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Optimal Economic Production Quantity Policies Considering the Holding Cost of Deteriorating Raw Materials under Two-Level Trade Credit and Limited Storage Capacity

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Abstract

The traditional economic production quantity (EPQ) model assumes that raw materials are supplied timely. But during the production and transport process of raw materials could change the holding cost of raw materials, therefore, they should be considered in the total relevant cost. [1] combined [2]'s concept of holding cost of raw materials and [3]'s two-level trade credit and limited storage capacity model to develop innovative and detailed EPQ model that consider the holding cost of non-deteriorating raw materials. It's closer to the real world. However, in reality, most of the raw materials are deteriorating; it also needs to be considered. Therefore, this research extends [1]'s model to consider the holding cost of deteriorating raw materials. We use cost minimization to develop the total relevant cost and determine the optimal cycle time by four theorems. Finally, we use sensitivity analyses to investigate the effects of the parameters on ordering policies.

Subject Areas

Business Analysis

Keywords

Economic Production Quantity, Deteriorating Raw Materials, Two-Level Trade Credit, Limited Storage Capacity

1. Introduction

The economic order quantity (EOQ) model [4] and the economic production quantity (EPQ) model [5] are widely used in the inventory management. Scholars studied many interesting issues, including the holding cost of production and sale process, but they usually assume the production of raw materials is timely. However, suppliers provide raw materials would interfere with other factors, such as climate changes affect insufficient production of raw materials, shipment delay because rise raw materials cost, etc., those affect the price volatility. Consequently, we should not only simply use the ordering cost to roughly calculate the total relevant cost, but also consider the holding cost of raw materials accurately to calculate. [2] modified the EPQ model to consider the holding cost of raw materials first, [6] and [7] have further discussion with the holding cost of raw materials.

[8] established a standard EOQ model for non-deteriorating items under the condition of permissible delay. [9] developed two-level trade credit to extend [8] which provide a fixed trade credit period M between supplier-retailer, and trade credit period N between retailer-customer ($M \geq N$). If a customer buys one item from the retailer at time $t \in [0, T]$, then the customer will have a trade credit period $N - t$ and make the payment at time N. This trade credit allows retailer not only have a maximal profit [10]-[19], but also extend an interesting issue, the retailer would store exceed quantities in a rented warehouse (RW) when owned warehouse (OW) full [20] [21] [22] [23] [24] [31]. And [3] developed a complete inventory model by incorporated two levels of trade credit and limited storage capacity together.

Recently, [1] combined [2]'s concept of holding cost of raw materials and [3]'s two-level trade credit and limited storage capacity model to develop innovative and detailed EPQ model that consider the holding cost of non-deteriorating raw materials. However, the raw materials are mostly energy, grain, metal, and fiber, etc., time-sensitive and volatile, therefore, these need to consider the deteriorating property of raw materials [25]. Therefore, we organize the relevant literatures on two-level trade credit, limited storage capacity, and raw materials, as shown in **Table 1**.

As mentioned above, studies that consider the impact of the holding cost of deteriorating raw materials in the total relevant cost are limited or nonexistent. Moreover, [1] developed a complete inventory model by incorporating the holding cost of non-deteriorating raw materials with two-level trade credit and limited storage capacity. Therefore, this research extends [1]'s model to develop a new inventory model by considering the holding cost of deteriorating raw materials to determine the optimal inventory policies, two-level trade credit and limited storage. According to cost-minimization policy, four theorems are developed to characterize the optimal solution. Finally, the sensitivity analyses are used to illustrate to find out the critical impact factors and draw the conclusions.

2. Notations and Assumptions

The mathematical model is developed based on the following.

Table 1. Summary of related literature for inventory models with two-level trade credit, limited storage capacity, and raw materials.

Author	Model	$N-t$	LSC	NRM	DRM
[10]	EOQ	V			
[3]	EPQ	V		V	
[26]	EOQ	V		V	
[11]	EPQ	V			
[27]	EPQ	V			
[28]	EPQ	V			
[29]	EOQ	V			
[30]	EPQ	V			
[31]	EPQ	V		V	
[32]	EOQ	V			
[33]	EPQ	V			
[22]	EPQ	V		V	
[2]	EPQ				V
[34]	EPQ	V			
[35]	EPQ	V			
[15]	EPQ	V			
[14]	EOQ	V			
[23]	EPQ	V		V	
[16]	EPQ	V			
[17]	EOQ	V			
[36]	EPQ	V			
[6]	EPQ				V
[7]	EPQ				V
[37]	EPQ	V			
[38]	EPQ	V			
[18]	EPQ	V			
[39]	EPQ	V			
[40]	EPQ	V			
[24]	EOQ			V	
[19]	EOQ	V	V	V	
[1]	EPQ	V	V	V	
This research	EPQ	V	V		V

Note: Column $N-t$ for [9]'s payment method, LSC for limited storage capacity, NRM for the holding cost of non-deteriorating raw materials, and DRM for the holding cost of deteriorating raw materials. For the answers, V for Yes and empty for No.

2.1. Notations

Q the order size.

P the production rate.

D the demand rate.

A the ordering cost.

T the cycle time.

$\rho \quad = 1 - \dfrac{D}{P} > 0$.

$L_{\max}$ the storage maximum.

$I_m(t)$ the inventory function for raw materials.

θ the deterioration rate, $0 \le \theta < 1$.

s the unit selling price per item.

c the unit purchasing price per item.

h_m, the unit holding cost per item for raw materials in a raw materials warehouse.

h_o the unit holding cost per item for product in an owned warehouse.

h_r the unit holding cost per item for product in a rented warehouse.

I_p the interest rate payable per \$ unit time (year).

I_e the interest rate earned per \$ unit time (year).

t_s time in years at which production stops.

M the manufacturer's trade credit period offered by the supplier.

N the customer's trade credit period offered by the manufacturer.

W the storage capacity of an owned warehouse.

tw_i the point in time when the inventory level increases to W when the production period is $\dfrac{W}{P-D}$.

tw_d the point in time when the inventory level decreases to W when the production cease period is $T - \dfrac{W}{D}$.

$tw_d - tw_i$ the time of rented warehouse is

$$\begin{cases} \dfrac{DT\rho - W}{P-D} + \dfrac{DT\rho - W}{D}, & \text{if } DT\rho > W \\ 0, & \text{if } DT\rho \le W. \end{cases}$$

$TRC(T)$ the total relevant cost per unit time of the model when $T > 0$.

T^* the optimal solution of $TRC(T)$.

2.2. Assumptions

1) Demand rate D is known and constant.

2) Production rate P is known and constant, $P > D$.

