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EPQ Inventory Model for Deteriorating Raw Materials with Two-Level Trade Credit and Limited Storage Capacity under Alternate Due Date of Payment

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Abstract

Economic production quantity (EPQ) research has typically focused on the cost of production processes, but has not employed accurate calculation to assess factors influencing ordering costs, because one of their assumptions is the raw materials that are product timely. However, the production and transport process of raw materials are influencing factors and increase the holding cost of raw materials, either by increasing or reducing the total relevant cost. [1] combined [2]'s concept of holding cost of raw materials and [3]'s two-level trade credit and limited storage capacity model to develop innovative and detailed EPQ model that considers the holding cost of non-deteriorating raw materials to closer to the real world. However, some raw materials have deteriorated should be considered. Therefore, this research extends [1]'s model to consider the holding cost of deteriorating raw materials. Four theorems for determining the optimal cycle time and the total relevant cost were developed using cost minimization. Finally, sensitivity analyses are used to find out the effects of the parameters to determine the ordering policies.

Subject Areas

Applied Statistical Mathematics, Business Analysis

Keywords

Economic Production Quantity, Deteriorating Raw Materials, Two-Level Trade Credit, Limited Storage Capacity

1. Introduction

[4] and [5] first developed the concepts of the economic order quantity (EOQ) and the economic production quantity (EPQ). These models facilitate the use of mathematical analysis for inventory management. For convenience, researchers have used various assumptions and parameters to account for unimportant factors; an example of such parameters is, the ordering cost, which involves the relative cost incurred during pre-production processes. A supply chain is consisting of suppliers, manufacturers, transporters, warehouses, retailers, and customers. When suppliers provide raw materials, the holding cost of raw materials can be affected by factors such as the shipment and acquisition prices; ultimately, the total relevant cost is affected. [2] modified the EPQ model to incorporate the holding cost of raw materials, thereby enhancing its practicality [6] [7].

[8] developed an EOQ model that includes the condition of permissible delay in payment (also called trade credit). [9] and [10] have extended the model developed by [8] to two-level trade credit, providing a fixed trade credit period M between a supplier and a retailer as well as, a trade credit period N between a retailer and a customer which is different between [9]'s and [10]'s payment terms as follows:

1) In [9]'s payment terms, if a customer buys one item from a retailer at time $t \in [0,T]$, then the customer receives a trade credit period $N-t$ and makes the payment at time N. Therefore, retailers allow a maximal trade credit period N for customers to settle the account [11]-[21].

2) In [10]'s payment terms, if a customer buys one item from the retailer at time $t \in [0,T]$, then the customer obtains a trade credit period N and makes the payment at time $N+t$. Therefore, retailers allow a maximal trade credit period N for customers to settle the account [22]-[27].

Trade credit stimulates retailers to purchase larger quantities of goods, as well as more storage capacity in which to store those goods. [28] developed an EOQ model for a two-warehouse solution: if an owned warehouse (OW) has insufficient storage capacity, then a rented warehouse (RW) can be used [3] [12] [29] [30] [31] [32].

[3] developed an EPQ model according to [10]'s payment terms (denoting it as "alternate due date of payment"), finite replenishment rates, and limited storage capacity. Furthermore, [1] combined [2]'s concept of holding cost of raw materials and [3]'s two-level trade credit and limited storage capacity model to develop innovative and detailed EPQ model that considers the holding cost of non-deteriorating raw materials. However, raw materials such as grain, metal, energy, and fiber are often volatile and time-sensitive, hence, the necessity of considering the tendency of raw materials to deteriorate [33]. Therefore, we organize the relevant literatures on two-level trade credit, limited storage capacity, and raw materials, as shown in **Table 1**.

As mentioned above, we found there is lack about the holding cost of deteriorating raw materials in the total relevant cost. Moreover, [1] developed a

Table 1. Summary of related literature for inventory models with two-level trade credit, limited storage capacity, and raw materials.

Author	Model	$N+t$	LSC	NRM	DRM
[35]	EPQ	V			
[22]	EPQ	V			
[12]	EPQ		V		
[34]	EOQ		V		
[14]	EOQ	V			
[23]	EOQ	V			
[36]	EPQ		V		
[24]	EPQ	V	V		
[30]	EPQ		V		
[37]	EOQ	V	V		
[2]	EPQ			V	
[25]	EPQ	V			
[3]	EPQ	V	V		
[6]	EPQ			V	
[7]	EPQ			V	
[26]	EOQ	V			
[27]	EPQ	V			
[10]	EOQ	V			
[38]	EOQ	V			
[39]	EPQ	V			
[40]	EOQ		V		
[20]	EOQ	V	V		
[41]	EPQ	V	V		
This research	EPQ	V	V	V	V

Note: Column $N+t$ for [10]'s payment method, LSC for limited storage capacity, NRM for the holding cost of non-deteriorating raw materials, and DRM for the holding cost of deteriorating raw materials. For the answers, V for Yes and empty for No.

complete inventory model by incorporating the holding cost of non-deteriorating raw materials with two-level trade credit and limited storage capacity. Therefore, this research extends [1]'s model to develop a new inventory model by considering the holding cost of deteriorating raw materials to determine the optimal inventory policies, two-level trade credit and limited storage. According to the cost-minimization strategy, four theorems are developed to characterize the optimal solution. Finally, sensitivity analyses are performed to determine the critical impact factors and draw the conclusions.

2. Notations and Assumptions

2.1. Notations

Q the order size.

P the production rate.

D the demand rate.

A the ordering cost.

T the cycle time.

$\rho = 1 - \frac{D}{P} > 0.$

$L_{\max}$ the storage maximum.

$I_m(t)$ the inventory function for raw materials.

θ the deterioration rate, $0 \le \theta < 1$.

s the unit selling price per item.

c the unit purchasing price per item.

h_m the unit holding cost per item for raw materials in a raw materials warehouse.

h_o the unit holding cost per item for product in an owned warehouse.

h_r the unit holding cost per item for product in a rented warehouse.

I_p the interest rate payable per \$ unit time (year).

I_e the interest rate earned per \$ unit time (year).

t_s time in years at which production stops.

M the manufacturer's trade credit period offered by the supplier.

N the customer's trade credit period offered by the manufacturer.

W the storage capacity of an owned warehouse.

tw_i the point in time when the inventory level increases to W when the production period is $\frac{W}{P-D}$.

tw_d the point in time when the inventory level decreases to W when the production cease period is $T - \frac{W}{D}$.

$tw_d - tw_i$ the time of rented warehouse is

$$\begin{cases} \frac{DT\rho - W}{P-D} + \frac{DT\rho - W}{D}, & \text{if } DT\rho > W \\ 0, & \text{if } DT\rho \le W. \end{cases}$$

$TRC(T)$ the total relevant cost per unit time of the model when $T > 0$.

T^* the optimal solution of $TRC(T)$.

2.2. Assumptions

1) Demand rate D is known and constant.

2) Production rate P is known and constant, $P > D$.

3) Shortages are not allowed.

4) Backlogging is not allowed.

5) A single item is considered.

6) Time period is infinite.

7) Replenishment rate is infinite.

8) $h_r \ge h_o \ge h_m$, $M \ge N$, and $s \ge c$.

9) Storage capacity of raw materials warehouse is unlimited.

10) If the order quantity is larger than the manufacturer's OW (owned warehouse) storage capacity, then the manufacturer will rent an RW (rented warehouse) with unlimited storage capacity. When demand occurs, it is first replenished from the RW which has storage that exceeds the items. The RW takes first in last out (FILO), and products in the OW or RW will not deteriorate.

11) During the period the account is not settled, generated sales revenue is deposited in and interest-bearing account.

a) When $M \le T$, the account is settled at $t = M$, the manufacturer pays off all units sold, keeps his or her profits, and starts paying for the higher interest payable on the items in stock with rate I_p.

b) When $T \le M$, the account is settled at $t = M$ and the manufacturer does not have to pay any interest payable.

12) If a customer buys an item from a manufacturer at time $t \in [0, T]$, then the customer receives a trade credit period N and makes the payment at time $N + t$.

13) The manufacturer can accumulate revenue and earn interest after his or her customer pays the amount of the purchasing cost to the manufacturer until the end of the trade credit period offered by the supplier. In other words, the manufacturer can accumulate revenue and earn interest during the period from N to M with rate I_e under the condition of trade credit.

14) The manufacturer keeps the profit for use in other activities.

2.3. Model

The model considers three stages of a supply chain system. It assumes that the supplier prepares the deteriorating raw materials for production, and the deteriorating raw materials are expected to decrease by the inventory function $I_m(t)$ with the deterioration rate θ (from time 0 to t_s). The quantity of products is expected to increase with time to the maximum inventory level (from 0 to t_s); the products are sold on demand at the same time. After production stops (at time t_s), the products are sold only on demand until the quantity reaches zero (at time T), as shown in **Figure 1**.

3. Annual Total Relevant Cost

The annual total relevant cost consists of the following element.

As shown in **Figure 1**, the raw material inventory level can be described by the following formulas, and we set the time in years at which production stops t_s, the optimal order size Q and storage maximum $L_{\max}$:

$$\frac{\mathrm{d}I_m(t)}{\mathrm{d}t} + \theta I_m(t) = -P,\ 0 \le t \le t_s. \tag{1}$$

By using the boundary condition $I_m(t_s) = 0$, we obtain

$$I_m(t) = \frac{P}{\theta}\left(\mathrm{e}^{\theta(t_s - t)} - 1\right),\ 0 \le t \le t_s. \tag{2}$$

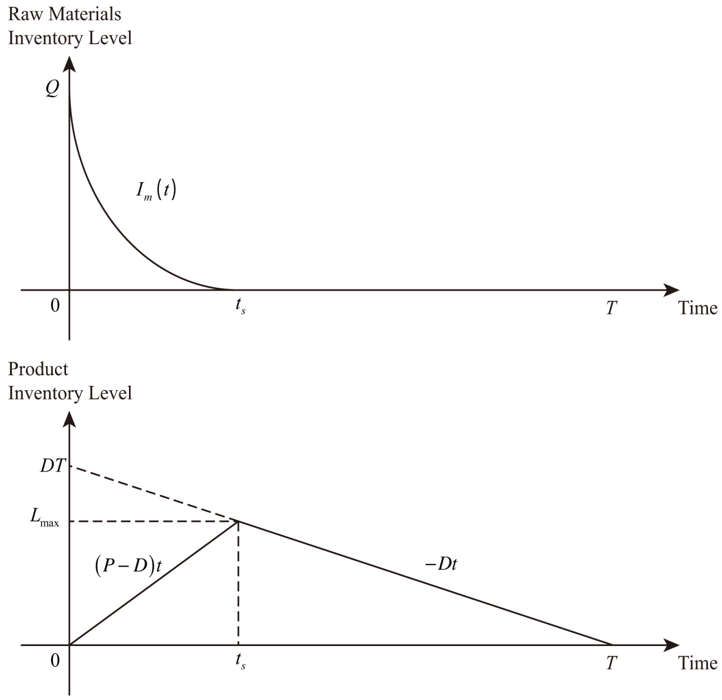

Figure 1. Raw materials and product inventory level.

