Open Journal of Applied Sciences, 2012, 2, 193-197
doi:10.4236/ojapps.2012.23029 Published Online September 2012 (http://www.SciRP.org/journal/ojapps)

Approximate Analytical Solutions of Fractional Coupled mKdV Equation by Homotopy Analysis Method

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Received June 1, 2012; revised July 3, 2012; accepted July 15, 2012

ABSTRACT

In this paper, the approximate analytical solutions of the fractional coupled mKdV equation are obtained by homotopy analysis method (HAM). The method includes an auxiliary parameter $\hbar$ which provides a convenient way of adjusting and controlling the convergence region of the series solution. The suitable value of auxiliary parameter $\hbar$ is determined and the obtained results are presented graphically.

Keywords: Homotopy Analysis Method; Approximate Analytical Solution; Fractional Coupled mKdV Equation

1. Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary characteristics of different materials and processes due to their non-locality characteristics. This is the main advantage of fractional derivatives in comparison with integer order model, in which such effects are in fact neglected [1]. Several definitions of fractional integration and derivation such as Riemann-Liouville's and Caputo's have been proposed. The Riemann-Liouville integral operator [1] having order $\alpha > 0$, which is a real number, is defined as

$$J^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \quad (x>0), \tag{1}$$

and as for $\alpha = 0$

$$J^0 f(x)=f(x)$$

where the real function $f(x)$, $x>0$ is said to be in the space $C_\mu, \mu\in R$, if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^n\in C_\mu$, $n\in N$. Its fractional derivative of order $\alpha>0$ is generally used

$$D^{\alpha} f(x)=\frac{\mathrm{d}^n}{\mathrm{d}x^n} J^{n-\alpha} f(x), \quad n-1<\alpha<n$$

where n is an arbitrary integer. The Riemann-Liouville integral operator has an important role for the development of the theory of both fractional derivatives and integrals. In spite of this fact, it has certain disadvantages when it comes to modelling real-world phenomena with fractional differential equations. This problem has been solved by M. Caputo first in his article [2] and then in his book [3]. Caputo definition, which is a modification of Riemann-Liouville definition, can be given as

$$\begin{aligned} D^{\alpha} f(x) &= J^{n-\alpha} D^n f(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t, \\ &(\alpha>0), \quad (n-1<\alpha<n). \end{aligned}$$

Note that Caputo derivative has the following two important properties

$$D^{\alpha} J^{\alpha} f(x)=f(x)$$

and

$$J^{\alpha} D^{\alpha} f(x)=f(x)-\sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x^k}{k!} \quad (n-1<\alpha<n).$$

In recent years, many important phenomena in various scientific and technological areas have been well described by fractional differential equations. In general, since these type equations are not exactly solved, their numerical solution techniques have become increasingly important. The HAM, a powerful tool for searching the approximate solutions which was first proposed by Liao [4,5], is one of such numerical solution techniques. Unlike perturbation techniques, the HAM is not limited to any small physical parameters in the considered equation. Therefore, the HAM can overcome the foregoing restricttions and limitations of perturbation techniques so it provides us with a powerful tool to analyze strongly nonlinear problems [6]. The HAM has been proposed and successfully applied to solve several fractional differential equations modeling problems arising in science and engineering by many authors [6-18] and the references therein. In this paper, we will apply the HAM to fractional coupled mKdV equation by using Caputo's definition of fractional differentiation.

2. HAM Solutions of the Time-Fractional Coupled mKdV Equation

In this section, we implement the HAM to the fractional coupled mKdV equation defined by

$$\begin{aligned} &D_t^\alpha u + 3u^2 u_x - 3(vv_x)_x - 3(uv^2) = 0 \\ &D_t^\alpha v + v_{xxx} + 3(vu_x)_x - 6uvu_x - 3(u^2 - v^2)v_x = 0 \\ &0 < \alpha \le 1, \quad t > 0 \end{aligned} \tag{2}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= \tanh(2x), \\ v(x,0) &= \tanh(2x). \end{aligned} \tag{3}$$

