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Friendship Decompositions of Graphs: The general problem

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Abstract – A friendship graph is a graph consisting of cliques sharing a common vertex. In this paper we investigate the maximum number of elements in an optimal friendship decomposition of graphs of order n. We obtain upper and lower bounds for this number. These bounds relate this problem with the classical Ramsey numbers.

Keywords: Graph Decompositions; Friendship Graph; Friendship Decompositions

1 Introduction

For notation and terminology the reader is referred to [1]. All graphs considered here are finite and simple, i.e., they have no loops or multiple edges.

Let $\mathcal{H}$ be a fixed family of graphs. An $\mathcal{H}$-*decomposition* of a given graph G is a partition of its edge set into subgraphs that are isomorphic to elements of $\mathcal{H}$. We define $\phi(G, \mathcal{H})$ as the minimum number of elements in an $\mathcal{H}$-decomposition of G. One of the main problems in graph decompositions is the one of finding the smallest number $\phi(n, \mathcal{H})$, such that, any graph of order n admits an $\mathcal{H}$-decomposition with at most $\phi(n, \mathcal{H})$ elements. This problem has been studied for some well known families of graphs, such as, cliques, bipartite graphs, cycles and paths.

A *clique* is a complete graph and a *clique decomposition* or a *clique partition* of a graph is a decomposition of its edge set into edge disjoint cliques. Erdős, Goodman and Pósa [2] showed that the edges of any graph on n vertices can be decomposed into at most $\lfloor n^2/4 \rfloor$ cliques. They also showed that a minimum clique decomposition has exactly $\lfloor n^2/4 \rfloor$ cliques if and only if the graph is precisely $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Further, in the decomposition they only need to use edges and triangles. Later Bollobás [3] generalized this result by showing that a graph of order n can be decomposed into at most $t_{p-1}(n)$ edge disjoint cliques of order p ($p \geq 4$) and single edges. By $t_{p-1}(n)$ we denote the number of edges in the Turán graph of order n, $T_{p-1}(n)$, which is the unique complete $(p-1)$-partite graph on n vertices that has the maximum number of edges. Furthermore, for $p \geq 4$ it was also proved that $T_{p-1}(n)$ is the only graph that cannot be decomposed into fewer than $t_{p-1}(n)$ edge disjoint cliques of order p and single edges.

Let $\mathscr{B}$ denote the family of all complete bipartite graphs. It is easy to see that the edges of the complete graph K_n can be decomposed into $n-1$ edge disjoint complete bipartite subgraphs, since K_n decomposes into edge disjoint copies of the stars $K_{1,n-1}, K_{1,n-2}, \ldots, K_{1,2}, K_{1,1}$. Graham and Pollak [4] proved that K_n cannot be partitioned into fewer than $n-1$ edge disjoint complete bipartite graphs. Therefore, $\phi(K_n, \mathscr{B}) = n-1$. This result of Graham and Pollak was motivated by an addressing problem in communications networks and simple proofs were found later by Tverberg [5] and Peck [6].

Consider now the family $\mathscr{C}$ of all cycles, then it is easy to see that a graph admits a cycle decomposition if an only if all its vertices have even degree. In this case, Hajós made the following conjecture.

Conjecture 1. *Every graph G on n vertices with all degrees even can be decomposed into at most $\lfloor n/2 \rfloor$ edge disjoint cycles.*

In this direction Lovász [7] proved the following theorem:

Theorem 1.1. *A graph on n vertices can be decomposed into at most $\lfloor n/2 \rfloor$ edge disjoint paths and cycles.*

This theorem includes the following two results concerning complete graphs: the complete graph on $2k$ vertices can be decomposed into k edge disjoint paths and the complete graph on $2k+1$ vertices can be decomposed into k edge disjoint Hamiltonian cycles. Lovász theorem is also a partial result towards the following conjecture of Gallai.

Conjecture 2. *A graph or order n can be decomposed into at most $\lfloor (n+1)/2 \rfloor$ paths.*

Let $\mathscr{P}$ be the family of all paths. Dean and Kouider [8] showed that for any graph G (possibly disconnected), $\phi(G, \mathscr{P}) \leq u(G)/2 + \left\lfloor \frac{2}{3} g(G) \right\rfloor$, where $u(G)$ denotes the number of odd vertices in G and $g(G)$ denotes the number of nonisolated even vertices in G.