We will then set the cycle time T and the optimal quantity Q.

$$(P-D)t_s - D(T-t_s) = 0,$$

$$t_s = \frac{D}{P}T. \tag{3}$$

$$Q = I_m(0) = \frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right). \tag{4}$$

3.1. Annual Ordering Cost

Annual ordering cost is

$$\frac{A}{T}. \tag{5}$$

3.2. Annual Purchasing Cost

Annual purchasing cost is

$$c \times Q \times \frac{1}{T} = \frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T} - 1\right). \tag{6}$$

3.3. Annual Holding Cost

Annual holding cost is

1) As shown in **Figure 1**, annual holding cost of raw materials

$$h_m \times \int_0^{t_s} I_m(t)\,dt \times \frac{1}{T} = \frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - \frac{D}{P}T\right]. \tag{7}$$

2) Two cases occur in annual holding costs of owned warehouse.

a) $DT\rho \le W$, as shown in **Figure 2**.

Annual holding cost in owned warehouse is

$$h_o \times \frac{T \times L_{\max}}{2} \times \frac{1}{T} = \frac{DTh_o\rho}{2}. \tag{8}$$

b) $W \le DT\rho$, as shown in **Figure 3**.

Annual holding cost in owned warehouse is

$$h_o \times \frac{\left[\left(tw_d - tw_i\right) + T\right]W}{2} \times \frac{1}{T} = Wh_o - \frac{W^2 h_o}{2DT\rho}. \tag{9}$$

3) Two cases occur in annual holding costs of rented warehouse.

a) $DT\rho \le W$, as shown in **Figure 2**.

Annual holding cost in rented warehouse is

$$0. \tag{10}$$

b) $W \le DT\rho$, as shown in **Figure 3**.

Annual holding cost in rented warehouse is

$$h_r \times \frac{\left(tw_d - tw_i\right) \times \left(L_{\max} - W\right)}{2} \times \frac{1}{T} = \frac{h_r\left(DT\rho - W\right)^2}{2DT\rho}. \tag{11}$$

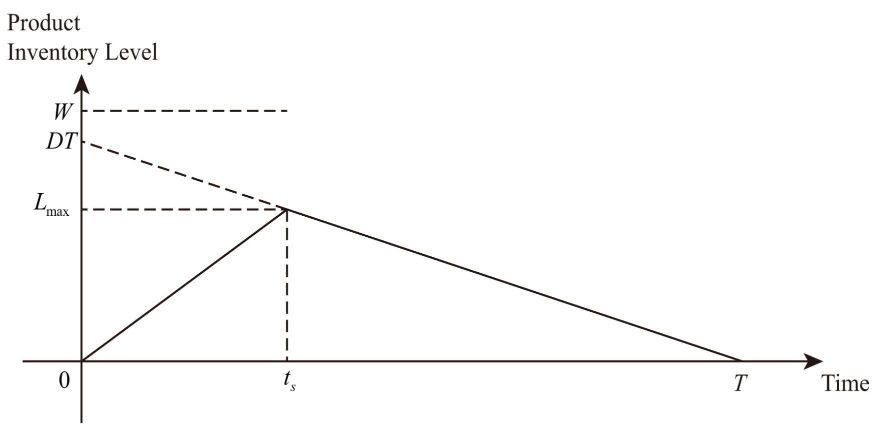

Figure 2. Annual holding cost when $DT\rho \le W$.

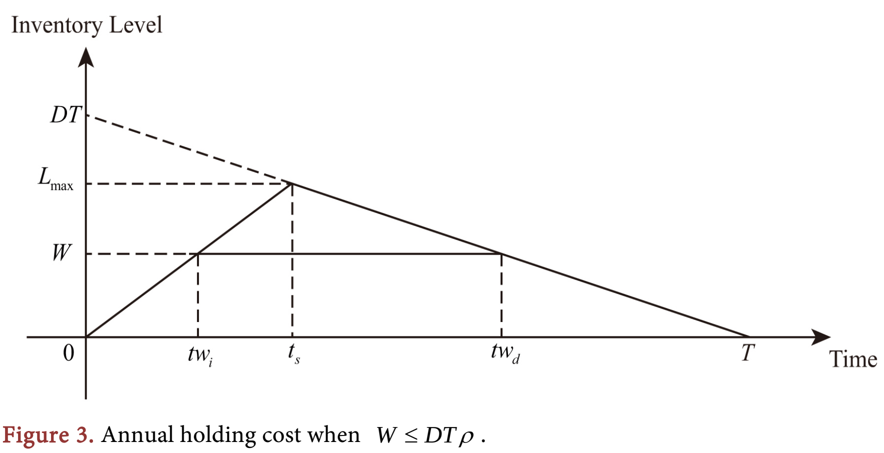

Figure 3. Annual holding cost when $W \le DT\rho$.

3.4. Annual Interest Payable

Four cases to occur in costs of annual interest payable for the items kept in stock.

1) $0 < T \le M - N$.

Annual interest payable is

$$0. \tag{12}$$

2) $M - N \le T \le M$.

Annual interest payable is

$$0. \tag{13}$$

3) $M \le T \le \frac{PM}{D}$, as shown in **Figure 4**.

Annual interest payable is

$$cI_p \times \left(\frac{(T-M)\times D(T-M)}{2}\right)\times\frac{1}{T} = \frac{cI_p D(T-M)^2}{2T}. \tag{14}$$

4) $M \le \frac{PM}{D} \le T$, as shown in **Figure 5**.

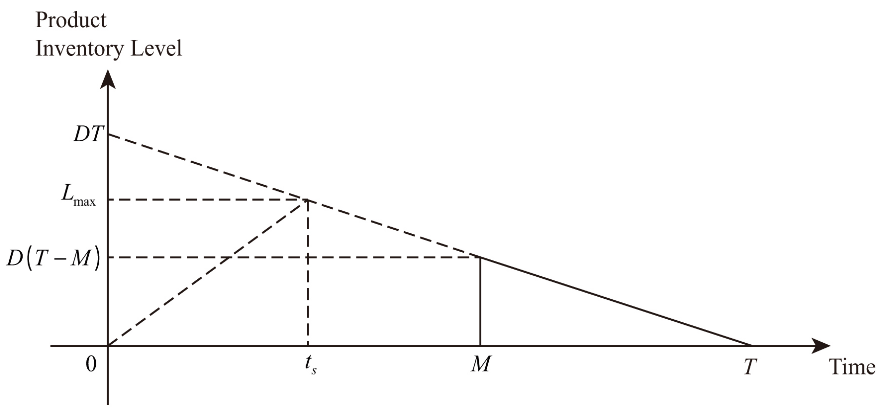

Figure 4. Annual interest payable when $M \le T \le \frac{PM}{D}$.

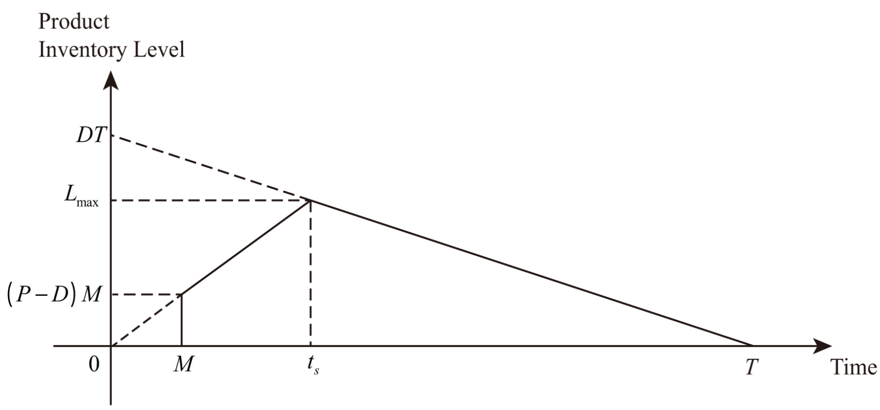

Figure 5. Annual interest payable when $M \le \frac{PM}{D} \le T$.

Annual interest payable is

$$cI_p \times \left(\frac{T \times DT\rho}{2} - \frac{M \times (P-D)M}{2} \right) \times \frac{1}{T} = \frac{cI_p \rho \left(DT^2 - PM^2 \right)}{2T}. \tag{15}$$

3.5. Annual Interest Earned

Five cases to occur in annual interest earned.

1) $0 < T \le N$ and $T \le M - N$, as shown in Figure 6.

Annual interest earned is

$$\begin{aligned} & sI_e \times \left\{ \frac{\left[(T+N) - N \right] \times DT}{2} + \left[M - (T+N) \right] \times DT \right\} \times \frac{1}{T} \\ & = \frac{sI_e D (2M - 2N - T)}{2}. \end{aligned} \tag{16}$$

2) $0 < T \le N$ and $M - N \le T$, as shown in Figure 7.

Annual interest earned is

$$sI_e \times \left[\frac{(M-N) \times D(M-N)}{2} \right] \times \frac{1}{T} = \frac{sI_e D (M-N)^2}{2T}. \tag{17}$$

3) $N \le T \le M$ and $T \le M - N$, as shown in Figure 8.

Annual interest earned is

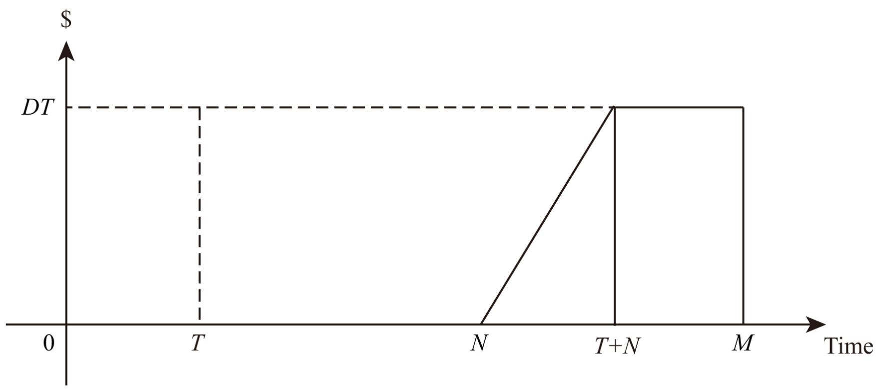

Figure 6. Annual interest earned when $0 < T \le N$ and $T \le M - N$.