3) Shortages are not allowed.

4) Backlogging is not allowed.

5) A single item is considered.

6) Time period is infinite.

7) Replenishment rate is infinite.

8) $h_r \ge h_o \ge h_m$, $M \ge N$, and $s \ge c$.

9) Storage capacity of raw materials warehouse is unlimited.

10) If the order quantity is larger than the manufacturer's OW (owned warehouse) storage capacity, then the manufacturer will rent an RW (rented

warehouse) with unlimited storage capacity. When demand occurs, it is first replenished from the RW which has storage that exceeds the items. The RW takes first in last out (FILO), and products in the OW or RW will not deteriorate.

11) During the period the account is not settled, generated sales revenue is deposited in and interest-bearing account.

a) When $M \le T$, the account is settled at $t = M$, the manufacturer pays off all units sold, keeps his or her profits, and starts paying for the higher interest payable on the items in stock with rate I_p.

b) When $T \le M$, the account is settled at $t = M$ and the manufacturer does not have to pay any interest payable.

12) If a customer buys an item from the manufacturer at time $t \in [0, T]$, then the customer will have a trade credit period $N - t$ and make the payment at time N.

13) The manufacturer can accumulate revenue and earn interest after his or her customer pays the amount of the purchasing cost to the manufacturer until the end of the trade credit period offered by the supplier. In other words, the manufacturer can accumulate revenue and earn interest during the period from N to M with rate I_e under the condition of trade credit.

14) The manufacturer keeps the profit for use in other activities.

2.3. Model

The model considers three stages of a supply chain system. It assumes that the supplier prepares the deteriorating raw materials for production, and the deteriorating raw materials are expected to decrease by the inventory function $I_m(t)$ with the deterioration rate θ (from time 0 to t_s). The quantity of products is expected to increase with time to the maximum inventory level (from 0 to t_s); the products are sold on demand at the same time. After production stops (at time t_s), the products are sold only on demand until the quantity reaches zero (at time T), as shown in Figure 1.

3. Annual Total Relevant Cost

The annual total relevant cost consists of the following element.

As shown in Figure 1, the raw material inventory level can be described by the following formulas, and we set the time in years at which production stops t_s, the optimal order size Q and storage maximum $L_{\max}$:

$$\frac{\mathrm{d}I_m(t)}{\mathrm{d}t} + \theta I_m(t) = -P,\ 0 \le t \le t_s. \tag{1}$$

By using the boundary condition $I_m(t_s) = 0$, we obtain

$$I_m(t) = \frac{P}{\theta}\left(\mathrm{e}^{\theta(t_s - t)} - 1\right),\ 0 \le t \le t_s. \tag{2}$$

We will then set the cycle time T and the optimal quantity Q.

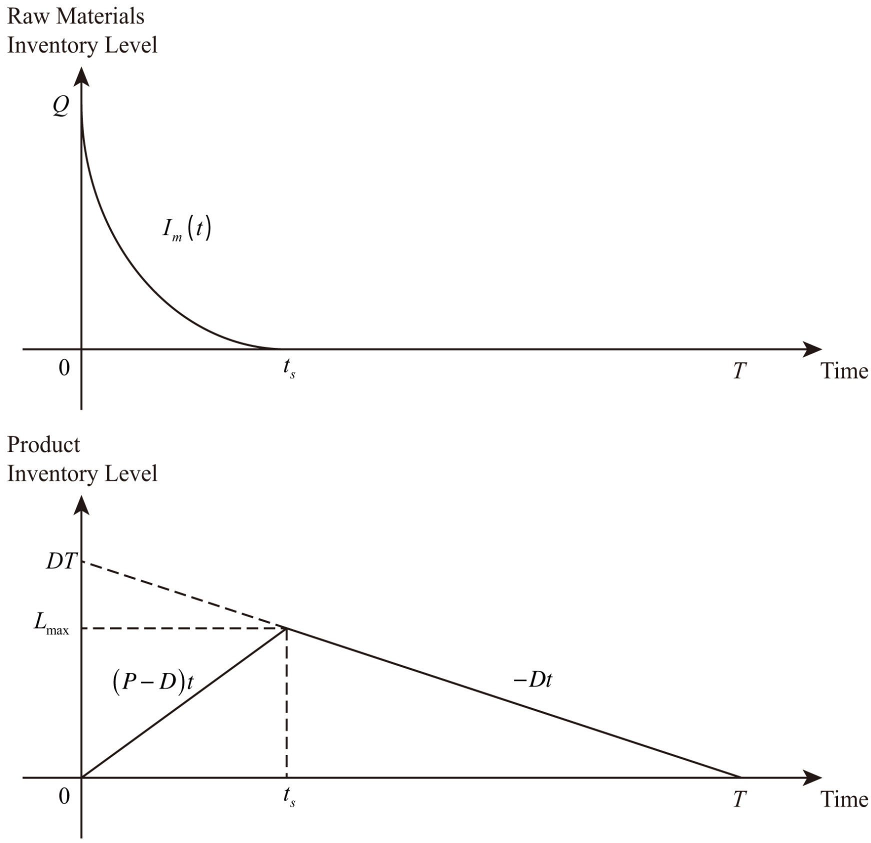

Figure 1. Raw materials and product inventory level.

$$(P-D)t_s - D(T-t_s) = 0,$$

$$t_s = \frac{D}{P}T. \tag{3}$$

$$Q = I_m(0) = \frac{P}{\theta}\left(e^{\theta \frac{D}{P}T} - 1\right). \tag{4}$$

3.1. Annual Ordering Cost

Annual ordering cost is

$$\frac{A}{T}. \tag{5}$$

3.2. Annual Purchasing Cost

Annual purchasing cost is

$$c \times Q \times \frac{1}{T} = \frac{cP}{\theta T}\left(e^{\theta \frac{D}{P}T} - 1\right). \tag{6}$$

3.3. Annual Holding Cost

Annual holding cost is

1) As shown in Figure 1, annual holding cost of raw materials

$$h_m \times \int_0^{t_s} I_m(t)\,\mathrm{d}t \times \frac{1}{T} = \frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(\mathrm{e}^{\theta \frac{D}{P} T} - 1\right) - \frac{D}{P} T\right]. \tag{7}$$

2) Two cases occur in annual holding costs of owned warehouse.

a) $DT\rho \le W$, as shown in **Figure 2**.

Annual holding cost in owned warehouse is

$$h_o \times \frac{T \times L_{\max}}{2} \times \frac{1}{T} = \frac{DTh_o\rho}{2}. \tag{8}$$

b) $W \le DT\rho$, as shown in **Figure 3**.