To investigate the HAM solutions of Equation (2) with the initial conditions given by Equation (3), we can choose the linear operator

$$\mathcal{L}[\phi_i(x,t;q)] = D_t^\alpha[\phi_i(x,t;q)], \quad i = 1,2$$

having the property

$$\mathcal{L}[c_i] = 0$$

where $c_i{}'s\,(i=1,2)$ are constants. From Equation (2), we can now define nonlinear operators as

$$\begin{aligned} &\mathcal{N}_1[\phi_1(x,t,q),\phi_2(x,t,q)] \\ &= \frac{\partial^\alpha \phi_1(x,t;q)}{\partial t^\alpha} - \frac{1}{2}\frac{\partial^3 \phi_1(x,t;q)}{\partial x^3} + 3(\phi_1(x,t;q))^2 \frac{\partial \phi_1(x,t;q)}{\partial x} \\ &-3\frac{\partial\left(\phi_2(x,t;q)\frac{\partial \phi_2(x,t;q)}{\partial x}\right)}{\partial x} \\ &-3\frac{\partial\left(\phi_1(x,t;q)(\phi_2(x,t;q))^2\right)}{\partial x}, \end{aligned}$$

$$\begin{aligned} &\mathcal{N}_2[\phi_1(x,t;q),\phi_2(x,t;q)] \\ &= \frac{\partial^\alpha \phi_2(x,t;q)}{\partial t^\alpha} + \frac{\partial^3 \phi_2(x,t;q)}{\partial x^3} + 3\frac{\partial\left(\phi_2(x,t;q)\frac{\partial \phi_1(x,t;q)}{\partial x}\right)}{\partial x} \\ &-6\phi_1(x,t;q)\phi_2(x,t;q)\frac{\partial \phi_1(x,t;q)}{\partial x} \\ &-3\left((\phi_1(x,t;q))^2 - (\phi_2(x,t;q))^2\right)\frac{\partial \phi_2(x,t;q)}{\partial x}. \end{aligned}$$

Therefore, we construct the zero-order deformation equation as follows

$$\begin{aligned} &(1-q)\mathcal{L}[\phi_1(x,t;q) - u_0(x,t)] \\ &= q\hbar_1 \mathcal{N}_1[\phi_1(x,t;q),\ \phi_2(x,t;q)], \end{aligned} \tag{4}$$

$$\begin{aligned} &(1-q)\mathcal{L}[\phi_2(x,t;q) - v_0(x,t)] \\ &= q\hbar_2 \mathcal{N}_2[\phi_1(x,t;q),\phi_2(x,t;q)]. \end{aligned} \tag{5}$$

If we choose $q = 0$ then we get

$$\begin{aligned} \phi_1(x,t;0) &= u_0(x,t) = u(x,0), \\ \phi_2(x,t;0) &= v_0(x,t) = v(x,0) \end{aligned}$$

and $q = 1$, we obtain

$$\phi_1(x,t;1) = u(x,t),\ \phi_2(x,t;1) = v(x,t).$$

Thus, as the embedding parameter q increases from 0 to 1, the solutions $\phi_i(x,t;q)(i=1,2)$ also change from the initial values $u_0(x,t)$ and $v_0(x,t)$ to the solutions $u(x,t)$ and $v(x,t)$. If we expand $\phi_i(x,t;q)$ for $i=1,2$ in Taylor series with respect to the embedding parameter q, we obtain

$$\phi_1(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m,$$

$$\phi_2(x,t;q) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t) q^m,$$

where

$$u_m(x,t) = \frac{1}{m!}\left.\frac{\partial^m \phi_1(x,t;q)}{\partial q^m}\right|_{q=0},$$

$$v_m(x,t) = \frac{1}{m!}\left.\frac{\partial^m \phi_2(x,t;q)}{\partial q^m}\right|_{q=0}.$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are properly chosen, as pointed out by Liao [5,8], the above series converges at $q = 1$, and one can have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{6}$$