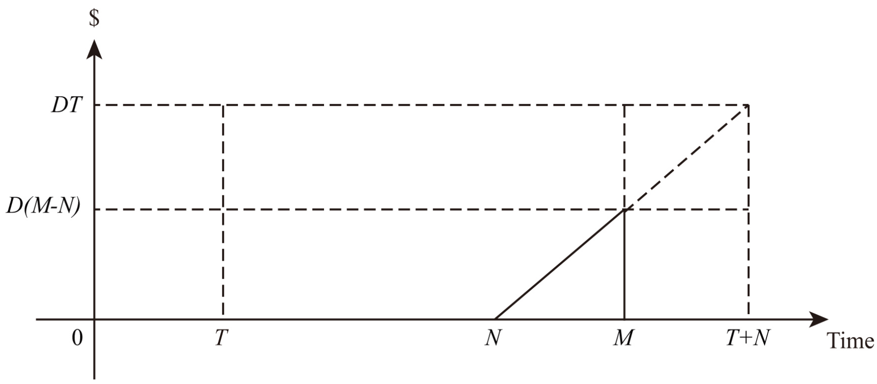

Figure 7. Annual interest earned when $0 < T \le N$ and $M - N \le T$.

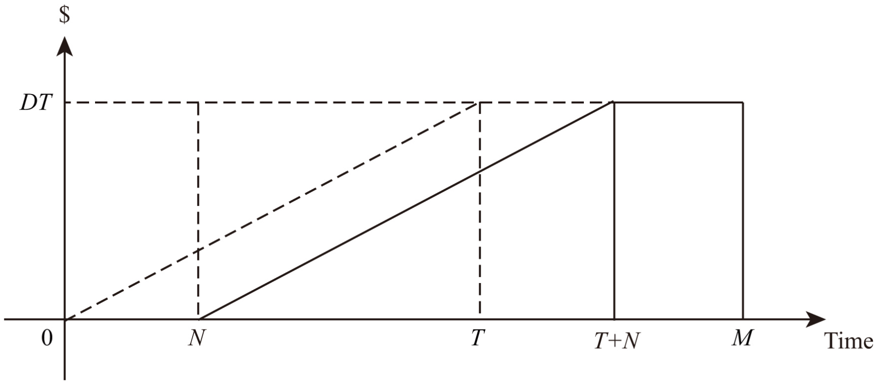

Figure 8. Annual interest earned when $N \le T < M$ and $T \le M - N$.

$$sI_e \times \left\{ \frac{\left[(T+N)-N\right] \times DT}{2} + \left[M-(T+N)\right] \times DT \right\} \times \frac{1}{T} = \frac{sI_e D(2M-2N-T)}{2}. \tag{18}$$

4) $N \le T \le M$ and $M - N \le T$, as shown in **Figure 9**.

Annual interest earned is

$$sI_e \times \left[\frac{(M-N) \times D(M-N)}{2} \right] \times \frac{1}{T} = \frac{sI_e D(M-N)^2}{2T}. \tag{19}$$

5) $N \le M \le T$, as shown in **Figure 10**.

Annual interest earned is

$$sI_e \times \left[\frac{(M-N) \times D(M-N)}{2} \right] \times \frac{1}{T} = \frac{sI_e D(M-N)^2}{2T}. \tag{20}$$

3.6. Annual Total Relevant Cost

From the above arguments, the annual total relevant cost for the manufacturer can be expressed as $TRC(T)$ = annual ordering cost + annual purchasing cost + annual holding cost + annual interest payable − annual interest earned.

Because storage capacity $W = DT\rho$, there are four cases arise:

1) $\frac{W}{D\rho} < M - N$,

2) $M - N \le \frac{W}{D\rho} < M$,

3) $M \le \frac{W}{D\rho} < \frac{PM}{D}$,

4) $\frac{PM}{D} \le \frac{W}{D\rho}$.

Case 1. $\frac{W}{D\rho} < M - N$.

According to Equations (1)-(20), *the total relevant cost* $TRC(T)$ *can be expressed by*

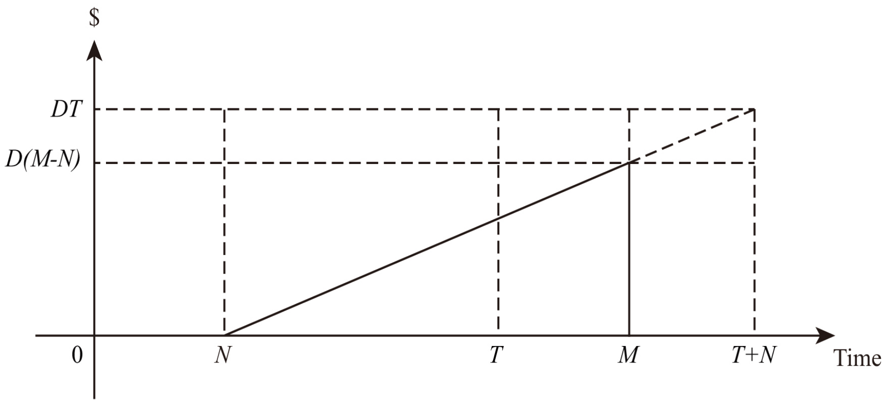

Figure 9. Annual interest earned when $N \le T \le M$ and $M - N \le T$.

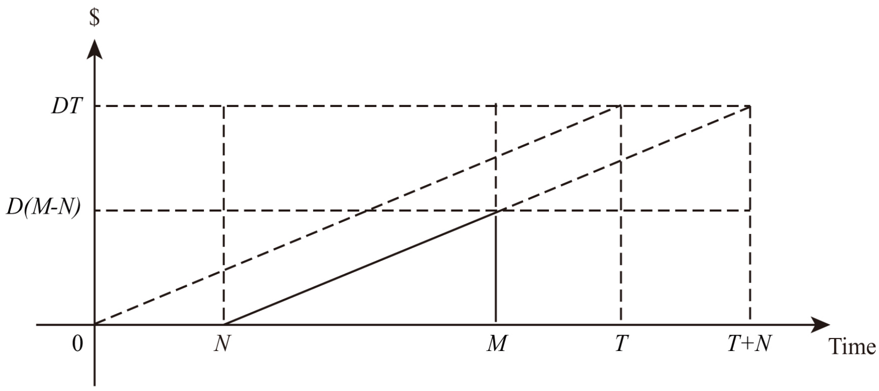

Figure 10. Annual interest earned when $M \le T$.

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } 0 < T < \dfrac{W}{D\rho} \quad (21a) \\ TRC_2(T), & \text{if } \dfrac{W}{D\rho} \le T < M - N \quad (21b) \\ TRC_3(T), & \text{if } M - N \le T < M \quad (21c) \\ TRC_4(T), & \text{if } M \le T < \dfrac{PM}{D} \quad (21d) \\ TRC_5(T), & \text{if } \dfrac{PM}{D} \le T \quad (21e) \end{cases}$$

where

$$\begin{aligned} TRC_1(T) = {} & \frac{A}{T} + \frac{cP}{\theta T}\left(e^{\theta \frac{D}{P} T} - 1\right) + \frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta \frac{D}{P} T} - 1\right) - \frac{D}{P}T\right] \\ & + \frac{DTh_o \rho}{2} - \frac{sI_e D(2M - 2N - T)}{2}, \end{aligned} \tag{22}$$

$$\begin{aligned} TRC_2(T) = {} & \frac{A}{T} + \frac{cP}{\theta T}\left(e^{\theta \frac{D}{P} T} - 1\right) + \frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta \frac{D}{P} T} - 1\right) - \frac{D}{P}T\right] \\ & + Wh_o - \frac{W^2 h_o}{2DT\rho} + \frac{h_r (DT\rho - W)^2}{2DT\rho} - \frac{sI_e D(2M - 2N - T)}{2}, \end{aligned} \tag{23}$$

$$TRC_3(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+Wh_o-\frac{W^2h_o}{2DT\rho}+\frac{h_r(DT\rho-W)^2}{2DT\rho}-\frac{sI_eD(M-N)^2}{2T}, \tag{24}$$

$$TRC_4(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+Wh_o-\frac{W^2h_o}{2DT\rho}+\frac{h_r(DT\rho-W)^2}{2DT\rho} \tag{25}$$
$$+\frac{cI_pD(M-N)^2}{2T}-\frac{sI_eD(M-N)^2}{2T},$$

$$TRC_5(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+Wh_o-\frac{W^2h_o}{2DT\rho}+\frac{h_r(DT\rho-W)^2}{2DT\rho} \tag{26}$$
$$+\frac{cI_p\rho(DT^2-PM^2)}{2T}-\frac{sI_eD(M-N)^2}{2T}.$$

$TRC(T)$ *is continuous at T,* $T\in[0,\infty)$ *because of* $TRC_1\left(\frac{W}{D\rho}\right)=TRC_2\left(\frac{W}{D\rho}\right)$, $TRC_2(M-N)=TRC_3(M-N)$, $TRC_3(M)=TRC_4(M)$, *and* $TRC_4\left(\frac{PM}{D}\right)=TRC_5\left(\frac{PM}{D}\right)$.

Case 2. $M-N\le\frac{W}{D\rho}<M$.

According to Equations (1)-(20), *the total relevant cost* $TRC(T)$ *can be expressed by*

$$TRC(T)=\begin{cases}TRC_1(T), & \text{if } 0<T<M-N & (27a)\\ TRC_6(T), & \text{if } M-N\le T<\frac{W}{D\rho} & (27b)\\ TRC_3(T), & \text{if } \frac{W}{D\rho}\le T<M & (27c)\\ TRC_4(T), & \text{if } M\le T<\frac{PM}{D} & (27d)\\ TRC_5(T), & \text{if } \frac{PM}{D}\le T & (25e)\end{cases}$$

where

$$TRC_6(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+\frac{DTh_o\rho}{2}-\frac{sI_eD(M-N)^2}{2T}. \tag{28}$$

$TRC(T)$ *is continuous at T,* $T \in [0,\infty)$ *because of* $TRC_1(M-N) = TRC_6(M-N)$, $TRC_6\left(\frac{W}{D\rho}\right) = TRC_3\left(\frac{W}{D\rho}\right)$, $TRC_3(M) = TRC_4(M)$, *and* $TRC_4\left(\frac{PM}{D}\right) = TRC_5\left(\frac{PM}{D}\right)$.

Case 3. $M \le \frac{W}{D\rho} < \frac{PM}{D}$.

According to Equations (1)-(20), *the total relevant cost* $TRC(T)$ *can be expressed by*

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } 0 < T < M-N & \text{(29a)} \\ TRC_6(T), & \text{if } M-N \le T < M & \text{(29b)} \\ TRC_7(T), & \text{if } M \le T < \frac{W}{D\rho} & \text{(29c)} \\ TRC_4(T), & \text{if } \frac{W}{D\rho} \le T < \frac{PM}{D} & \text{(29d)} \\ TRC_5(T), & \text{if } \frac{PM}{D} \le T & \text{(29e)} \end{cases}$$

where

$$\begin{aligned} TRC_7(T) = {} & \frac{A}{T} + \frac{cP}{\theta T}\left(e^{\theta \frac{D}{P}T} - 1\right) + \frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta \frac{D}{P}T} - 1\right) - \frac{D}{P}T\right] \\ & + \frac{DTh_o\rho}{2} + \frac{cI_p D(T-M)^2}{2T} - \frac{sI_e D(M-N)^2}{2T}. \end{aligned} \tag{30}$$

$TRC(T)$ *is continuous at T,* $T \in [0,\infty)$ *because of* $TRC_1(M-N) = TRC_6(M-N)$, $TRC_6(M) = TRC_7(M)$, $TRC_7\left(\frac{W}{D\rho}\right) = TRC_4\left(\frac{W}{D\rho}\right)$, *and* $TRC_4\left(\frac{PM}{D}\right) = TRC_5\left(\frac{PM}{D}\right)$.