Annual holding cost in owned warehouse is

$$h_o \times \frac{\left[\left(tw_d - tw_i\right) + T\right]W}{2} \times \frac{1}{T} = Wh_o - \frac{W^2 h_o}{2DT\rho}. \tag{9}$$

3) Two cases occur in annual holding costs of rented warehouse.

a) $DT\rho \le W$, as shown in **Figure 2**.

Annual holding cost in rented warehouse is

$$0. \tag{10}$$

b) $W \le DT\rho$, as shown in **Figure 3**.

Annual holding cost in rented warehouse is

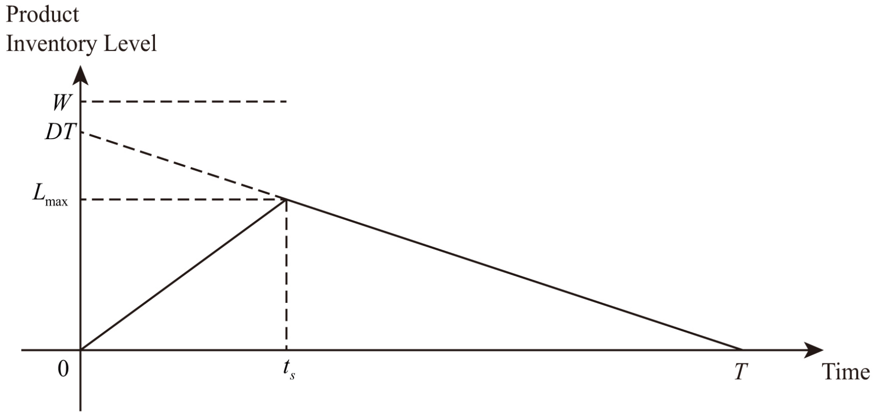

Figure 2. Annual holding cost when $DT\rho \le W$.

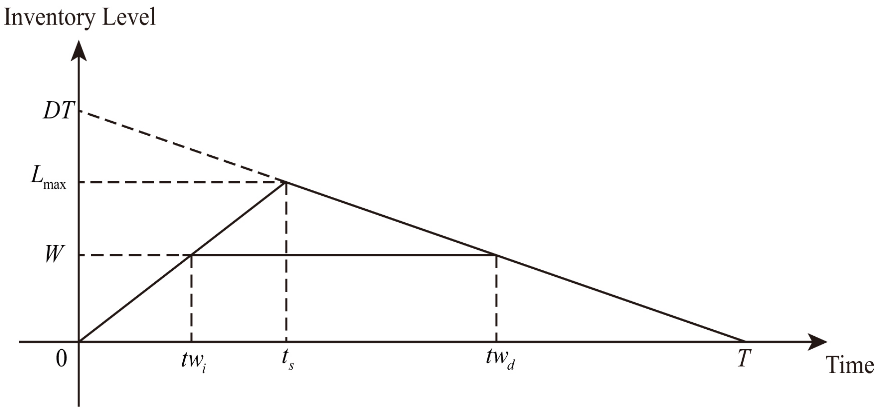

Figure 3. Annual holding cost when $W \le DT\rho$.

$$h_r \times \frac{\left(tw_d - tw_i\right)\times\left(L_{\max} - W\right)}{2} \times \frac{1}{T} = \frac{h_r\left(DT\rho - W\right)^2}{2DT\rho}. \tag{11}$$

3.4. Annual Interest Payable

Four cases to occur in costs of annual interest payable for the items kept in stock.

1) $0 < T \le N$.

Annual interest payable is

$$0. \tag{12}$$

2) $N \le T \le M$.

Annual interest payable is

$$0. \tag{13}$$

3) $M \le T \le \frac{PM}{D}$, as shown in Figure 4.

Annual interest payable is

$$cI_p \times \left[\frac{\left(T-M\right)\times D\left(T-M\right)}{2}\right] \times \frac{1}{T} = \frac{cI_p D\left(T-M\right)^2}{2T}. \tag{14}$$

4) $M \le \frac{PM}{D} \le T$, as shown in Figure 5.

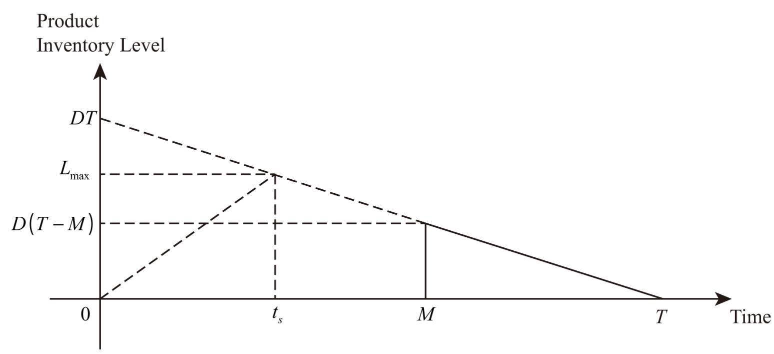

Figure 4. Annual interest payable when $M \le T \le \frac{PM}{D}$.

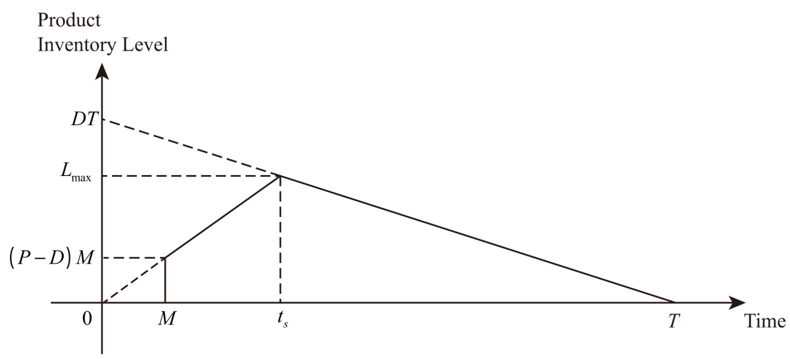

Figure 5. Annual interest payable when $M \le \frac{PM}{D} \le T$.

Annual interest payable is

$$cI_p \times \left[\frac{T \times DT\rho}{2} - \frac{M \times (P-D)M}{2}\right] \times \frac{1}{T} = \frac{cI_p\rho\left(DT^2 - PM^2\right)}{2T}. \tag{15}$$

3.5. Annual Interest Earned

Three cases to occur in annual interest earned.

1) $0 < T \le N$, as shown in Figure 6.

Annual interest earned is

$$sI_e \times DT(M-N) \times \frac{1}{T} = sI_eD(M-N). \tag{16}$$

2) $N \le T \le M$, as shown in Figure 7.