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t) \tag{7}$$

which should be one of the solutions of the original equation. Let's define the following vectors

$$\begin{aligned} \boldsymbol{u}_{m-1} &= \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}, \\ \boldsymbol{v}_{m-1} &= \{v_0(x,t), v_1(x,t), \cdots, v_n(x,t)\}. \end{aligned}$$

By differentiating Equations (4) and (5) m times with respect to the embedding parameter q, we obtain the mth-order deformation equations as follows

$$\mathcal{L}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar_1 R_{1,m}\left(\boldsymbol{u}_{m-1},\boldsymbol{v}_{m-1}\right) \quad (8)$$

$$\mathcal{L}\left[v_m(x,t)-\chi_m v_{m-1}(x,t)\right]=\hbar_2 R_{2,m}\left(\boldsymbol{u}_{m-1},\boldsymbol{v}_{m-1}\right) \quad (9)$$

subject to the initial conditions

$$u_m(x,0)=\tanh(2x),$$
$$v_m(x,0)=\tanh(2x)$$

where

$$\begin{aligned}
&R_{1,m}\left(\boldsymbol{u}_{m-1},\boldsymbol{v}_{m-1}\right)\\
&=\frac{\partial^\alpha u_{m-1}(x,t)}{\partial t^\alpha}-\frac{1}{2}\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}\\
&-3\frac{\partial}{\partial x}\left(\sum_{n=0}^{m-1} v_n(x,t)\frac{\partial v_{m-1-n}(x,t)}{\partial x}\right)\\
&+3\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} u_k(x,t)u_{n-k}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x}\\
&-3\sum_{n=0}^{m-1}\frac{\partial}{\partial x}\left(\sum_{k=0}^{n} u_k(x,t)v_{n-k}(x,t)\right)v_{m-1-n}(x,t),
\end{aligned}$$

$$\begin{aligned}
&R_{2,m}\left(\boldsymbol{u}_{m-1},\boldsymbol{v}_{m-1}\right)\\
&=\frac{\partial^\alpha v_{m-1}(x,t)}{\partial t^\alpha}+\frac{\partial^3 v_{m-1}(x,t)}{\partial x^3}\\
&+3\frac{\partial}{\partial x}\left(\sum_{n=0}^{m-1} v_n(x,t)\frac{\partial u_{m-1-n}(x,t)}{\partial x}\right)\\
&-6\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n} u_k(x,t)v_{n-k}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x}\\
&-3\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n}\left(u_k(x,t)u_{n-k}(x,t)-v(x,t)v_{n-k}(x,t)\right)\right)\\
&\times\frac{\partial v_{m-1-n}(x,t)}{\partial x}
\end{aligned}$$

and
$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases}$$

By applying the operator J^α given by Equation (1), which is the inverse of the operator D_t^α, to the both sides of the *m*th-order deformation Equations (8) and (9) for $m\ge 1,$ we obtain

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar_1 J_t^\alpha\left[R_{1,m}\left(\boldsymbol{u}_{m-1},\boldsymbol{v}_{m-1}\right)\right], \quad (10)$$

$$v_m(x,t)=\chi_m v_{m-1}(x,t)+\hbar_2 J_t^\alpha\left[R_{2,m}\left(\boldsymbol{u}_{m-1},\boldsymbol{v}_{m-1}\right)\right]. \quad (11)$$

For the purpose of simplicity, setting $\hbar_1=\hbar_2=\hbar$ and by using Equations (10) and (11) with the initial conditions given by Equation (3), we successively obtain

$$u_0(x,t)=\tanh(2x),$$

$$u_1(x,t)=-\frac{4\hbar t^\alpha}{\Gamma(\alpha+1)\cos h^2(2x)},$$

$$\begin{aligned}u_2(x,t)=&-\frac{4\hbar(1+\hbar)t^\alpha}{\Gamma(\alpha+1)\cos h^2(2x)}\\&-\frac{32\hbar^2 t^{2\alpha}\tanh(2x)}{\Gamma(2\alpha+1)\cos h^2(2x)},\end{aligned}$$