Case 4. $\frac{PM}{D} \le \frac{W}{D\rho}$.

According to Equations (1)-(20), *the total relevant cost* $TRC(T)$ *can be expressed by*

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } 0 < T < M-N & \text{(31a)} \\ TRC_6(T), & \text{if } M-N \le T < M & \text{(31b)} \\ TRC_7(T), & \text{if } M \le T < \frac{PM}{D} & \text{(31c)} \\ TRC_8(T), & \text{if } \frac{PM}{D} \le T < \frac{W}{D\rho} & \text{(31d)} \\ TRC_5(T), & \text{if } \frac{W}{D\rho} \le T & \text{(31e)} \end{cases}$$

where

$$\begin{aligned} TRC_8(T) = {} & \frac{A}{T} + \frac{cP}{\theta T}\left(e^{\theta \frac{D}{P}T} - 1\right) + \frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta \frac{D}{P}T} - 1\right) - \frac{D}{P}T\right] \\ & + \frac{DTh_o\rho}{2} + \frac{cI_p\rho(DT^2 - PM^2)}{2T} - \frac{sI_e D(M-N)^2}{2T}. \end{aligned} \tag{32}$$

$TRC(T)$ is continuous at T, $T\in[0,\infty)$ because of
$TRC_1(M-N)=TRC_6(M-N)$, $TRC_6(M)=TRC_7(M)$,
$TRC_7\left(\frac{PM}{D}\right)=TRC_8\left(\frac{PM}{D}\right)$, *and* $TRC_8\left(\frac{W}{D\rho}\right)=TRC_5\left(\frac{W}{D\rho}\right)$.

For convenience, all $TRC_i(T)(i=1\sim8)$ are defined on $T>0$.

4. The Convexity of $TRC_i(T)(i=1\sim8)$

Equations (22)-(26), (28), (30), and (32) yield the first order and second-order derivatives as follows.

$$TRC_1'(T)=\frac{1}{T^2}\left\{-A-\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]+\frac{D(h_o\rho+sI_e)}{2}T^2\right\}, \quad (33)$$

$$TRC_1''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]\right\}, \quad (34)$$

$$\begin{aligned}TRC_2'(T)=\frac{1}{2T^2}&\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]\right.\\&\left.+\frac{W^2(h_o-h_r)}{D\rho}+D(h_r\rho+sI_e)T^2\right\},\end{aligned} \quad (35)$$

$$\begin{aligned}TRC_2''(T)=\frac{1}{T^3}&\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}\right.\right.\\&\left.\left.+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]+\frac{W^2(h_r-h_o)}{D\rho}\right\},\end{aligned} \quad (36)$$

$$\begin{aligned}TRC_3'(T)=\frac{1}{2T^2}&\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]\right.\\&\left.+\frac{W^2(h_o-h_r)}{D\rho}+sI_eD(M-N)^2+Dh_r\rho T^2\right\},\end{aligned} \quad (37)$$

$$\begin{aligned}TRC_3''(T)=\frac{1}{T^3}\{&2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]\\&\left.+\frac{W^2(h_r-h_o)}{D\rho}-sI_eD(M-N)^2\right\},\end{aligned} \quad (38)$$

$$\begin{aligned}TRC_4'(T)=\frac{1}{2T^2}&\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]\right.\\&\left.+\frac{W^2(h_o-h_r)}{D\rho}-cI_pDM^2+sI_eD(M-N)^2+D(h_r\rho+cI_p)T^2\right\},\end{aligned} \quad (39)$$

$$\begin{aligned}TRC_4''(T)=\frac{1}{T^3}&\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]\right.\\&\left.+\frac{W^2(h_r-h_o)}{D\rho}+cI_pDM^2-sI_eD(M-N)^2\right\},\end{aligned} \quad (40)$$

$$TRC_5'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]+\frac{W^2(h_o-h_r)}{D\rho}\right. \\ \left.+cI_p(P-D)M^2+sI_eD(M-N)^2+D\rho(h_r+cI_p)T^2\right\}, \tag{41}$$

$$TRC_5''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]\right. \\ \left.+\frac{W^2(h_r-h_o)}{D\rho}-cI_p(P-D)M^2-sI_eD(M-N)^2\right\}, \tag{42}$$

$$TRC_6'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]\right. \\ \left.+sI_eD(M-N)^2+Dh_o\rho T^2\right\}, \tag{43}$$

$$TRC_6''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}\right.\right. \\ \left.\left.+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]-sI_eD(M-N)^2\right\}, \tag{44}$$

$$TRC_7'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]\right. \\ \left.-cI_pDM^2+sI_eD(M-N)^2+D(h_o\rho+cI_p)T^2\right\}, \tag{45}$$

$$TRC_7''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}\right.\right. \\ \left.\left.+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]+cI_pDM^2-sI_eD(M-N)^2\right\}, \tag{46}$$

$$TRC_8'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]\right. \\ \left.+cI_p(P-D)M^2+sI_eD(M-N)^2+D\rho(h_o+cI_p)T^2\right\}, \tag{47}$$

and

$$TRC_8''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}\right.\right. \\ \left.\left.+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]-cI_p(P-D)M^2-sI_eD(M-N)^2\right\}. \tag{48}$$

Let

$$G_1=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right], \tag{49}$$

$$G_2 = 2A + \left(c + \frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - 2DTe^{\theta\frac{D}{P}T} + \theta\frac{D^2}{P}T^2 e^{\theta\frac{D}{P}T}\right] + \frac{W^2(h_r - h_o)}{D\rho}, \tag{50}$$

$$G_3 = 2A + \left(c + \frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - 2DTe^{\theta\frac{D}{P}T} + \theta\frac{D^2}{P}T^2 e^{\theta\frac{D}{P}T}\right] + \frac{W^2(h_r - h_o)}{D\rho} - sI_e D(M-N)^2, \tag{51}$$

$$G_4 = 2A + \left(c + \frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - 2DTe^{\theta\frac{D}{P}T} + \theta\frac{D^2}{P}T^2 e^{\theta\frac{D}{P}T}\right] + \frac{W^2(h_r - h_o)}{D\rho} + cI_p DM^2 - sI_e D(M-N)^2, \tag{52}$$

$$G_5 = 2A + \left(c + \frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - 2DTe^{\theta\frac{D}{P}T} + \theta\frac{D^2}{P}T^2 e^{\theta\frac{D}{P}T}\right] + \frac{W^2(h_r - h_o)}{D\rho} - cI_p(P-D)M^2 - sI_e D(M-N)^2, \tag{53}$$

$$G_6 = 2A + \left(c + \frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - 2DTe^{\theta\frac{D}{P}T} + \theta\frac{D^2}{P}T^2 e^{\theta\frac{D}{P}T}\right] - sI_e D(M-N)^2, \tag{54}$$

$$G_7 = 2A + \left(c + \frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - 2DTe^{\theta\frac{D}{P}T} + \theta\frac{D^2}{P}T^2 e^{\theta\frac{D}{P}T}\right] + cI_p DM^2 - sI_e D(M-N)^2, \tag{55}$$

and

$$G_8 = 2A + \left(c + \frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T} - 1\right) - 2DTe^{\theta\frac{D}{P}T} + \theta\frac{D^2}{P}T^2 e^{\theta\frac{D}{P}T}\right] - cI_p(P-D)M^2 - sI_e D(M-N)^2. \tag{56}$$

Equations (49)-(56) imply

$$G_4 > G_3 > G_5 > G_8, \tag{57}$$

$$G_4 > G_7 > G_6 > G_8, \tag{58}$$

and

$$G_2 > G_1 > G_6 > G_8. \tag{59}$$

Equations (33)-(48) reveal the following results.

Lemma 1. *$TRC_i'(T)$ is increasing on $T > 0$ if $G_i > 0$ for all $i = 1 \sim 8$. That is, $TRC_i(T)$ is convex on $T > 0$ if $G_i > 0$.*

$$TRC_i'(T) = \begin{cases} < 0, & \text{if } 0 < T < T_i^* & \text{(60a)} \\ = 0, & \text{if } T = T_i^* & \text{(60b)} \\ > 0, & \text{if } T_i^* < T < \infty & \text{(60c)} \end{cases}$$

Equations (60a)-(60c) *imply that* $TRC_i(T)$ *is decreasing on* $(0,T_i^*]$ *and increasing on* $[T_i^*,\infty)$ *for all* $i=1\sim 8$. *Solving optimal cycle* $T_i^*(T)(i=1\sim 8)$ *by* $TRC_i'(T)=0(i=1\sim 8)$.

5. The Values of Δ_{ij} under Different Cases

Case 1. $\frac{W}{D\rho}<M-N$.

Equations (33), (35), (37), (39), *and* (41) *yield*

$$TRC_1'\left(\frac{W}{D\rho}\right)=TRC_2'\left(\frac{W}{D\rho}\right)=\frac{\Delta_{12}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{61}$$

$$TRC_2'(M-N)=TRC_3'(M-N)=\frac{\Delta_{23}}{2(M-N)^2}, \tag{62}$$

$$TRC_3'(M)=TRC_4'(M)=\frac{\Delta_{34}}{2M^2}, \tag{63}$$

$$TRC_4'\left(\frac{PM}{D}\right)=TRC_5'\left(\frac{PM}{D}\right)=\frac{\Delta_{45}}{2\left(\frac{PM}{D}\right)^2}, \tag{64}$$

where

$$\begin{aligned}\Delta_{12}=&-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right]\\&+D(h_o\rho+sI_e)\left(\frac{W}{D\rho}\right)^2,\end{aligned} \tag{65}$$

$$\begin{aligned}\Delta_{23}=&-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}(M-N)}-1\right)-D(M-N)e^{\theta\frac{D}{P}(M-N)}\right]\\&+\frac{W^2(h_o-h_r)}{D\rho}+D(h_r\rho+sI_e)(M-N)^2,\end{aligned} \tag{66}$$

$$\begin{aligned}\Delta_{34}=&-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}M}-1\right)-DMe^{\theta\frac{D}{P}M}\right]\\&+\frac{W^2(h_o-h_r)}{D\rho}+sI_eD(M-N)^2+Dh_r\rho M^2,\end{aligned} \tag{67}$$

$$\begin{aligned}\Delta_{45}=&-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}-1\right)-D\left(\frac{PM}{D}\right)e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}\right]\\&+\frac{W^2(h_o-h_r)}{D\rho}-cI_pDM^2+sI_eD(M-N)^2+D(h_r\rho+cI_p)\left(\frac{PM}{D}\right)^2.\end{aligned} \tag{68}$$

Equations (65)-(68) *imply*

$$\Delta_{12}<\Delta_{23}<\Delta_{34}<\Delta_{45}. \tag{69}$$

Case 2. $M-N \le \frac{W}{D\rho} < M$.