Annual interest earned is

$$sI_e \times \left[\frac{(DN+DT)(T-N)}{2} + DT(M-T)\right] \times \frac{1}{T} = \frac{sI_eD\left(2MT - N^2 - T^2\right)}{2T}. \tag{17}$$

3) $N \le M \le T$, as shown in Figure 8.

Annual interest earned is

$$sI_e \times \left[\frac{(DN+DM)(M-N)}{2}\right] \times \frac{1}{T} = \frac{sI_eD\left(M^2 - N^2\right)}{2T}. \tag{18}$$

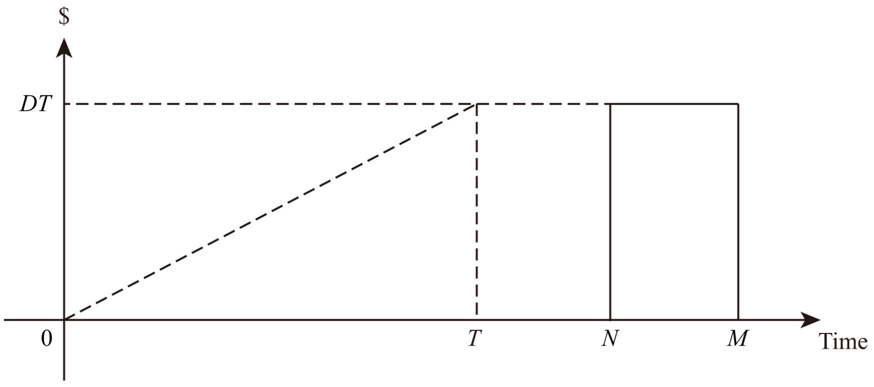

Figure 6. Annual interest earned when $0 < T \le N$.

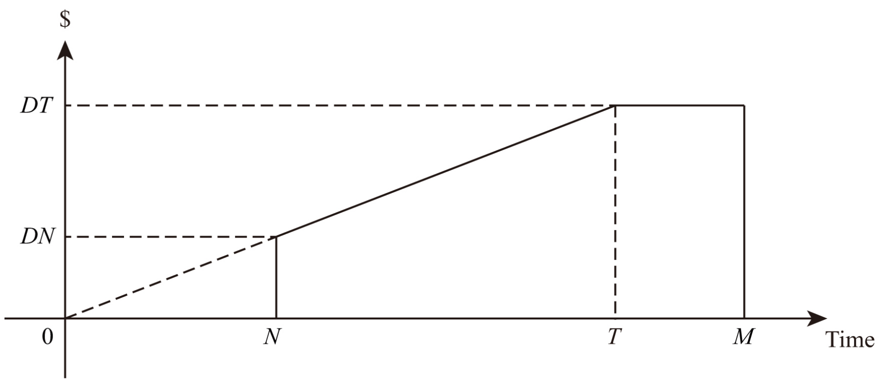

Figure 7. Annual interest earned when $N \le T \le M$.

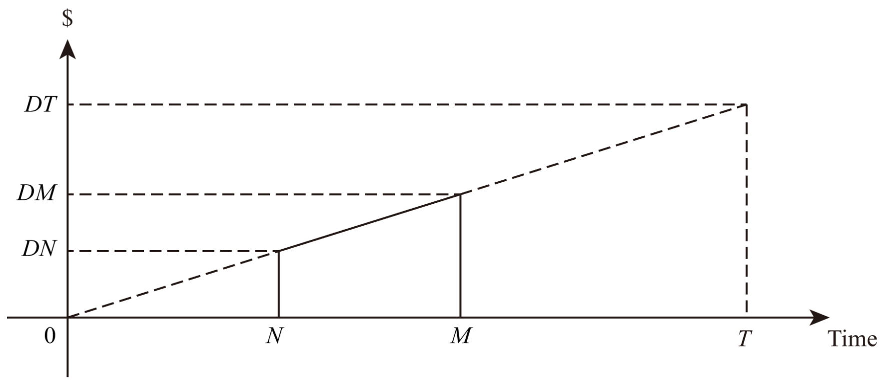

Figure 8. Annual interest earned when $N \le M \le T$.

3.6. Annual Total Relevant Cost

From the above arguments, the annual total relevant cost for the manufacturer can be expressed as $TRC(T)$ = annual ordering cost + annual purchasing cost + annual holding cost + annual interest payable − annual interest earned.

Because storage capacity $W = DT\rho$, there are four cases arise:

1) $\frac{W}{D\rho} < N$,

2) $N \le \frac{W}{D\rho} < M$,

3) $M \le \frac{W}{D\rho} < \frac{PM}{D}$,

4) $\frac{PM}{D} \le \frac{W}{D\rho}$.

Case 1. $\frac{W}{D\rho} < N$.

According to Equations (1)-(18), *the total relevant cost* $TRC(T)$ *can be expressed by*

$$TRC(T) = \begin{cases} TRC_1(T), & \text{if } 0 < T < \frac{W}{D\rho} & (19a) \\ TRC_2(T), & \text{if } \frac{W}{D\rho} \le T < N & (19b) \\ TRC_3(T), & \text{if } N \le T < M & (19c) \\ TRC_4(T), & \text{if } M \le T < \frac{PM}{D} & (19d) \\ TRC_5(T), & \text{if } \frac{PM}{D} \le T & (19e) \end{cases}$$

where

$$TRC_1(T) = \frac{A}{T} + \frac{cP}{\theta T}\left(e^{\theta \frac{D}{P} T} - 1\right) + \frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta \frac{D}{P} T} - 1\right) - \frac{D}{P}T\right] + \frac{DTh_o\rho}{2} - sI_e D(M - N), \quad (20)$$

$$TRC_2(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+Wh_o-\frac{W^2h_o}{2DT\rho}+\frac{h_r(DT\rho-W)^2}{2DT\rho}-sI_eD(M-N), \tag{21}$$

$$TRC_3(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]+Wh_o$$
$$-\frac{W^2h_o}{2DT\rho}+\frac{h_r(DT\rho-W)^2}{2DT\rho}-\frac{sI_eD(2MT-N^2-T^2)}{2T}, \tag{22}$$

$$TRC_4(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+Wh_o-\frac{W^2h_o}{2DT\rho}+\frac{h_r(DT\rho-W)^2}{2DT\rho} \tag{23}$$
$$+\frac{cI_pD(T-M)^2}{2T}-\frac{sI_eD(M^2-N^2)}{2T},$$

$$TRC_5(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+Wh_o-\frac{W^2h_o}{2DT\rho}+\frac{h_r(DT\rho-W)^2}{2DT\rho} \tag{24}$$
$$+\frac{cI_p\rho(DT^2-PM^2)}{2T}-\frac{sI_eD(M^2-N^2)}{2T}.$$

$TRC(T)$ is continuous at T, $T\in[0,\infty)$ because of $TRC_1\left(\frac{W}{D\rho}\right)=TRC_2\left(\frac{W}{D\rho}\right)$, $TRC_2(N)=TRC_3(N)$, $TRC_3(M)=TRC_4(M)$, *and* $TRC_4\left(\frac{PM}{D}\right)=TRC_5\left(\frac{PM}{D}\right)$.

