schemes with higher arity and more degree of freedom for curve designing. We decided to investigate schemes with an odd number of control points, specifically 3-point schemes. This led to a more general investigation of higher arity subdivision schemes.

In this paper 3-point subdivision schemes are extended to α -ary 3-point approximating subdivision schemes for any integer $a \geq 2$. In a -ary 3-point approximating subdivision schemes, we introduced new families of subdivision schemes for curve design. The first family is binary approximation, second is ternary approximation and onto a -ary approximation. A general formula for the mask of a -ary 3-point approximating subdivision scheme is defined as follows

$$\alpha_3^a(z) = \frac{1}{(2a)^2} \left(\frac{1-z^a}{1-z} \right)^3 \left(\sum_{i=0}^2 \binom{2}{i} z^i \right), \quad (1)$$

where “ a ” represents the arity.

In this paper we recall basic definitions and preliminary results in Section II. The family of a -ary 3-point approximating scheme is presented in Section III. Comparison with the existing 3-point scheme, basic properties of the limit function, error analysis and effect of parameters a -ary 3-point schemes are discussed in Section IV. Conclusion is also discussed in Section IV.

I. ANALYSIS OF THE GENERAL α -ARY SCHEME

A general form of univariate α -ary subdivision scheme S which maps a polygon $f^k = \{f_i^k\}_{i \in Z}$ to a refined polygon

$f^{k+1} = \{f_i^{k+1}\}_{i \in Z}$ is defined by

$$f_i^{k+1} = \sum_{j \in Z} \alpha_{aj-i} f_j^k, \quad i \in Z \quad (2)$$

where the set $\alpha = \{\alpha_i; i \in Z\}$ of coefficients is called the mask at k -th level of refinement. A necessary condition for the uniform convergence of subdivision scheme (2) is

$$\sum_{i \in Z} \alpha_{aj} = \sum_{i \in Z} \alpha_{aj+1} = \sum_{i \in Z} \alpha_{aj+2} = \dots = \sum_{i \in Z} \alpha_{aj+a-1} = 1. \quad (3)$$

A subdivision scheme is uniformly convergent if for any initial data $f^0 = \{f_i^0; i \in Z\}$, there exists a continuous function f such that for any closed interval $I \subset R$, it satisfies

$$\lim_{k \rightarrow \infty} \sup_{i \in a^k I} |f^k - f(a^{-k}i)| = 0.$$

Obviously, $f = S^\infty f^0$.

Introducing a symbol called Laurent polynomial

$$\alpha(z) = \sum_{i \in Z} \alpha_i z^i, \quad (4)$$

of the mask $\alpha = \{\alpha_i; i \in Z\}$ which play an efficient role to analyze the convergence and smoothness of subdivision scheme. From (3) and (4) the Laurent polynomial of convergent subdivision scheme satisfies

$$\alpha(e^{4ih\pi/a}) = 0, \quad h \in Z \cap (0, a) \text{ and } \alpha(1) = a. \quad (5)$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of an associated Laurent polynomial $\alpha(z)$

$$\alpha^{(1)}(z) = az^{a-1} \left(\frac{1-z}{1-z^a} \right) \alpha(z).$$

The subdivision scheme S_1 with Laurent polynomial $\alpha^{(1)}(z)$, is related to the scheme S with Laurent polynomial $\alpha(z)$ by the following theorem.

Theorem 1[5] *Let S denote a subdivision scheme with Laurent polynomial $\alpha(z)$ satisfying (5). Then there exists a subdivision scheme S_1 with the property*

$$\Delta f^k = S_1 \Delta f^{k-1},$$

where $f^k = S^k f^0$ and $\Delta f^k = (\nabla f^k)_i = a^k (f_{i+1}^k - f_i^k); i \in Z$. Furthermore, S is a uniformly convergent if and only if $\frac{1}{S_1}$ converges uniformly to zero function for all initial data f^0 , in the sense that

$$\lim_{k \rightarrow \infty} \left(\frac{1}{a} S_1 \right)^k f^0 = 0.$$

The above theorem indicates that for any given scheme S , with the mask α satisfying (3), we can prove the uniform convergence of S by deriving the mask of $\frac{1}{a} S_1$ and