Equations (33), (37), (39), (41), *and* (43) *yield*

$$TRC_1'(M-N) = TRC_6'(M-N) = \frac{\Delta_{16}}{2(M-N)^2}, \tag{70}$$

$$TRC_6'\left(\frac{W}{D\rho}\right) = TRC_3'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{63}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{71}$$

$$TRC_3'(M) = TRC_4'(M) = \frac{\Delta_{34}}{2M^2}, \tag{72}$$

$$TRC_4'\left(\frac{PM}{D}\right) = TRC_5'\left(\frac{PM}{D}\right) = \frac{\Delta_{45}}{2\left(\frac{PM}{D}\right)^2}, \tag{73}$$

where

$$\begin{aligned}\Delta_{16} &= -2A - 2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}(M-N)}-1\right) - D(M-N)e^{\theta\frac{D}{P}(M-N)}\right] \\ &\quad + D(h_o\rho + sI_e)(M-N)^2,\end{aligned} \tag{74}$$

$$\begin{aligned}\Delta_{63} &= -2A - 2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right) - D\left(\frac{W}{D\rho}\right)e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] \\ &\quad + sI_e D(M-N)^2 + Dh_o\rho\left(\frac{W}{D\rho}\right)^2.\end{aligned} \tag{75}$$

Equations (67), (68), (74), *and* (75) *imply*

$$\Delta_{16} \le \Delta_{63} < \Delta_{34} < \Delta_{45}. \tag{76}$$

Case 3. $M \le \frac{W}{D\rho} < \frac{PM}{D}$.

Equations (33), (39), (41), (43), *and* (45) *yield*

$$TRC_1'(M-N) = TRC_6'(M-N) = \frac{\Delta_{16}}{2(M-N)^2}, \tag{77}$$

$$TRC_6'(M) = TRC_7'(M) = \frac{\Delta_{67}}{2M^2}, \tag{78}$$

$$TRC_7'\left(\frac{W}{D\rho}\right) = TRC_4'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{74}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{79}$$

$$TRC_4'\left(\frac{PM}{D}\right) = TRC_5'\left(\frac{PM}{D}\right) = \frac{\Delta_{45}}{2\left(\frac{PM}{D}\right)^2}, \tag{80}$$

where

$$\Delta_{67} = -2A - 2\left(c + \frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta \frac{D}{P} M} - 1\right) - DMe^{\theta \frac{D}{P} M}\right] + sI_e D(M-N)^2 + Dh_o \rho M^2, \tag{81}$$

$$\Delta_{74} = -2A - 2\left(c + \frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta \frac{D}{P}\left(\frac{W}{D\rho}\right)} - 1\right) - D\left(\frac{W}{D\rho}\right)e^{\theta \frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] - cI_p DM^2 + sI_e D(M-N)^2 + D(h_o\rho + cI_p)\left(\frac{W}{D\rho}\right)^2. \tag{82}$$

Equations (68), (74), (81), *and* (82) *imply*

$$\Delta_{16} \le \Delta_{67} \le \Delta_{74} < \Delta_{45}. \tag{83}$$

Case 4. $\frac{PM}{D} \le \frac{W}{D\rho}$.

Equations (33), (41), (43), (45), *and* (47) *yield*

$$TRC_1'(M-N) = TRC_6'(M-N) = \frac{\Delta_{16}}{2(M-N)^2}, \tag{84}$$

$$TRC_6'(M) = TRC_7'(M) = \frac{\Delta_{67}}{2M^2}, \tag{85}$$

$$TRC_7'\left(\frac{PM}{D}\right) = TRC_8'\left(\frac{PM}{D}\right) = \frac{\Delta_{78}}{2\left(\frac{PM}{D}\right)^2}, \tag{86}$$

$$TRC_8'\left(\frac{W}{D\rho}\right) = TRC_5'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{45}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{87}$$

where

$$\Delta_{78} = -2A - 2\left(c + \frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta \frac{D}{P}\left(\frac{PM}{D}\right)} - 1\right) - D\left(\frac{PM}{D}\right)e^{\theta \frac{D}{P}\left(\frac{PM}{D}\right)}\right] - cI_p DM^2 + sI_e D(M-N)^2 + D(h_o\rho + cI_p)\left(\frac{PM}{D}\right)^2, \tag{88}$$

$$\Delta_{85} = -2A - 2\left(c + \frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta \frac{D}{P}\left(\frac{W}{D\rho}\right)} - 1\right) - D\left(\frac{W}{D\rho}\right)e^{\theta \frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] + cI_p(P-D)M^2 + sI_e D(M-N)^2 + D\rho(h_o + cI_p)\left(\frac{W}{D\rho}\right)^2. \tag{89}$$

Equations (74), (81), (88), *and* (89) *imply*

$$\Delta_{16} \le \Delta_{67} \le \Delta_{78} \le \Delta_{85}. \tag{90}$$

Based on the above arguments, the following results holds.

Lemma 2.

1) *If* $\Delta_{12} \le 0$, *then*

a) $G_1 > 0$ *and* $G_2 > 0$,

b) T_1^* *and* T_2^* *exist,*

c) $TRC_1(T)$ *and* $TRC_2(T)$ *are convex on* $T > 0$.

2) *If* $\Delta_{16} \le 0$, *then*

a) $G_1 > 0$ *and* $G_6 > 0$,

b) T_1^* *and* T_6^* *exist,*

c) $TRC_1(T)$ *and* $TRC_6(T)$ *are convex on* $T > 0$.

3) *If* $\Delta_{34} \le 0$, *then*

a) $G_3 > 0$ *and* $G_4 > 0$,

b) T_3^* *and* T_4^* *exist,*

c) $TRC_3(T)$ *and* $TRC_4(T)$ *are convex on* $T > 0$.

4) *If* $\Delta_{85} \le 0$, *then*

a) $G_5 > 0$ and $G_8 > 0$,

b) T_5^* *and* T_8^* *exist,*

c) $TRC_5(T)$ *and* $TRC_8(T)$ *are convex on* $T > 0$.

5) If $\Delta_{45} \le 0$, *then*

a) $G_4 > 0$ *and* $G_5 > 0$,

b) T_4^* *and* T_5^* *exist,*

c) $TRC_4(T)$ *and* $TRC_5(T)$ *are convex on* $T > 0$.

6) *If* $\Delta_{78} \le 0$, *then*

a) $G_7 > 0$ *and* $G_8 > 0$,

b) T_7^* *and* T_8^* *exist,*

c) $TRC_7(T)$ *and* $TRC_8(T)$ *are convex on* $T > 0$.

7) *If* $\Delta_{74} \le 0$, *then*

a) $G_4 > 0$ *and* $G_7 > 0$,

b) T_4^* *and* T_7^* *exist,*

c) $TRC_4(T)$ *and* $TRC_7(T)$ *are convex on* $T > 0$.

8) *If* $\Delta_{67} \le 0$, *then*

a) $G_6 > 0$ *and* $G_7 > 0$,

b) T_6^* *and* T_7^* *exist,*

c) $TRC_6(T)$ *and* $TRC_7(T)$ *are convex on* $T > 0$.

Proof. 1. (a) If $\Delta_{12} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] + D\left(h_o\rho+sI_e\right)\left(\frac{W}{D\rho}\right)^2. \tag{91}$$

Equation (91) implies

$$G_1 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+h_o\rho+sI_e\right] > 0. \tag{92}$$

$$G_2 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+h_r\rho+sI_e\right] > 0. \tag{93}$$

Equations (59), (92), and (93) demonstrate $G_2 > G_1 > 0$.

b) Lemma 1 implies that T_1^* and T_2^* exist.

c) Equations (34), (36), and lemma 1 imply that $TRC_1(T)$ and $TRC_2(T)$ are convex on $T > 0$.

2. a) If $\Delta_{16} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}(M-N)}-1\right)-D(M-N)e^{\theta\frac{D}{P}(M-N)}\right] + D(h_o\rho+sI_e)(M-N)^2. \tag{94}$$

Equation (94) implies

$$G_1 \ge D(M-N)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}(M-N)}+h_o\rho+sI_e\right]>0. \tag{95}$$

$$G_6 \ge D(M-N)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}(M-N)}+h_o\rho\right]>0. \tag{96}$$

Equations (59), (95), and (96) demonstrate $G_1 > G_6 > 0$.

b) Lemma 1 implies that T_1^* and T_6^* exist.

c) Equations (34), (44), and lemma 1 imply that $TRC_1(T)$ and $TRC_6(T)$ are convex on $T > 0$.

3. a) If $\Delta_{34} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}M}-1\right)-DMe^{\theta\frac{D}{P}M}\right] + \frac{W^2(h_o-h_r)}{D\rho}+sI_eD(M-N)^2+Dh_r\rho M^2. \tag{97}$$

Equation (97) implies

$$G_3 \ge DM^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}M}+h_r\rho\right]>0. \tag{98}$$

$$G_4 \ge DM^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}M}+h_r\rho+cI_p\right]>0. \tag{99}$$

Equations (57), (98), and (99) demonstrate $G_4 > G_3 > 0$.

b) Lemma 1 implies that T_3^* and T_4^* exist.

c) Equations (38), (40), and lemma 1 imply that $TRC_3(T)$ and $TRC_4(T)$ are convex on $T > 0$.

4. a) If $\Delta_{85} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] + cI_p(P-D)M^2+sI_eD(M-N)^2+D\rho(h_o+cI_p)\left(\frac{W}{D\rho}\right)^2. \tag{100}$$

Equation (100) implies

$$G_5 \geq D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+\rho\left(h_o+cI_p\right)\right]+\frac{W^2\left(h_r-h_o\right)}{D\rho}>0. \tag{101}$$

$$G_8 \geq D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+\rho\left(h_o+cI_p\right)\right]>0. \tag{102}$$

Equations (57), (101), and (102) demonstrate $G_5 > G_8 > 0$.

b) Lemma 1 implies that T_5^* and T_8^* exist.

c) Equations (42), (48), and lemma 1 imply that $TRC_5(T)$ and $TRC_8(T)$ are convex on $T > 0$.

5. a) If $\Delta_{45} \leq 0$, then

$$\begin{aligned}2A \geq & -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}-1\right)-D\left(\frac{PM}{D}\right)e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}\right] \\ & +\frac{W^2(h_o-h_r)}{D\rho}-cI_pDM^2+sI_eD(M-N)^2+D\left(h_r\rho+cI_p\right)\left(\frac{PM}{D}\right)^2.\end{aligned} \tag{103}$$

Equation (103) implies

$$G_4 \geq D\left(\frac{PM}{D}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}+h_r\rho+cI_p\right]>0. \tag{104}$$

$$G_5 \geq D\left(\frac{PM}{D}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}+\rho(h_r+cI_p)\right]>0. \tag{105}$$

Equations (57), (104), and (105) demonstrate $G_4 > G_5 > 0$.

b) Lemma 1 implies that T_4^* and T_5^* exist.

c) Equations (40), (42), and lemma 1 imply that $TRC_4(T)$ and $TRC_5(T)$ are convex on $T > 0$.