$$\vdots$$

and

$$v_0(x,t)=\tanh(2x),$$

$$v_1(x,t)=-\frac{4\hbar t^\alpha}{\Gamma(\alpha+1)\cos h^2(2x)},$$

$$\begin{aligned}v_2(x,t)=&-\frac{4\hbar(1+\hbar)t^\alpha}{\Gamma(\alpha+1)\cos h^2(2x)}\\&-\frac{32\hbar^2 t^{2\alpha}\tanh(2x)}{\Gamma(2\alpha+1)\cos h^2(2x)},\end{aligned}$$

$$\vdots$$

etc. Therefore, the series solutions expressed by the HAM given in Equations (6) and (7) can be written in the following forms

$$\begin{aligned}u(x,t)&=u_0(x,t)+u_1(x,t)+u_2(x,t)+\dots,\\ v(x,t)&=v_0(x,t)+v_1(x,t)+v_2(x,t)+\dots.\end{aligned} \quad (12)$$

To demonstrate the efficiency of the method, we compare the HAM solutions of fractional coupled mKdV equation given by Equation (12) for $\alpha=0$ with its exact solutions [19]

$$u(x,t)=\tanh(2x+4t),$$
$$v(x,t)=\tanh(2x+4t).$$

The fact that HAM solution series contains the auxiliary parameter $\hbar$ providing us with a simply way to adjust and control the convergence of the solution series should be noted. To obtain an appropriate range for $\hbar$, we consider the so-called $\hbar$-curve to choose a proper value of $\hbar$ which ensures that the solution series is convergent, as pointed by Liao [5], by finding out the valid region of $\hbar$ corresponding to the line segments nearly parallel to the horizontal axis. The $\hbar$-curves of $u(0,0.01)$ and $v(0,0.01)$ are given by 3th-order HAM solution given by Equation (12) for various α parameters in **Figure 1**. It can be seen from the figure that the valid range of $\hbar$ lies in approximately $-1.3\le\hbar\le-0.7.$

Figure 2 shows the numerical solutions of $u(x,t)$ and $v(x,t)$ at $x = 2$ from $t = 0$ to $t = 0.5$ for $\hbar$ = –0.7, –1 and –1.3 obtained by 3th-order HAM for $\alpha=1$ and analytical solutions, respectively. Between $t = 0$ and $t = 0.5$, it can be seen from the figure that the choice of $\hbar$ = –0.7 is an appropriate value.

Figure 3 shows the numerical solutions of $u(x,t)$ and $v(x,t)$ at $x = 2$ during $0\le t\le 0.5$ for $\hbar$ = –0.7 obtained by 3th-order HAM for $\alpha=0.9$ and $\alpha=0.8$, respectively.

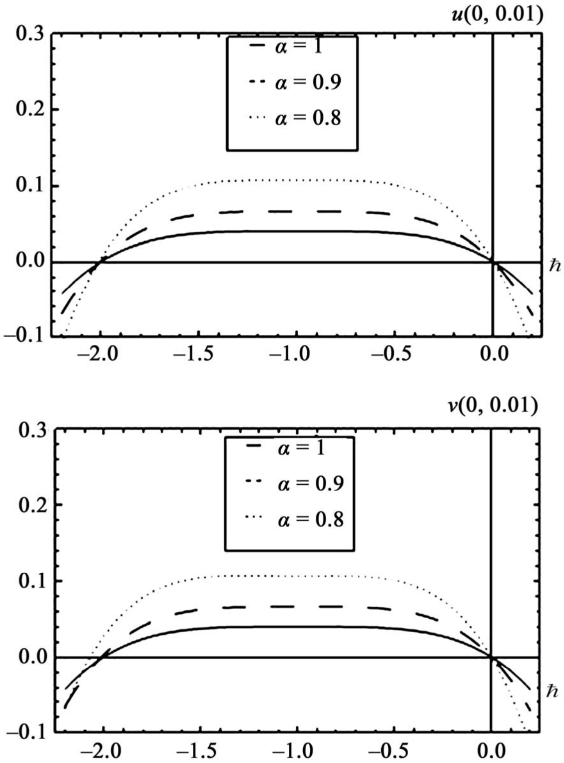

Figure 1. The $\hbar$-curves of 3th-order approximate solutions obtained by the HAM.