In this paper we will study decompositions of graphs into friendship graphs. A *friendship graph* is a graph consisting of cliques sharing a common vertex. For $t \geq 2$, a *t-friendship* graph is a friendship graph that consists of exactly t cliques sharing a common vertex. Let $\mathscr{F}_t$ be the set of all t-friendship graphs. Sousa [9] determined the exact value of the function $\phi(n, \mathscr{F}_t)$ for $t = 2$ and $t = 3$, and its asymptotic value for $t \geq 4$. More precisely, Sousa [9] proved the following theorems.

Theorem 1.2 (Sousa [9])**.** *Let $\mathscr{F}_2$ be the set of all 2-friendship graphs. We have,*

$$\phi(n, \mathscr{F}_2) = \begin{cases} \left\lceil \frac{n^2}{8} \right\rceil & \text{if } n \text{ is even,} \\ \frac{n^2-1}{8} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.3 (Sousa [9])**.** *Let $\mathscr{F}_3$ be the set of all 3-friendship graphs. We have,*

$$\phi(n, \mathscr{F}_3) = \begin{cases} \left\lceil \frac{n^2}{12} \right\rceil & \text{if } n \text{ is even,} \\ \left\lceil \frac{n^2-1}{12} \right\rceil & \text{if } n \text{ is odd and } n \neq 5, \\ 3 & \text{if } n = 5. \end{cases}$$

Theorem 1.4 (Sousa [9])**.** *Let $t \geq 4$ and let $\mathscr{F}_t$ be the set of all t-friendship graphs. We have,*

$$\phi(n, \mathscr{F}_t) = \Big(\frac{1}{4t} + o(1) \Big) n^2.$$

Let $\mathscr{F}$ be the set of all friendship graphs. Our goal is to study the function $\phi(n, \mathscr{F})$, which is the smallest number such that any graph of order n admits a friendship decomposition with at most $\phi(n, \mathscr{F})$ elements. The exact value of the function $\phi(n, \mathscr{F})$ is still unknown. However, for n sufficiently large, we were able to obtain lower and upper bounds (see Theorem 1.5 below). These results relate the function $\phi(n, \mathscr{F})$ with the classical Ramsey numbers, which might be an indication that the problem of finding the exact value of the function $\phi(n, \mathscr{F})$ might be deep and hard.

Theorem 1.5. *Let $\mathscr{F}$ be the set of all friendship graphs. There exists n_0 such that for all $n \geq n_0$ we have*

$$n - 9\sqrt{n \log n} \leq \phi(n, \mathscr{F}) \leq n - \frac{1}{2} \log_2 \frac{n}{4}.$$

2 Friendship Decompositions

in this section we will prove Theorem 1.5. We start with some definitions and auxiliary results.

Definition 2.1. *Let G be a graph.*

(a) A vertex cover *of G is a set $S \subset V(G)$ such that every edge of G is incident with a vertex of S. The minimum size of a vertex cover of G is denoted by $\alpha_0(G)$.*

(b) A subset $A \subset V(G)$ is said to be independent *if no two vertices of A are adjacent in G. The* independence number *of a graph G is the maximum size of an independent set of vertices and is denoted by $\alpha(G)$.*

Observation 2.2. *In a graph G, $S \subset V(G)$ is a vertex cover if and only if $V(G) \backslash S$ is an independent set, and hence $\alpha_0(G) + \alpha(G) = v(G)$.*

Lemma 2.3. *Let G be a graph. We have,*

(a) $\phi(G, \mathscr{F}) \leq \alpha_0(G)$;

(b) If, in addition, G is triangle free then $\phi(G, \mathscr{F}) = \alpha_0(G)$.

Proof. *(a)* Let $\mathcal{C}$ be a vertex cover of G such that $|\mathcal{C}| = \alpha_0(G)$ and assume that its elements are ordered, i.e., $\mathcal{C} = \{v_1, \ldots, v_m\}$, where $m = \alpha_0(G)$. Let $\mathcal{S} = \{S_{v_i} | \ i = 1, \ldots, m\}$ where S_{v_i} denotes the star with center v_i and edge set

$$E(S_{v_i}) = \big\{\{v_i, x\} \mid \{v_i, x\} \in E(G) - \cup_{j=1}^{i-1} E(S_{v_j})\big\}.$$

Clearly $\mathcal{S}$ is a friendship decomposition of G, hence $\phi(G, \mathscr{F}) \le |\mathcal{S}| = \alpha_0(G)$.

(b) It suffices to see that $\alpha_0(G) \le \phi(G, \mathscr{F})$. In this case all friendship subgraphs of G are stars. Thus, if $\mathcal{S}$ is a friendship decomposition of G with $|\mathcal{S}| = \phi(G, \mathscr{F})$ then the set of vertices $\{ \ v \mid v \text{ is the center of } F, \ F \in \mathcal{S}\}$ is a vertex cover for G. Therefore $\alpha_0(G) \le |\mathcal{S}| = \phi(G, \mathscr{F})$. □

We will also need some results from Ramsey Theory. We start with the definition of the Ramsey numbers.