6. a) If $\Delta_{78} \leq 0$, then

$$\begin{aligned}2A \geq & -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}-1\right)-D\left(\frac{PM}{D}\right)e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}\right] \\ & -cI_pDM^2+sI_eD(M-N)^2+D\left(h_o\rho+cI_p\right)\left(\frac{PM}{D}\right)^2.\end{aligned} \tag{106}$$

Equation (106) implies

$$G_7 \geq D\left(\frac{PM}{D}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}+h_o\rho+cI_p\right]>0. \tag{107}$$

$$G_8 \geq D\left(\frac{PM}{D}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}+\rho\left(h_o+cI_p\right)\right]>0. \tag{108}$$

Equations (58), (107), and (108) demonstrate $G_7 > G_8 > 0$.

b) Lemma 1 implies that T_7^* and T_8^* exist.

c) Equations (46), (48), and lemma 1 imply that $TRC_7(T)$ and $TRC_8(T)$ are convex on $T > 0$.

7. a) If $\Delta_{74} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] -cI_pDM^2+sI_eD(M-N)^2+D(h_o\rho+cI_p)\left(\frac{W}{D\rho}\right)^2. \quad (109)$$

Equation (109) implies

$$G_4 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+(h_o\rho+cI_p)\right]+\frac{W^2(h_r-h_o)}{D\rho}>0. \quad (110)$$

$$G_7 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+(h_o\rho+cI_p)\right]>0. \quad (111)$$

Equations (58), (110), and (111) demonstrate $G_4 > G_7 > 0$.

b) Lemma 1 implies that T_4^* and T_7^* exist.

c) Equations (40), (46), and lemma 1 imply that $TRC_4(T)$ and $TRC_7(T)$ are convex on $T>0$.

8. a) If $\Delta_{67} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}M}-1\right)-DMe^{\theta\frac{D}{P}M}\right]+sI_eD(M-N)^2+Dh_o\rho M^2. \quad (112)$$

Equation (112) implies

$$G_6 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+h_o\rho\right]>0. \quad (113)$$

$$G_7 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+(h_o\rho+cI_p)\right]>0. \quad (114)$$

Equations (58), (113), and (114) demonstrate $G_7 > G_6 > 0$.

b) Lemma 1 implies that T_6^* and T_7^* exist.

c) Equations (44), (46), and lemma 1 imply that $TRC_6(T)$ and $TRC_7(T)$ are convex on $T>0$.

Incorporate the above arguments, we have completed the proof of Lemma 2. □

6. The Determination of the Optimal Cycle Time T^* of $TRC(T)$

Theorem 1. *Suppose* $\frac{W}{D\rho} < M-N$.

1) *If* $0<\Delta_{12}$, *then* $TRC(T^*)=TRC_1(T_1^*)$ *and* $T^*=T_1^*$.

2) *If* $\Delta_{12}\le 0<\Delta_{23}$, *then* $TRC(T^*)=TRC_2(T_2^*)$ *and* $T^*=T_2^*$.

3) *If* $\Delta_{23}\le 0<\Delta_{34}$, *then* $TRC(T^*)=TRC_3(T_3^*)$ *and* $T^*=T_3^*$.

4) *If* $\Delta_{34}\le 0<\Delta_{45}$, *then* $TRC(T^*)=TRC_4(T_4^*)$ *and* $T^*=T_4^*$.

5) *If* $\Delta_{45}\le 0$, *then* $TRC(T^*)=TRC_5(T_5^*)$ *and* $T^*=T_5^*$.

Proof. 1) If $0<\Delta_{12}$, then $0<\Delta_{12}<\Delta_{23}<\Delta_{34}<\Delta_{45}$. So, lemmas 1, 2, and

Equations (60a)-(60c) imply

a) $TRC_1(T)$ is decreasing on $\left(0, T_1^*\right]$ and increasing on $\left[T_1^*, \frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is increasing on $\left[\frac{W}{D\rho}, M-N\right]$.

c) $TRC_3(T)$ is increasing on $[M-N, M]$.

d) $TRC_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (21a)-(21e) and 1.1-1.5 reveal that $TRC(T)$ is decreasing on $\left(0, T_1^*\right]$ and increasing on $\left[T_1^*, \infty\right)$. Hence, $T^* = T_1^*$ and $TRC\left(T^*\right) = TRC_1\left(T_1^*\right)$.

2) If $\Delta_{12} \le 0 < \Delta_{23}$, then $\Delta_{12} \le 0 < \Delta_{23} < \Delta_{34} < \Delta_{45}$. So, lemmas 1, 2, and Equations (60a)-(60c) imply

a) $TRC_1(T)$ is decreasing on $\left[0, \frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, T_2^*\right]$ and increasing on $\left[T_2^*, M-N\right]$.

c) $TRC_3(T)$ is increasing on $[M-N, M]$.

d) $TRC_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (21a)-(21e) and 2.1-2.5 reveal that $TRC(T)$ is decreasing on $\left(0, T_2^*\right]$ and increasing on $\left[T_2^*, \infty\right)$. Hence, $T^* = T_2^*$ and $TRC\left(T^*\right) = TRC_2\left(T_2^*\right)$.

3) If $\Delta_{23} \le 0 < \Delta_{34}$, then $\Delta_{12} < \Delta_{23} \le 0 < \Delta_{34} < \Delta_{45}$. So, lemmas 1, 2, and Equations (60a)-(60c) imply

a) $TRC_1(T)$ is decreasing on $\left[0, \frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, M-N\right]$.

c) $TRC_3(T)$ is decreasing on $\left[M-N, T_3^*\right]$ and increasing on $\left[T_3^*, M\right]$.

d) $TRC_4(T)$ is increasing on $\left[M, \frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (21a)-(21e) and 3.1-3.5 reveal that $TRC(T)$ is decreasing on $\left(0, T_3^*\right]$ and increasing on $\left[T_3^*, \infty\right)$. Hence, $T^* = T_3^*$ and $TRC\left(T^*\right) = TRC_3\left(T_3^*\right)$.

4) If $\Delta_{34} \le 0 < \Delta_{45}$, then $\Delta_{12} < \Delta_{23} < \Delta_{34} \le 0 < \Delta_{45}$. So, lemmas 1, 2, and Equations (60a)-(60c) imply

a) $TRC_1(T)$ is decreasing on $\left[0, \frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, M-N\right]$.

c) $TRC_3(T)$ is decreasing on $[M-N, M]$.

d) $TRC_4(T)$ is decreasing on $\left[M, T_4^*\right]$ and increasing on $\left[T_4^*, \frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (21a)-(21e) and 4a-4e reveal that $TRC(T)$ is decreasing on $\left(0, T_4^*\right]$ and increasing on $\left[T_4^*, \infty\right)$. Hence, $T^* = T_4^*$ and $TRC\left(T^*\right) = TRC_4\left(T_4^*\right)$.

5) If $\Delta_{45} \le 0$, then $\Delta_{12} < \Delta_{23} < \Delta_{34} < \Delta_{45} \le 0$. So, lemmas 1, 2, and Equations (60a)-(60c) imply

a) $TRC_1(T)$ is decreasing on $\left[0, \frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, M-N\right]$.

c) $TRC_3(T)$ is decreasing on $[M-N, M]$.

d) $TRC_4(T)$ is decreasing on $\left[M, T_4^*\right]$.

e) $TRC_5(T)$ is decreasing on $\left[\frac{PM}{D}, T_5^*\right]$ and increasing on $\left[T_5^*, \infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (21a)-(21e) and 5.1-5.5 reveal that $TRC(T)$ is decreasing on $\left(0, T_5^*\right]$ and increasing on $\left[T_5^*, \infty\right)$. Hence, $T^* = T_5^*$ and $TRC\left(T^*\right) = TRC_5\left(T_5^*\right)$.

Incorporating all argument above arguments, we have completed the proof of theorem 1. □

Applying lemmas 1, 2, and Equations (27a)-(27e), the following results hold.

Theorem 2. *Suppose* $M - N \le \frac{W}{D\rho} < M$.

1) *If* $0 < \Delta_{16}$, *then* $TRC\left(T^*\right) = TRC_1\left(T_1^*\right)$ *and* $T^* = T_1^*$.

2) *If* $\Delta_{16} \le 0 < \Delta_{63}$, *then* $TRC\left(T^*\right) = TRC_6\left(T_6^*\right)$ *and* $T^* = T_6^*$.

3) *If* $\Delta_{63} \le 0 < \Delta_{34}$, *then* $TRC\left(T^*\right) = TRC_3\left(T_3^*\right)$ *and* $T^* = T_3^*$.

4) *If* $\Delta_{34} \le 0 < \Delta_{45}$, *then* $TRC\left(T^*\right) = TRC_4\left(T_4^*\right)$ *and* $T^* = T_4^*$.

5) *If* $\Delta_{45} \le 0$, *then* $TRC\left(T^*\right) = TRC_5\left(T_5^*\right)$ *and* $T^* = T_5^*$.

Applying lemmas 1, 2, and Equations (29a)-(29e), the following results hold.

Theorem 3. *Suppose* $M \le \frac{W}{D\rho} < \frac{PM}{D}$.

1) *If* $0 < \Delta_{16}$, *then* $TRC\left(T^*\right) = TRC_1\left(T_1^*\right)$ *and* $T^* = T_1^*$.

2) *If* $\Delta_{16} \le 0 < \Delta_{67}$, *then* $TRC\left(T^*\right) = TRC_6\left(T_6^*\right)$ *and* $T^* = T_6^*$.

3) *If* $\Delta_{67} \le 0 < \Delta_{74}$, *then* $TRC\left(T^*\right) = TRC_7\left(T_7^*\right)$ *and* $T^* = T_7^*$.

4) *If* $\Delta_{74} \le 0 < \Delta_{45}$, *then* $TRC\left(T^*\right) = TRC_4\left(T_4^*\right)$ *and* $T^* = T_4^*$.

5) *If* $\Delta_{45} \le 0$, *then* $TRC\left(T^*\right) = TRC_5\left(T_5^*\right)$ *and* $T^* = T_5^*$.

Applying lemmas 1, 2, and Equations (31a)-(31e), the following results hold.

Theorem 4. *Suppose* $\frac{PM}{D} \le \frac{W}{D\rho}$.

1) *If* $0 < \Delta_{16}$, *then* $TRC(T^*) = TRC_1(T_1^*)$ *and* $T^* = T_1^*$.