computing $\left\| \left(\frac{1}{a} S_1 \right)^i \right\|_\infty$ for $i = 1, 2, 3, \dots, L$, where L is the

first integer for which $\left\| \left(\frac{1}{a} S_1 \right)^L \right\|_\infty < 1$. If such an L exists, then S converges uniformly. Since there are a rules for computing the values at the next refinement level, so we define the norm

$$\|S\|_\infty = \max \left\{ \sum_{i \in Z} |\alpha_{aj}|, \sum_{i \in Z} |\alpha_{aj+1}|, \sum_{i \in Z} |\alpha_{aj+2}|, \dots, \sum_{i \in Z} |\alpha_{aj+a-1}| \right\}, \quad (6)$$

and

$$\left\| \left(\frac{1}{a} S_1 \right)^L \right\|_\infty = \max \left\{ \sum_{i \in Z} |b_{i+a^L j}^{n,L}|; i = 0, 1, 2, \dots, a^L - 1 \right\}, \quad (7)$$

where

$$b^{[n, L]} = \frac{1}{a^L} \prod_{j=0}^{L-1} \alpha^{(n)}(z^{a^j}), \quad (8)$$

and

$$\begin{aligned} \alpha^{(n)}(z) &= az^{a-1} \left(\frac{1-z}{1-z^a} \right) \alpha^{(n-1)}(z) \\ &= az^{a-1} \left(\frac{1-z}{1-z^a} \right) \alpha(z), \quad n \geq 1. \end{aligned}$$

Definition 1. *The number of points inserted at the level $k + 1$ between two consecutive points from a level k is called arity of the scheme. In the case when number of points inserted are 2, 3, ..., a , the subdivision schemes are called binary, ternary, ..., a -ary, respectively.*



Figure 1: (a), (b) and (c) represent binary, ternary and quaternary refinement of coarse polygons using (1) for $n = 2, 3, 4$, respectively.

2. Family of the general a -ary 3-point approximating scheme

In this section, we are introducing a family of 3-point a -ary approximating subdivision schemes for curve design for any integer $a \geq 2$, which is an extension of "B-spline". We have proved this family by using Chaikin [2], Hassan and Dodgson [4]. The Chaikin's algorithm for curve design introduced in 1974 is given by

$$\begin{cases} f_{2i}^{k+1} = \frac{3}{4} f_i^k + \frac{1}{4} f_{i+1}^k, \\ f_{2i+1}^{k+1} = \frac{1}{4} f_i^k + \frac{3}{4} f_{i+1}^k. \end{cases} \quad (9)$$

About twenty seven years later, it was extended to the 3-point scheme by Hassan and Dodgson and is given by

$$\begin{cases} f_{2i}^{k+1} = \frac{5}{16} f_{i-1}^k + \frac{10}{16} f_i^k + \frac{1}{16} f_{i+1}^k, \\ f_{2i+1}^{k+1} = \frac{1}{16} f_{i-1}^k + \frac{10}{16} f_i^k + \frac{5}{16} f_{i+1}^k. \end{cases} \quad (10)$$

The Laurent polynomials of (9) and (10) are

$$\begin{cases} \alpha_2^2(z) = \frac{1}{4} \left(\frac{1-z^2}{1-z} \right)^2 \left(\sum_{i=0}^1 \binom{1}{i} z^i \right), \\ \alpha_3^2(z) = \frac{1}{16} \left(\frac{1-z^2}{1-z} \right)^3 \left(\sum_{i=0}^2 \binom{2}{i} z^i \right). \end{cases} \quad (11)$$

If " a " represents the arity, then by generalizing, we get

$$\alpha_3^a(z) = \frac{1}{(2a)^2} \left(\frac{1-z^a}{1-z} \right)^3 \left(\sum_{i=0}^2 \binom{2}{i} z^i \right). \quad (12)$$

where integers $a \geq 2$. From the coefficients of Laurent polynomial (12), we get the mask α_3^a of family of 3-point a -ary approximating subdivision schemes for curve design for any integer $a \geq 2$.