Case 2. $N\le\frac{W}{D\rho}<M$.

According to Equations (1)-(18), *the total relevant cost* $TRC(T)$ *can be expressed by*

$$TRC(T)=\begin{cases}TRC_1(T), & \text{if } 0<T<N & (25a)\\ TRC_6(T), & \text{if } N\le T<\frac{W}{D\rho} & (25b)\\ TRC_3(T), & \text{if } \frac{W}{D\rho}\le T<M & (25c)\\ TRC_4(T), & \text{if } M\le T<\frac{PM}{D} & (25d)\\ TRC_5(T), & \text{if } \frac{PM}{D}\le T & (25e)\end{cases}$$

where

$$TRC_6(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+\frac{DTh_o\rho}{2}-\frac{sI_eD\left(2MT-N^2-T^2\right)}{2T}. \tag{26}$$

$TRC(T)$ is continuous at T, $T\in[0,\infty)$ because of $TRC_1(N)=TRC_6(N)$, $TRC_6\left(\frac{W}{D\rho}\right)=TRC_3\left(\frac{W}{D\rho}\right)$, $TRC_3(M)=TRC_4(M)$, *and* $TRC_4\left(\frac{PM}{D}\right)=TRC_5\left(\frac{PM}{D}\right)$.

Case 3. $M\le\frac{W}{D\rho}<\frac{PM}{D}$.

According to Equations (1)-(18), *the total relevant cost* $TRC(T)$ *can be expressed by*

$$TRC(T)=\begin{cases}TRC_1(T), & \text{if } 0<T<N & (27a)\\ TRC_6(T), & \text{if } N\le T<M & (27b)\\ TRC_7(T), & \text{if } M\le T<\frac{W}{D\rho} & (27c)\\ TRC_4(T), & \text{if } \frac{W}{D\rho}\le T<\frac{PM}{D} & (27d)\\ TRC_5(T), & \text{if } \frac{PM}{D}\le T & (27e)\end{cases}$$

where

$$TRC_7(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(e^{\theta\frac{D}{P}T}-1\right)+\frac{h_mP}{\theta T}\left[\frac{1}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+\frac{DTh_o\rho}{2}+\frac{cI_pD(T-M)^2}{2T}-\frac{sI_eD\left(M^2-N^2\right)}{2T}. \tag{28}$$

$TRC(T)$ is continuous at T, $T\in[0,\infty)$ because of $TRC_1(N)=TRC_6(N)$, $TRC_6(M)=TRC_7(M)$, $TRC_7\left(\frac{W}{D\rho}\right)=TRC_4\left(\frac{W}{D\rho}\right)$, and $TRC_4\left(\frac{PM}{D}\right)=TRC_5\left(\frac{PM}{D}\right)$.

Case 4. $\frac{PM}{D}\le\frac{W}{D\rho}$.

According to Equations (1)-(18), *the total relevant cost* $TRC(T)$ *can be expressed by*

$$TRC(T)=\begin{cases}TRC_1(T), & \text{if } 0<T<N & (29a)\\ TRC_6(T), & \text{if } N\le T<M & (29b)\\ TRC_7(T), & \text{if } M\le T<\frac{PM}{D} & (29c)\\ TRC_8(T), & \text{if } \frac{PM}{D}\le T<\frac{W}{D\rho} & (29d)\\ TRC_5(T), & \text{if } \frac{W}{D\rho}\le T & (29e)\end{cases}$$

where

$$TRC_8(T)=\frac{A}{T}+\frac{cP}{\theta T}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)+\frac{h_m P}{\theta T}\left[\frac{1}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-\frac{D}{P}T\right]$$
$$+\frac{DTh_o\rho}{2}+\frac{cI_p\rho\left(DT^2-PM^2\right)}{2T}-\frac{sI_eD\left(M^2-N^2\right)}{2T}. \tag{30}$$

$TRC(T)$ is continuous at T, $T\in[0,\infty)$ because of $TRC_1(N)=TRC_6(N)$, $TRC_6(M)=TRC_7(M)$, $TRC_7\left(\frac{PM}{D}\right)=TRC_8\left(\frac{PM}{D}\right)$, *and* $TRC_8\left(\frac{W}{D\rho}\right)=TRC_5\left(\frac{W}{D\rho}\right)$.

For convenience, all $TRC_i(T)(i=1\text{-}8)$ are defined on $T>0$.

4. The Convexity of $TRC_i(T)(i=1\text{-}8)$

Equations (20)-(24), (26), (28), and (30) yield the first order and second-order derivatives as follows.

$$TRC_1'(T)=\frac{1}{T^2}\left\{-A-\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]+\frac{Dh_o\rho}{2}T^2\right\}, \tag{31}$$

$$TRC_1'(T)=\frac{1}{T^3}\left\{2A+2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]+\theta\frac{D^2}{P}T^2\mathrm{e}^{\theta\frac{D}{P}T}\right\}, \tag{32}$$

$$TRC_2'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+\frac{W^2\left(h_o-h_r\right)}{D\rho}+Dh_r\rho T^2\right\}, \tag{33}$$

$$TRC_2''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-2DT\mathrm{e}^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2\mathrm{e}^{\theta\frac{D}{P}T}\right].\right.$$
$$\left.+\frac{W^2\left(h_r-h_o\right)}{D\rho}\right\}, \tag{34}$$

$$TRC_3'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+\frac{W^2\left(h_o-h_r\right)}{D\rho}-sI_eDN^2+D\left(h_r\rho+sI_e\right)T^2\right\}, \tag{35}$$

$$TRC_3''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-2DT\mathrm{e}^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+\frac{W^2\left(h_r-h_o\right)}{D\rho}+sI_eDN^2\right\}, \tag{36}$$

$$TRC_4'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+\frac{W^2(h_o-h_r)}{D\rho}+DM^2(sI_e-cI_p)-sI_eDN^2+D(h_r\rho+sI_e)T^2\right\}, \tag{37}$$