2) *If* $\Delta_{16} \le 0 < \Delta_{67}$, *then* $TRC(T^*) = TRC_6(T_6^*)$ *and* $T^* = T_6^*$.

3) *If* $\Delta_{67} \le 0 < \Delta_{78}$, *then* $TRC(T^*) = TRC_7(T_7^*)$ *and* $T^* = T_7^*$.

4) *If* $\Delta_{78} \le 0 < \Delta_{85}$, *then* $TRC(T^*) = TRC_8(T_8^*)$ *and* $T^* = T_8^*$.

5) *If* $\Delta_{85} \le 0$, *then* $TRC(T^*) = TRC_5(T_5^*)$ *and* $T^* = T_5^*$.

7. Sensitivity Analyses

To find out the critical parameters in this research, [1], and [3] models, we use Maple 18.00 to execute the sensitivity analyses and increasing and decreasing 25% and 50% of the parameters to determine the unique solution T_i^* when $TRC_i'(T^*) = 0, i = 1 \sim 8$. We give $P = 9000$ units/year, $D = 5500$ units/year, $W = 800$ units, $A = \$1000$/order, $s = \$14$/unit, $c = \$6$/unit, $\theta = 0.1$, $h_m = \$0.7$/unit/year, $h_o = \$1.5$/unit/year, $h_r = \$4.5$/unit/year, $M = 120$ days $= 120/365$ year, $N = 65$ days $= 65/365$ year, $I_p = \$0.4$/year, and $I_e = \$0.21$/year.

From the computational outcomes, we can determine T^* and $TRC(T^*)$ from the sensitivity analyses for for this research, [1], and [3] models as shown in Table 2 and Table 3, and derive a relative comparison of the impact of the parameters on T^* and $TRC(T^*)$ in the sensitivity analyses as shown in Figures 11-16.

According to Table 2 and Table 3 and Figures 11-16, it can be seen the variables impact order cycle time T^* for this research, [1], and [3] models:

1) this research model

a) Positive & Major: the ordering cost *A*.

b) Positive & Minor: the unit holding cost per item for product in a rented warehouse h_r.

c) Negative & Minor: the unit selling price per item *s*, the unit holding cost per item for raw materials in a raw materials warehouse h_m, the unit holding cost per item for product in an owned warehouse h_o, the interest rate payable I_p, and the interest rate earned I_e.

d) Negative & Major: the unit purchasing price per item *c* and the deterioration rate θ.

2) [1]'s model

a) Positive & Major: the ordering cost *A*.

b) Positive & Minor: none.

c) Negative & Minor: the unit selling price per item *s*, the unit purchasing price per item *c*, the unit holding cost per item for raw materials in a raw materials warehouse h_m, the unit holding cost per item for product in an owned warehouse h_o, the unit holding cost per item for product in a rented warehouse h_r, the interest rate payable I_p, and the interest rate earned I_e.

Table 2. The sensitivity analyses for T^* of this research, [1], and [3] models.

Parameters	+/−	this research	[1]	[3]
A	−50%	0.323796413	0.336873359	0.353809894
	−25%	0.356136518	0.370559305	0.389189423
	0%	0.380988522	0.396450476	0.416382291
	+25%	0.404310371	0.420751426	0.441904987
	+50%	0.426353536	0.443723500	0.466031997
s	−50%	0.389714048	0.405541903	0.425930795
	−25%	0.385376100	0.401021954	0.421183603
	0%	0.380988522	0.396450476	0.416382291
	+25%	0.376549626	0.391825666	0.411524966
	+50%	0.372057574	0.387145613	0.406609619
c	−50%	0.406384273	0.417864966	0.447136879
	−25%	0.391824025	0.405680770	0.429428145
	0%	0.380988522	0.396450476	0.416382291
	+25%	0.372601550	0.389210740	0.406358844
	+50%	0.365912858	0.383377581	0.398410010
h_m	−50%	0.389613329	0.406049944	
	−25%	0.385227856	0.401164096	
	0%	0.380988522	0.396450476	
	+25%	0.376887341	0.391899198	
	+50%	0.372916925	0.387501152	
h_o	−50%	0.391628615	0.407536772	0.428025957
	−25%	0.386345341	0.402031840	0.422244261
	0%	0.380988522	0.396450476	0.416382291
	+25%	0.375555056	0.390789406	0.410436607
	+50%	0.370041463	0.385045114	0.404403516
h_r	−50%	0.382462299	0.401566710	0.426988203
	−25%	0.381654883	0.398746393	0.421093248
	0%	0.380988522	0.396450476	0.416382291
	+25%	0.380429265	0.394545054	0.412530499
	+50%	0.379953137	0.392938258	0.409322090
I_p	−50%	0.396194694	0.417864966	0.447136879
	−25%	0.387619621	0.405680770	0.429428145
	0%	0.380988522	0.396450476	0.416382291
	+25%	0.375705573	0.389210740	0.406358844
	+50%	0.371396477	0.383377581	0.398410010
I_e	−50%	0.389714048	0.405541903	0.425930795
	−25%	0.385376100	0.401021954	0.421183603
	0%	0.380988522	0.396450476	0.416382291
	+25%	0.376549626	0.391825666	0.411524966
	+50%	0.372057574	0.387145613	0.406609619
θ	−50%	0.388544380		
	−25%	0.384723975		
	0%	0.380988522		
	+25%	0.377335518		
	+50%	0.373762524		

Table 3. The sensitivity analyses for $TRC\left(T^{*}\right)$ of this research, [1], and [3] models.

Parameters	+/−	this research	[1]	[3]
A	−50%	35267.01504	34992.21730	1613.304009
	−25%	36021.43281	35691.12590	2266.223520
	0%	36650.11429	36273.54970	2810.323113
	+25%	37278.79577	36855.97352	3354.422706
	+50%	37907.47725	37438.39734	3898.522299
s	−50%	36880.93796	36487.38963	3010.092101
	−25%	36765.52613	36380.46967	2910.207607
	0%	36650.11429	36273.54970	2810.323113
	+25%	36534.70246	36166.62974	2710.438620
	+50%	36419.29063	36059.70978	2610.554126
c	−50%	19908.61022	19695.93987	2687.620477
	−25%	28279.36227	27984.74479	2748.971795
	0%	36650.11429	36273.54970	2810.323113
	+25%	45020.86635	44562.35461	2871.674431
	+50%	53391.61838	52851.15952	2933.025750
h_m	−50%	36414.30792	36021.07272	
	−25%	36532.21111	36147.31121	
	0%	36650.11429	36273.54970	
	+25%	36768.01747	36399.78820	
	+50%	36885.92065	36526.02669	
h_o	−50%	36332.28614	35934.95967	2454.531969
	−25%	36491.20021	36104.25469	2632.427541
	0%	36650.11429	36273.54970	2810.323113
	+25%	36809.02837	36442.84472	2988.218684
	+50%	36967.94245	36612.13974	3166.114256
h_r	−50%	36646.73504	36256.45940	2772.085535
	−25%	36648.42467	36265.00455	2791.204324
	0%	36650.11429	36273.54970	2810.323113
	+25%	36651.80392	36282.09484	2829.441902
	+50%	36653.49354	36290.63999	2848.560691
I_p	−50%	36610.72995	36195.93987	2687.620477
	−25%	36630.42212	36234.74479	2748.971795
	0%	36650.11429	36273.54970	2810.323113
	+25%	36669.80647	36312.35461	2871.674431
	+50%	36689.49864	36351.15952	2933.025750
I_e	−50%	36880.93796	36487.38963	3010.092101
	−25%	36765.52613	36380.46967	2910.207607
	0%	36650.11429	36273.54970	2810.323113
	+25%	36534.70246	36166.62974	2710.438620
	+50%	36419.29063	36059.70978	2610.554126
θ	−50%	36445.25871		
	−25%	36547.47934		
	0%	36650.11429		
	+25%	36753.16317		
	+50%	36856.63035		

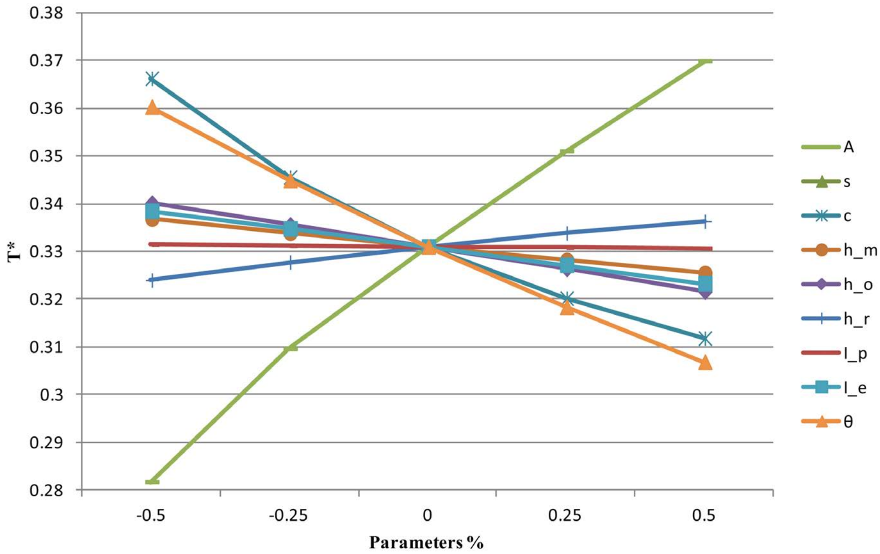

Figure 11. The sensitivity analyses for T^* of this research model.

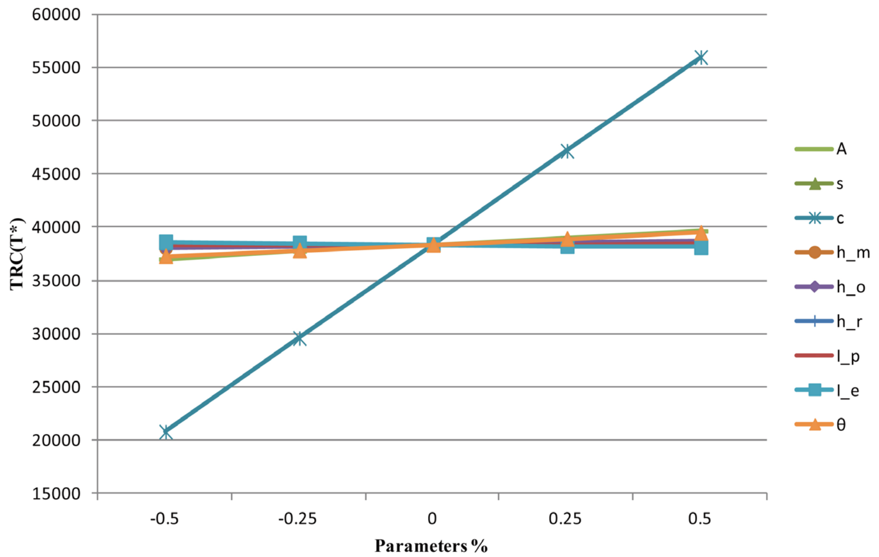

Figure 12. The sensitivity analyses for $TRC(T^*)$ of this research model.

d) Negative & Major: none.