By adjusting the shape parameter in eq (12), we get the 3-point a -ary parametric approximating subdivision scheme

$$\alpha_3^a(z) = \frac{1}{(2a)^2} \left(\frac{1-z^a}{1-z} \right)^3 \left(\sum_{i=0}^2 \binom{2}{i} \mu_i z^i \right), \quad (13)$$

and

$$\sum_{i=0}^2 \frac{a}{2^2} \binom{2}{i} \mu_i = a, \mu_j = \mu_{2-j}, j=0,1. \quad (14)$$

From the coefficients of Laurent polynomial (13) and using (14), we get the mask α_3^a of a family of the 3-point a -ary parametric approximating subdivision schemes for curve design for any integer $a \geq 2$.

Remark: For $a = 2, 3, 4$, in (12), we get the mask of the following 3-point binary, ternary and quaternary schemes, respectively,

$$\begin{cases} \alpha_3^2 = \frac{1}{16} \{1, 5, 10, 10, 5, 1\}, \\ \alpha_3^3 = \frac{1}{36} \{1, 5, 13, 22, 26, 22, 13, 5, 1\}, \\ \alpha_3^4 = \frac{1}{64} \{1, 5, 25, 38, 46, 46, 38, 25, 5, 1\}. \end{cases}$$

For $a = 2, 3, 4$, in (12) and using (13) we get the mask of the following 3-point binary, ternary and quaternary schemes, respectively,

$$\begin{cases} \alpha_3^a = \frac{1}{16} \{ \mu_0, 4 + \mu_0, 12 - 2\mu_0, 12 - 2\mu_0, 4 + \mu_0, \mu_0 \}, \\ \alpha_3^a = \frac{1}{36} \{ \mu_0, 4 + \mu_0, 12 + \mu_0, 24 - 2\mu_0, 28 - 2\mu_0, \\ \quad 24 - 2\mu_0, 12 + \mu_0, 4 + \mu_0, \mu_0 \}, \\ \alpha_3^a = \frac{1}{64} \{ \mu_0, 4 + \mu_0, 12 + \mu_0, 24 + \mu_0, 40 - 2\mu_0, 48 - 2\mu_0, \\ \quad 40 - 2\mu_0, 24 + \mu_0, 12 + \mu_0, 4 + \mu_0, \mu_0 \}. \end{cases} \quad (15)$$

3. Comparison with existing approximating schemes

In this section, we will show that the popular existing Chaikin scheme and 3-point schemes are special cases of our proposed family of schemes. Here we will also present support of the basic limit function and compare the error bounds between the limit curve and the control polygon after the k -fold subdivision of the 3-point schemes.

A. Special cases

Here we see that the existing symmetric schemes are the special cases of our scheme (15).

- By taking $\mu_0 = 0, \mu_0 = -48\omega, \mu_0 = 1, \mu_0 = -\frac{3}{2}, \mu_0 = \frac{2}{3} + 4\omega, \mu_0 = 1/2$ and $\mu_0 = -\frac{3}{2} + 16\mu$ in α_3^2 , we get the 3-point binary scheme of [2,3,4,6,11,14,15], respectively
- By setting $\mu_0 = u - \frac{1}{3}, \mu_0 = \frac{4}{3}, \mu_0 = -4, \mu_0 = 1/3 + 4\omega$ and $\mu_0 = 1/2 + 36\mu$ in α_3^3 , we get

the 3-point ternary scheme of Aslam et al. [1, 4, 10, 15], respectively.

B. Support of basic limit function

The basic function of a subdivision scheme is the limit function of the proposed scheme for the following data

$$f_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases} \quad (16)$$

Theorem 2. The basic limit functions of a_{ϕ_3} proposed a -ary 3-point approximating schemes have support width $s = \frac{3a-1}{a-1}$, for $a = 2, 3, 4, \dots, m$, which implies that it vanishes outside the interval $\left[-\frac{3a-1}{2(a-1)}, \frac{3a-1}{2(a-1)} \right]$.