$$TRC_4''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-2DT\mathrm{e}^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+\frac{W^2(h_r-h_o)}{D\rho}+DM^2(cI_p-sI_e)+sI_eDN^2\right\}, \tag{38}$$

$$TRC_5'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$+\frac{W^2(h_o-h_r)}{D\rho}+DM^2(sI_e-cI_p)-sI_eDN^2 \tag{39}$$
$$\left.+cI_pPM^2+D\rho(h_r+cI_p)T^2\right\},$$

$$TRC_5''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-2DT\mathrm{e}^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+\frac{W^2(h_r-h_o)}{D\rho}+DM^2(cI_p-sI_e)+sI_eDN^2-cI_pPM^2\right\}, \tag{40}$$

$$TRC_6'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.-sI_eDN^2+D(h_r\rho+sI_e)T^2\right\}, \tag{41}$$

$$TRC_6''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-2DT\mathrm{e}^{\theta\frac{D}{P}T}\right.\right.$$
$$\left.\left.+\theta\frac{D^2}{P}T^2\mathrm{e}^{\theta\frac{D}{P}T}\right]+sI_eDN^2\right\}, \tag{42}$$

$$TRC_7'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-DT\mathrm{e}^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+DM^2(sI_e-cI_p)-sI_eDN^2+D(h_r\rho+cI_p)T^2\right\}, \tag{43}$$

$$TRC_7''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}T}-1\right)-2DT\mathrm{e}^{\theta\frac{D}{P}T}\right.\right.$$
$$\left.\left.+\theta\frac{D^2}{P}T^2\mathrm{e}^{\theta\frac{D}{P}T}\right]+DM^2(cI_p-sI_e)+sI_eDN^2\right\}, \tag{44}$$

$$TRC_8'(T)=\frac{1}{2T^2}\left\{-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-DTe^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+DM^2\left(sI_e-cI_p\right)-sI_eDN^2+cI_pPM^2+D\rho\left(h_o+cI_p\right)T^2\right\}, \tag{45}$$

and

$$TRC_8''(T)=\frac{1}{T^3}\left\{2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]\right.$$
$$\left.+DM^2\left(cI_p-sI_e\right)+sI_eDN^2-cI_pPM^2\right\}. \tag{46}$$

Let

$$G_1=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right], \tag{47}$$

$$G_2=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]$$
$$+\frac{W^2\left(h_r-h_o\right)}{D\rho}, \tag{48}$$

$$G_3=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]$$
$$+\frac{W^2\left(h_r-h_o\right)}{D\rho}+sI_eDN^2, \tag{49}$$

$$G_4=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]$$
$$+\frac{W^2\left(h_r-h_o\right)}{D\rho}+DM^2\left(cI_p-sI_e\right)+sI_eDN^2, \tag{50}$$

$$G_5=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]$$
$$+\frac{W^2\left(h_r-h_o\right)}{D\rho}+DM^2\left(cI_p-sI_e\right)+sI_eDN^2-cI_pPM^2, \tag{51}$$

$$G_6=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]+sI_eDN^2, \tag{52}$$

$$G_7=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]$$
$$+DM^2\left(cI_p-sI_e\right)+sI_eDN^2, \tag{53}$$

and

$$G_8=2A+\left(c+\frac{h_m}{\theta}\right)\left[2\frac{P}{\theta}\left(e^{\theta\frac{D}{P}T}-1\right)-2DTe^{\theta\frac{D}{P}T}+\theta\frac{D^2}{P}T^2e^{\theta\frac{D}{P}T}\right]$$
$$+DM^2\left(cI_p-sI_e\right)+sI_eDN^2-cI_pPM^2. \tag{54}$$

Equations (47)-(54) imply

$$G_4 > G_5 > G_8, \tag{55}$$

$$G_4 > G_7 > G_8, \tag{56}$$

$$G_3 > G_2 > G_1, \tag{57}$$

and

$$G_3 > G_6 > G_1. \tag{58}$$

Equations (31)-(46) reveal the following results.

Lemma 1. *$TRC_i'(T)$ is increasing on $T > 0$ if $G_i > 0$ for all $i = 1\text{-}8$. That is, $TRC_i(T)$ is convex on $T > 0$ if $G_i > 0$.*

$$TRC_i'(T) = \begin{cases} < 0, & \text{if } 0 < T < T_i^* \quad (59a) \\ = 0, & \text{if } T = T_i^* \quad (59b) \\ > 0, & \text{if } T_i^* < T < \infty \quad (59c) \end{cases}$$

Equations (59a)-(59c) *imply that* $TRC_i(T)$ *is decreasing on* $(0, T_i^*]$ *and increasing on* $[T_i^*, \infty)$ *for all* $i = 1\text{-}8$. *Solving optimal cycle* $T_i^*(T)(i = 1\text{-}8)$ *by* $TRC_i'(T) = 0(i = 1\text{-}8)$.

5. The Values of δ_{ij} under Different Cases

Case 1. $\dfrac{W}{D\rho} < N$.

Equations (31), (33), (35), (37), *and* (39) *yield*

$$TRC_1'\left(\frac{W}{D\rho}\right) = TRC_2'\left(\frac{W}{D\rho}\right) = \frac{\Delta_{12}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{60}$$

$$TRC_2'(N) = TRC_3'(N) = \frac{\Delta_{23}}{2N^2}, \tag{61}$$

$$TRC_3'(M) = TRC_4'(M) = \frac{\Delta_{34}}{2M^2}, \tag{62}$$

$$TRC_4'\left(\frac{PM}{D}\right) = TRC_5'\left(\frac{PM}{D}\right) = \frac{\Delta_{45}}{2\left(\frac{PM}{D}\right)^2}, \tag{63}$$

where

$$\begin{aligned} \Delta_{12} = {} & -2A - 2\left(c + \frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)} - 1\right) - D\left(\frac{W}{D\rho}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] \\ & + Dh_o\rho\left(\frac{W}{D\rho}\right)^2, \end{aligned} \tag{64}$$

$$\begin{aligned} \Delta_{23} = {} & -2A - 2\left(c + \frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}N} - 1\right) - DN\mathrm{e}^{\theta\frac{D}{P}N}\right] \\ & + \frac{W^2(h_o - h_r)}{D\rho} + Dh_r\rho N^2, \end{aligned} \tag{65}$$

$$\Delta_{34}=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}M}-1\right)-DM\mathrm{e}^{\theta\frac{D}{P}M}\right]+\frac{W^2\left(h_o-h_r\right)}{D\rho}-sI_eDN^2+D\left(h_r\rho+sI_e\right)M^2, \tag{66}$$

$$\Delta_{45}=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}-1\right)-D\left(\frac{PM}{D}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}\right]+\frac{W^2\left(h_o-h_r\right)}{D\rho}+DM^2\left(sI_e-cI_p\right)-sI_eDN^2+D\left(h_r\rho+sI_e\right)\left(\frac{PM}{D}\right)^2. \tag{67}$$

Equations (64)-(67) *imply*

$$\Delta_{12}<\Delta_{23}<\Delta_{34}<\Delta_{45}. \tag{68}$$

Case 2. $N\le\frac{W}{D\rho}<M$.