Definition 2.4. *Let s and t be natural numbers. The* Ramsey number of s and t, *denoted by $R(s,t)$, is the smallest integer n, such that, in any graph on n or more vertices, there exists either a clique of s vertices or an independent set of t vertices.*

The existence of these Ramsey numbers is a simple consequence of a theorem proved by Ramsey [10]. The problem of estimating $R(s,t)$ or even $R(s,s)$ has proved to be very difficult and the best known bounds are still quite far apart. Erdős and Szekeres [11] showed that $R(s,t) \le \binom{s+t-2}{s-1}$ which implies that

$$R(s,s) \le \binom{2s-2}{s-1} \le 2^{2s-2} \qquad (2.1)$$

The best known upper bound on $R(s,s)$ was proved by Thomason in [12] where he shows that $R(s,s) \le s^{(-\frac{1}{2}+o(1))}\binom{2s-2}{s-1}$. In one of the first applications of the probabilistic method Erdős [?] proved an exponential lower bound on $R(s,s)$ using random colorings. These bounds were improved later by Spencer in [13] where he uses the probabilistic method and Lovász Local Lemma to obtain $R(s,s) > \frac{\sqrt{2}}{e}\big(1+o(1)\big)s2^{s/2}$ and this is the best known lower bound for $R(s,s)$.

The gap between these bounds is still large, and in recent years, relatively little progress has been made. Tight bounds are known only for $s = 3$. We have,

$$c_1 \frac{t^2}{\log t} \le R(3,t) \le c_2 \frac{t^2}{\log t}.$$

The upper bound is due to Ajtai, Komlós and Szemerédi [14] and the lower bound was proved by Kim [15] using a probabilistic argument. The main result of Kim, which will be used in the proof of Theorem 1.5, is the following upper bound on the independence number of triangle free graphs.

Theorem 2.5. *Let $G_n^{(3)}$ denote a triangle free graph on n vertices. Then every sufficiently large n has a $G_n^{(3)}$ for which*

$$\alpha(G_n^{(3)}) \le 9\sqrt{n \log n},$$

where log *denotes the natural logarithm.*

We are now able to prove Theorem 1.5

Proof of Theorem 1.5. Let n be sufficiently large and let $G_n^{(3)}$ be as in Theorem 2.5. From Lemma 2.3 we know that $\phi(G_n^{(3)}, \mathscr{F}) = \alpha_0(G_n^{(3)})$ and from Observation 2.2 we have $\alpha_0(G_n^{(3)}) = n - \alpha(G_n^{(3)})$. Therefore,

$$\phi(G_n^{(3)}, \mathscr{F}) \ge n - 9\sqrt{n \log n}$$

by Theorem 2.5 and this implies the lower bound. It remains to prove the upper bound. Let G be a graph of order n and let $t = \lfloor \frac{1}{2}\log_2 n + 1 \rfloor$. Then, by (2.1) we have $R(t,t) \le n$ and this implies that G contains either an independent set of t vertices or a clique of t vertices.

(a) Suppose that G contains an independent set of t vertices, say T and let $X := V(G) - T$. Assume that the elements of X are ordered and let $X = \{v_1, \ldots, v_m\}$, where $m = n - t$. Let $\mathcal{S} = \{S_{v_i} | \ i = 1, \ldots, m\}$, where S_{v_i} denotes the star with center v_i and edge set

$$E(S_{v_i}) = \big\{\{v_i, x\} \mid \{v_i, x\} \in E(G) - \cup_{j=1}^{i-1} E(S_{v_j})\big\}.$$

Clearly $\mathcal{S}$ is a friendship decomposition of G, hence $\phi(G, \mathscr{F}) \le |\mathcal{S}| = |X| = n - t$.

(b) Now suppose that G contains a clique of t vertices, say $G[X]$, with $|X| = t$. Let G_1 be the graph obtained from G after the deletion of the edges of $G[X]$. Now G_1 contains an independent set of t vertices, so $\phi(G_1, \mathscr{F}) \le n - t$ by part (a), therefore $\phi(G, \mathscr{F}) \le n - t + 1$.

We have $n - t + 1 \le n - \frac{1}{2}\log_2 n - 1 + 2 = n - \frac{1}{2}\log_2 \frac{n}{4}$, as required. □

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