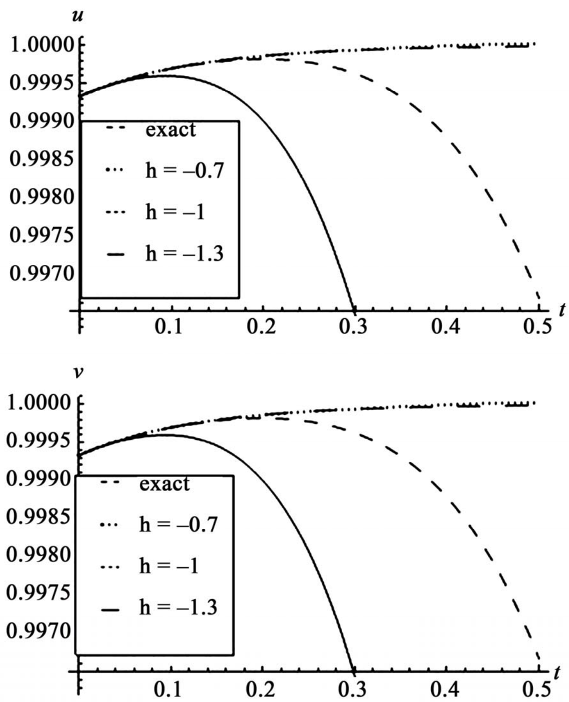

Figure 2. The results obtained by the HAM for α = 1 and various $\hbar$ by 3th-order approximate solution in comparison with the exact solution at x = 2.

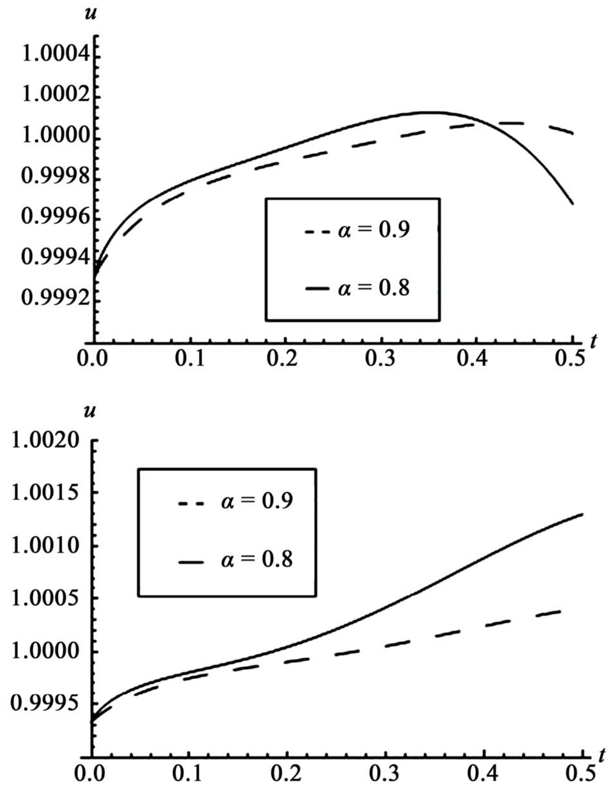

Figure 3. The results obtained by the HAM for α = 0.9, α = 0.8 and $\hbar$ = – 0.7 by 3th-order approximate solution at x = 2.

3. Conclusion

In this paper, the HAM has been successfully applied to obtain approximate analytical solution of fractional coupled mKdV equation. It has been also seen that the HAM solution of the problem converges very rapidly to the exact one by choosing an appropriate auxiliary parameter $\hbar$ whose valid range is determined using $\hbar$-curves presented by Liao. In conclusion, this study shows that the HAM is a powerful and efficient technique in finding the approximate analytical solution of fractional coupled mKdV equation and also many other fractional evolution equations arising in various areas.

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