Equations (31), (35), (37), (39), *and* (41) *yield*

$$TRC_1'\left(N\right)=TRC_6'\left(N\right)=\frac{\Delta_{16}}{2N^2}, \tag{69}$$

$$TRC_6'\left(\frac{W}{D\rho}\right)=TRC_3'\left(\frac{W}{D\rho}\right)=\frac{\Delta_{63}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{70}$$

$$TRC_3'\left(M\right)=TRC_4'\left(M\right)=\frac{\Delta_{34}}{2M^2}, \tag{71}$$

$$TRC_4'\left(\frac{PM}{D}\right)=TRC_5'\left(\frac{PM}{D}\right)=\frac{\Delta_{45}}{2\left(\frac{PM}{D}\right)^2}, \tag{72}$$

where

$$\Delta_{16}=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}N}-1\right)-DN\mathrm{e}^{\theta\frac{D}{P}N}\right]+Dh_o\rho N^2, \tag{73}$$

$$\Delta_{63}=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right]-sI_eDN^2+D\left(h_r\rho+sI_e\right)\left(\frac{W}{D\rho}\right)^2. \tag{74}$$

Equations (66), (67), (73), *and* (74) *imply*

$$\Delta_{16}\le\Delta_{63}<\Delta_{34}<\Delta_{45}. \tag{75}$$

Case 3. $M\le\frac{W}{D\rho}<\frac{PM}{D}$.

Equations (31), (37), (39), (41), *and* (43) *yield*

$$TRC_1'\left(N\right)=TRC_6'\left(N\right)=\frac{\Delta_{16}}{2N^2}, \tag{76}$$

$$TRC_6'\left(M\right)=TRC_7'\left(M\right)=\frac{\Delta_{67}}{2M^2}, \tag{77}$$

$$TRC_7'\left(\frac{W}{D\rho}\right)=TRC_4'\left(\frac{W}{D\rho}\right)=\frac{\Delta_{74}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{78}$$

$$TRC_4'\left(\frac{PM}{D}\right)=TRC_5'\left(\frac{PM}{D}\right)=\frac{\Delta_{45}}{2\left(\frac{PM}{D}\right)^2}, \tag{79}$$

where

$$\begin{aligned}\Delta_{67}&=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}M}-1\right)-DM\mathrm{e}^{\theta\frac{D}{P}M}\right]\\&\quad-sI_eDN^2+D\left(h_r\rho+sI_e\right)M^2,\end{aligned} \tag{80}$$

$$\begin{aligned}\Delta_{74}&=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right]\\&\quad+DM^2\left(sI_e-cI_p\right)-sI_eDN^2+D\left(h_r\rho+cI_p\right)\left(\frac{W}{D\rho}\right)^2.\end{aligned} \tag{81}$$

Equations (67), (73), (79), *and* (80) *imply*

$$\Delta_{16}\le\Delta_{67}\le\Delta_{74}<\Delta_{45}. \tag{82}$$

Case 4. $\frac{PM}{D}\le\frac{W}{D\rho}$.

Equations (31), (39), (41), (43), *and* (45) *yield*

$$TRC_1'(N)=TRC_6'(N)=\frac{\Delta_{16}}{2N^2}, \tag{83}$$

$$TRC_6'(M)=TRC_7'(M)=\frac{\Delta_{67}}{2M^2}, \tag{84}$$

$$TRC_7'\left(\frac{PM}{D}\right)=TRC_8'\left(\frac{PM}{D}\right)=\frac{\Delta_{78}}{2\left(\frac{PM}{D}\right)^2}, \tag{85}$$

$$TRC_8'\left(\frac{W}{D\rho}\right)=TRC_5'\left(\frac{W}{D\rho}\right)=\frac{\Delta_{45}}{2\left(\frac{W}{D\rho}\right)^2}, \tag{86}$$

where

$$\begin{aligned}\Delta_{78}&=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}-1\right)-D\left(\frac{PM}{D}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}\right]\\&\quad+DM^2\left(sI_e-cI_p\right)-sI_eDN^2+D\left(h_r\rho+cI_p\right)\left(\frac{PM}{D}\right)^2,\end{aligned} \tag{87}$$

$$\begin{aligned}\Delta_{85}&=-2A-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right]\\&\quad+DM^2\left(sI_e-cI_p\right)-sI_eDN^2+cI_pPM^2+D\rho\left(h_o+cI_p\right)\left(\frac{W}{D\rho}\right)^2.\end{aligned} \tag{88}$$

Equations (73), (80), (87), *and* (88) *imply*

$$\Delta_{16} \le \Delta_{67} \le \Delta_{78} \le \Delta_{85}. \tag{89}$$

Based on the above arguments, the following results holds.

Lemma 2

1) If $\Delta_{12} \le 0$, then

a) $G_1 > 0$ and $G_2 > 0$,

b) T_1^* and T_2^* exist,

c) $TRC_1(T)$ and $TRC_2(T)$ are convex on $T > 0$.

2) If $\Delta_{23} \le 0$, then

a) $G_2 > 0$ and $G_3 > 0$,

b) T_2^* and T_3^* exist,

c) $TRC_2(T)$ and $TRC_3(T)$ are convex on $T > 0$.

3) If $\Delta_{16} \le 0$, then

a) $G_1 > 0$ and $G_6 > 0$,

b) T_1^* and T_6^* exist,

c) $TRC_1(T)$ and $TRC_6(T)$ are convex on $T > 0$.

4) If $\Delta_{63} \le 0$, then

a) $G_3 > 0$ and $G_6 > 0$,

b) T_3^* and T_6^* exist,

c) $TRC_3(T)$ and $TRC_6(T)$ are convex on $T > 0$.

5) If $\Delta_{85} \le 0$, then

a) $G_5 > 0$ and $G_8 > 0$,

b) T_5^* and T_8^* exist,

c) $TRC_5(T)$ and $TRC_8(T)$ are convex on $T > 0$.

6) If $\Delta_{45} \le 0$, then

a) $G_4 > 0$ and $G_5 > 0$,

b) T_4^* and T_5^* exist,

c) $TRC_4(T)$ and $TRC_5(T)$ are convex on $T > 0$.