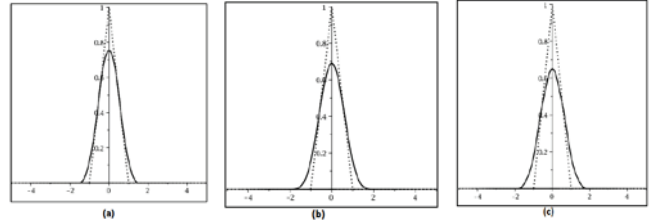


Figure 2: In this figure (a), (b), and (c) represent the basic limit functions of 3-point binary, ternary and quaternary schemes, respectively.

C. Error bounds

In Table 1 by using [13], with $\chi = 0.1$, we have computed the error bounds between limit the curve and the control polygon after the k -fold subdivision of the 3-point schemes. It is clear from Table 1 and Fig. 3 that the error bounds of the 3-point schemes (13) at each subdivision level decrease by increasing the arity of the schemes. Moreover, the support, computational cost and error bounds of higher arity schemes are better than the lower arity schemes.

TABLE 1: ERROR BOUNDS OF 3-POINT SCHEME WITH VARYING ARITY:

k/a	1	2	3	4	5
binary	0.075000	0.037500	0.018750	0.009375	0.004687
ternary	0.033333	0.011111	0.003704	0.001235	0.000412
quaternary	0.020833	0.005208	0.001302	0.000326	0.000081

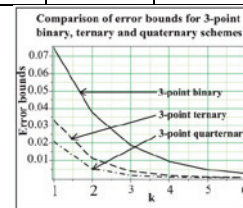


Figure 3: Comparison: Error bounds between the k -th level control polygons and the limit curves of 3-point schemes of varying arity.

4. Effects of parameters in proposed

We will discuss the three major effects/upshots of parameter in schemes (13). Effects of parameters in other schemes can be discussed analogously.

D. Continuity

The effects/upshots of the parameter μ in schemes (3.7) on order of continuity are shown in Table 2. One can easily find the order of continuity over the parametric intervals by using the approach of [5].

TABLE 2: THE ORDER OF CONTINUITY OF PROPOSED 3-POINT BINARY, TERNARY AND QUATERNARY APPROXIMATING SCHEMES FOR CERTAIN RANGES OF THE PARAMETER:

Scheme	Parameter	C^n	Scheme	Parameter	C^n
binary	$-2 \leq \mu_0 \leq 6$	C^0	ternary	$-6 \leq \mu_0 \leq 12$	C^0
.....	$-1 \leq \mu_0 \leq 3$	C^1	$-4 \leq \mu_0 \leq 8$	C^1
.....	$0 \leq \mu_0 \leq 2$	C^2	$0 \leq \mu_0 \leq 4$	C^2
.....	$\mu_0 = 1$	C^3			
quaternary	$-12 \leq \mu_0 \leq 20$	C^0			
.....	$-6 \leq \mu_0 \leq 10$	C^1			
.....	$0 \leq \mu_0 \leq 4$	C^2			

E. Shapes of limit curves

In Figure 4 the effect of the parameter in (13) on the graph and continuity of the limit curves is shown. This figure is exposed to show the role of the free parameter when 3-point schemes (14) applied on discrete data points. From these figures, we see that the behavior of the limiting curve acts as tightness/looseness when the values of free parameter vary.

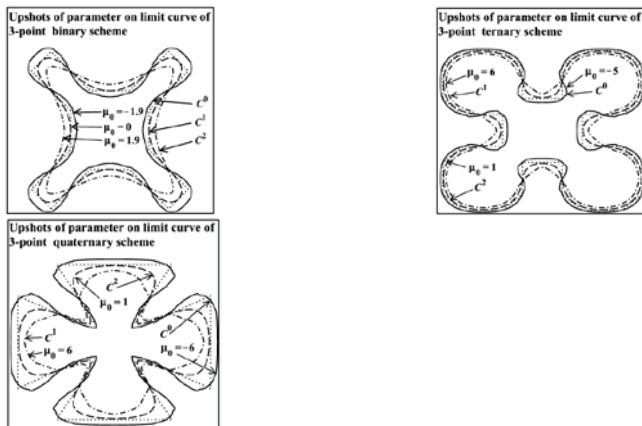


Figure 4: The initial polygons and effect of the parameter on limit curves of the 3-point binary, ternary and quaternary schemes.