3) [3]'s model

a) Positive & Major: the ordering cost A.

b) Positive & Minor: none.

c) Negative & Minor: the unit selling price per item s, the unit holding cost per item for product in an owned warehouse h_o, the unit holding cost per item for product in a rented warehouse h_r, and the interest rate earned I_e.

d) Negative & Major: the unit purchasing price per item c and the interest rate payable I_p.

Therefore, when making decisions on the order cycle time, variables with a relatively large influence must be considered as priority, while those with a small

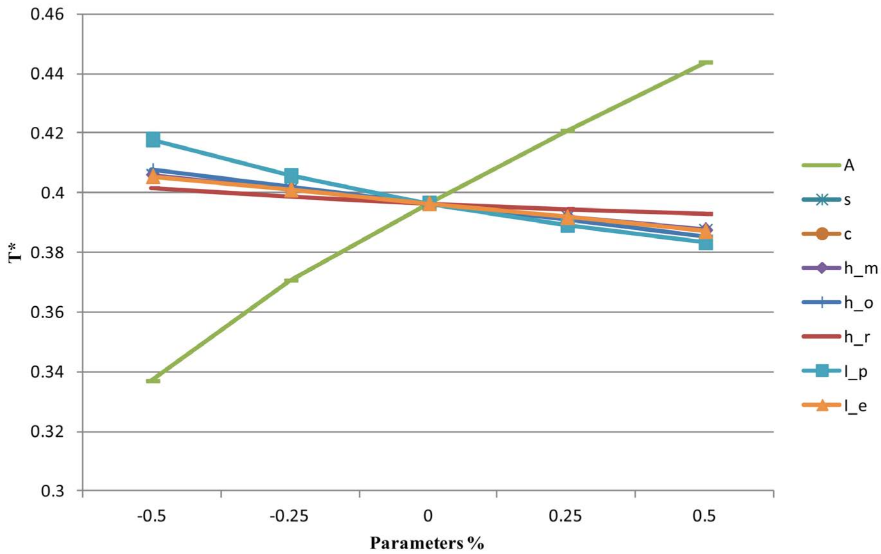

Figure 13. The sensitivity analyses for T^* of [1]'s model.

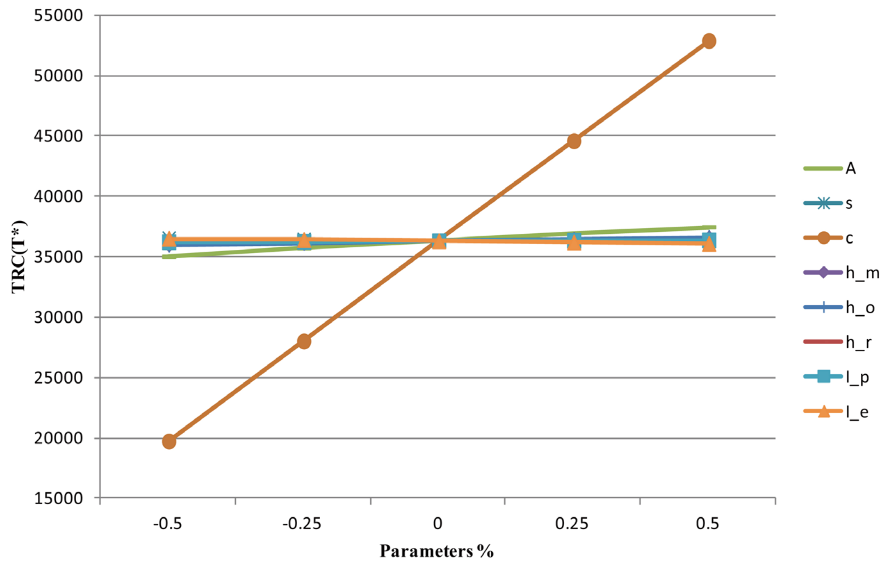

Figure 14. The sensitivity analyses for $TRC\left(T^*\right)$ of [1]'s model.

influence can be processed later.

On the other hand, it is seen that the variables impact the annual total relevant cost $TRC\left(T^*\right)$ for this research, [1], and [3] models:

1) this research model

a) Positive & Major: the unit purchasing price per item c.

b) Positive & Minor: the ordering cost A, the unit holding cost per item for raw materials in a raw materials warehouse h_m, the unit holding cost per item for product in an owned warehouse h_o, and the interest rate payable I_p.

c) Negative & Minor: the unit holding cost per item for product in a rented warehouse h_r, the interest rate earned I_e, and the deterioration rate θ.

d) Negative & Major: none.

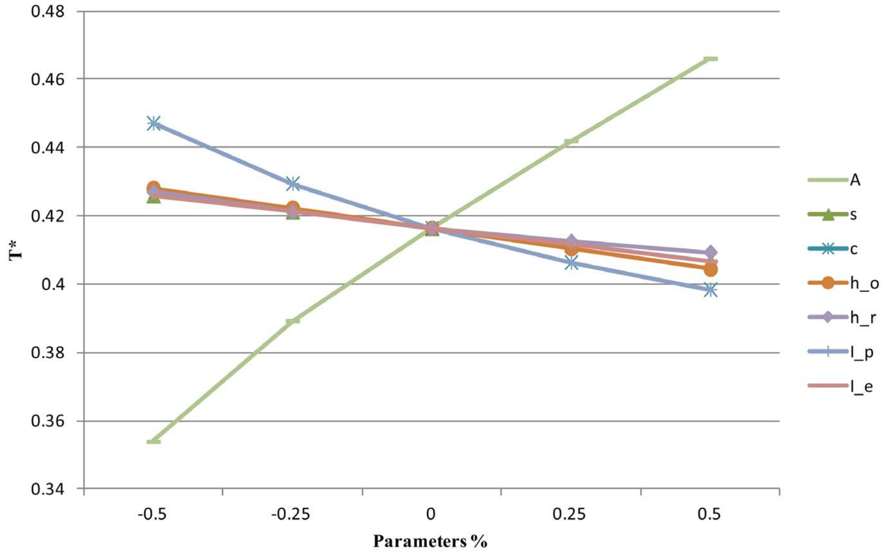

Figure 15. The sensitivity analyses for T^* of [3]'s model.

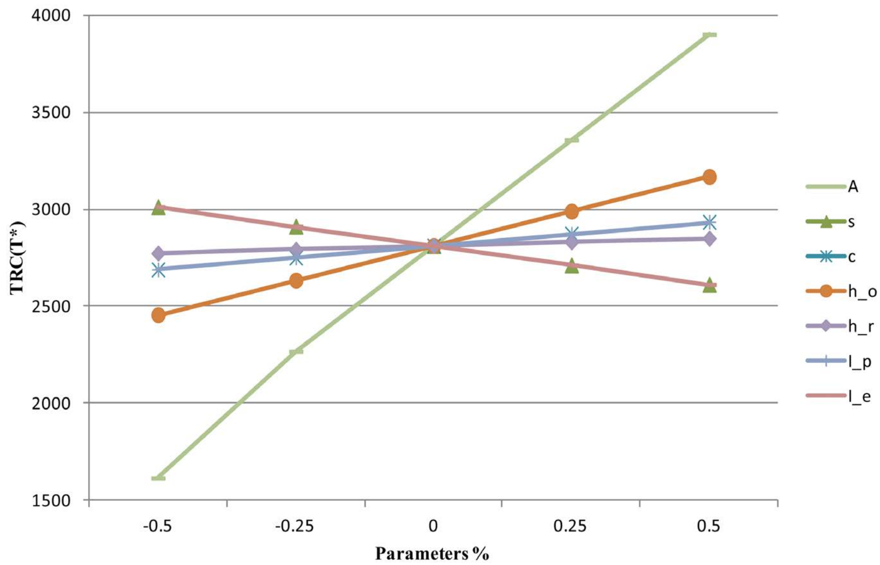

Figure 16. The sensitivity analyses for $TRC(T^*)$ of [3]'s model.

2) [1]'s model

a) Positive & Major: the unit purchasing price per item c.

b) Positive & Minor: the ordering cost A, the unit holding cost per item for raw materials in a raw materials warehouse h_m, the unit holding cost per item for product in an owned warehouse h_o, the unit holding cost per item for product in a rented warehouse h_r, and the interest rate payable I_p.

c) Negative & Minor: the unit selling price per item s and the interest rate earned I_e.

d) Negative & Major: none.

3) [3]'s model

a) Positive & Major: the ordering cost A and the unit holding cost per item for product in an owned warehouse h_o.

b) Positive & Minor: the unit purchasing price per item c, the unit holding cost per item for product in a rented warehouse h_r, and the interest rate payable I_p.

c) Negative & Minor: the interest rate earned I_e.

d) Negative & Major: the unit selling price per item s.

Therefore, when making decisions on the annual total relevant cost, variables with a relatively large influence can be considered as priority, while those with a small influence can be processed later.

We can organize the relative parameters impact to T^* and $TRC(T^*)$ for this research, [1], and [3] models, as shown in **Table 4** and **Table 5**.

8. Conclusions

One of traditional EPQ model's assumptions is that the raw materials required for production are timely, so that the holding cost of raw materials will be ignored. [2] pointed out the importance of the holding cost of raw materials will affect the total relevant cost, and [1] combined [2]'s the concept of holding cost of raw materials and [3]'s two-level trade credit and limited storage capacity model to present an inventory model with the holding cost of non-deteriorating raw materials. And this research further develops with the holding cost of deteriorating raw materials.

We reach the following conclusions in management practice after the sensitivity analyses:

1) When making decisions on the order cycle time T^* under limited resources, it gives priority order to the ordering cost A and the unit purchasing price per item c.

Table 4. Comparison of the relative parameters impact to T^* of this research, [1], and [3] models in the sensitivity analyses.

Impact	this research	[1]	[3]
Positive & Major	A	A	A
Positive & Minor	h_r		
Negative & Minor	s, h_m, h_o, I_p, I_e	$s, c, h_m, h_o, h_r, I_p, I_e$	s, h_o, h_r, I_e
Negative & Major	c, θ		c, I_p

Table 5. Comparison of the relative parameters impact to $TRC(T^*)$ of this research, [1], and [3] models in the sensitivity analyses.

Impact	this research	[1]	[3]
Positive & Major	c	c	A, h_o
Positive & Minor	c, h_m, h_o, I_p	c, h_m, h_o, h_r, I_p	c, h_r, I_p
Negative & Minor	h_r, I_e, θ	s, I_e	I_e
Negative & Major			s

2) When making decisions on the annual total relevant cost $TRC\left(T^*\right)$ under limited resources, it only considers the unit purchasing price per item *c*.

This research provides more precise decisions for practical business decisions. Although adding the holding cost of raw materials increases the complexity of the model, but it's useful and contributes to the field of industrial management.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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