F. Error bounds

The effects of parameter on error bounds at different subdivision levels of the control polygon and the limit curves are shown in Figure 5 and Table 3. From Table 3 and Figure 5, we conclude that: In the case of the 3-point binary scheme, the continuity is maximum over $0 < \mu_0 < 2$ and the error bound is minimum over $0 < \mu_0 < 4$. On each side of the interval $0 < \mu_0 < 2$, the continuity decreases while error bound increases on each side of the interval $0 < \mu_0 < 4$. In the cases of the 3-point ternary and quaternary schemes, the

continuity is maximum over $0 < \mu_0 < 4$, while the error bounds are minimum over $0 < \mu_0 < 6$ and $0 < \mu_0 < 8$, respectively. On each side of the interval $0 < \mu_0 < 4$, the continuity decreases while the error bound increases on each side of the interval $0 < \mu_0 < 6$ and $0 < \mu_0 < 8$, respectively.

TABLE 3: ERROR BOUNDS FOR 3-POINT BINARY, TERNARY AND QUATERNARY SCHEMES:

Scheme	Parameter	k=1	k=2	k=3	k=4
binary	$\mu_0 = 4$	0.075000	0.037500	0.018750	0.009375
...	$\mu_0 = \frac{-1}{3}$	0.110833	0.064653	0.037714	0.022000
...	$\mu_0 = \frac{-2}{3}$	0.166667	0.111111	0.074074	0.049383
...	$\mu_0 = -1$	0.262500	0.196875	0.147656	0.110742
ternary	$\mu_0 = 6$	0.033333	0.011111	0.003704	0.001235
...	$\mu_0 = 7$	0.053333	0.023704	0.010535	0.004682
...	$\mu_0 = \frac{-3}{2}$	0.075000	0.037500	0.018750	0.009375
...	$\mu_0 = -2$	0.097222	0.054012	0.030007	0.016670
quaternary	$\mu_0 = 8$	0.020833	0.005208	0.001302	0.000326
...	$\mu_0 = 9$	0.025068	0.007050	0.001983	0.000558
...	$\mu_0 = \frac{-1}{2}$	0.028409	0.008878	0.002774	0.000867
...	$\mu_0 = \frac{-5}{4}$	0.032431	0.010641	0.003492	0.001146

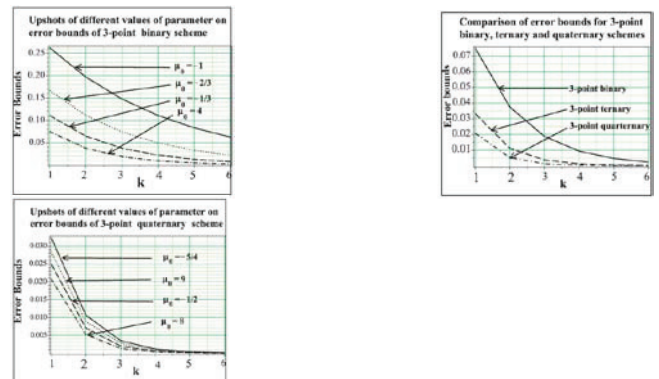


Figure 5: Comparison: Error bounds between k -th level control polygons and limit curves generated by 3-point schemes (13).

G. Conclusions and future research

We have shown that the 3-point approximating subdivision schemes [1,2,3,4,6,11,12,14,15] can be derived from a-ary 3-point approximating subdivision scheme. In context of binary and ternary subdivisions, we exploited a constructive method for generating 3-point schemes. As observed, our approach is more universal because it allows us to present general formula for 3-point approximating schemes and additionally it is applied to schemes of arbitrary arity. Therefore, we conclude that 3-point schemes with higher arity are better than lower arity schemes in the sense of support, computational cost and error bounds. These advantages motivate us to extend the proposed result to surface subdivision.

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