7) If $\Delta_{78} \le 0$, then

a) $G_7 > 0$ and $G_8 > 0$,

b) T_7^* and T_8^* exist,

c) $TRC_7(T)$ and $TRC_8(T)$ are convex on $T > 0$.

8) If $\Delta_{74} \le 0$, then

a) $G_4 > 0$ and $G_7 > 0$,

b) T_4^* and T_7^* exist,

c) $TRC_4(T)$ and $TRC_7(T)$ are convex on $T > 0$.

Proof. 1. a) If $\Delta_{12} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right]+Dh_o\rho\left(\frac{W}{D\rho}\right)^2. \tag{90}$$

Equation (90) implies

$$G_1 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+h_o\rho\right] > 0. \tag{91}$$

$$G_2 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+h_o\rho\right]+\frac{W^2\left(h_r-h_o\right)}{D\rho}>0. \tag{92}$$

Equations (54), (91), and (92) demonstrate $G_2 > G_1 > 0$.

b) Lemma 1 implies that T_1^* and T_2^* exist.

c) Equations (32), (34), and lemma 1 imply that $TRC_1(T)$ and $TRC_2(T)$ are convex on $T>0$.

2. a) If $\Delta_{23} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}N}-1\right)-DN\mathrm{e}^{\theta\frac{D}{P}N}\right]+\frac{W^2\left(h_o-h_r\right)}{D\rho}+Dh_r\rho N^2. \tag{93}$$

Equation (93) implies

$$G_2 \ge DN^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}N}+h_r\rho\right]>0. \tag{94}$$

$$G_3 \ge DN^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}N}+\left(h_r\rho+sI_e\right)\right]>0. \tag{95}$$

Equations (54), (94), and (95) demonstrate $G_3 > G_2 > 0$.

b) Lemma 1 implies that T_2^* and T_3^* exist.

c) Equations (34), (36), and lemma 1 imply that $TRC_2(T)$ and $TRC_3(T)$ are convex on $T>0$.

3. a) If $\Delta_{16} \le 0$, then

$$2A \ge -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}N}-1\right)-DN\mathrm{e}^{\theta\frac{D}{P}N}\right]+Dh_o\rho N^2. \tag{96}$$

Equation (96) implies

$$G_1 \ge DN^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}N}+h_o\rho\right]>0. \tag{97}$$

$$G_6 \ge DN^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}N}+\left(h_o\rho+sI_e\right)\right]>0. \tag{98}$$

Equations (55), (97), and (98) demonstrate $G_6 > G_1 > 0$.

b) Lemma 1 implies that T_1^* and T_6^* exist.

c) Equations (32), (42), and lemma 1 imply that $TRC_1(T)$ and $TRC_6(T)$ are convex on $T>0$.

4. a) If $\Delta_{63} \le 0$, then

$$\begin{aligned}2A \ge &-2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right]\\&-sI_eDN^2+D\left(h_r\rho+sI_e\right)\left(\frac{W}{D\rho}\right)^2.\end{aligned} \tag{99}$$

Equation (99) implies

$$G_3 \ge D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+\left(h_o\rho+sI_e\right)\right]+\frac{W^2\left(h_r-h_o\right)}{D\rho}>0. \tag{100}$$

$$G_6 \geq D\left(\frac{W}{D\rho}\right)^2 \left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)} + \left(h_o\rho + sI_e\right)\right] > 0. \tag{101}$$

Equations (55), (100), and (101) demonstrate $G_3 > G_6 > 0$.

b) Lemma 1 implies that T_3^* and T_6^* exist.

c) Equations (36), (42), and lemma 1 imply that $TRC_3(T)$ and $TRC_6(T)$ are convex on $T > 0$.

5. a) If $\Delta_{85} \leq 0$, then

$$\begin{aligned} 2A \geq & -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right) - D\left(\frac{W}{D\rho}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right] \\ & + DM^2\left(sI_e - cI_p\right) - sI_eDN^2 + cI_pPM^2 + D\rho\left(h_o + cI_p\right)\left(\frac{W}{D\rho}\right)^2. \end{aligned} \tag{102}$$

Equation (102) implies

$$G_5 \geq D\left(\frac{W}{D\rho}\right)^2 \left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)} + \rho\left(h_o + cI_p\right)\right] + \frac{W^2\left(h_r - h_o\right)}{D\rho} > 0. \tag{103}$$

$$G_8 \geq D\left(\frac{W}{D\rho}\right)^2 \left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)} + \rho\left(h_o + cI_p\right)\right] > 0. \tag{104}$$

Equations (51), (103), and (104) demonstrate $G_5 > G_8 > 0$.

b) Lemma 1 implies that T_5^* and T_8^* exist.

c) Equations (40), (46), and lemma 1 imply that $TRC_5(T)$ and $TRC_8(T)$ are convex on $T > 0$.

6. a) If $\Delta_{45} \leq 0$, then

$$\begin{aligned} 2A \geq & -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}-1\right) - D\left(\frac{PM}{D}\right)\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}\right] \\ & + \frac{W^2\left(h_o - h_r\right)}{D\rho} + DM^2\left(sI_e - cI_p\right) - sI_eDN^2 + D\left(h_r\rho + sI_e\right)\left(\frac{PM}{D}\right)^2. \end{aligned} \tag{105}$$

Equation (105) implies

$$G_4 \geq D\left(\frac{PM}{D}\right)^2 \left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)} + \left(h_r\rho + cI_p\right)\right] > 0. \tag{106}$$

$$G_5 \geq D\left(\frac{PM}{D}\right)^2 \left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}\mathrm{e}^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)} + \left(h_r\rho + cI_p\right)\right] - cI_pPM^2 > 0. \tag{107}$$

Equations (51), (106), and (107) demonstrate $G_4 > G_5 > 0$.

b) Lemma 1 implies that T_4^* and T_5^* exist.

c) Equations (38), (40), and lemma 1 imply that $TRC_4(T)$ and $TRC_5(T)$ are convex on $T > 0$.

7. a) If $\Delta_{78} \leq 0$, then

$$2A \geq -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}-1\right)-D\left(\frac{PM}{D}\right)e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}\right]$$
$$+DM^2\left(sI_e-cI_p\right)-sI_eDN^2+D\left(h_r\rho+cI_p\right)\left(\frac{PM}{D}\right)^2. \tag{108}$$

Equation (108) implies

$$G_7 \geq D\left(\frac{PM}{D}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}+\left(h_o\rho+cI_p\right)\right]>0. \tag{109}$$

$$G_8 \geq D\left(\frac{PM}{D}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{PM}{D}\right)}+\left(h_o\rho+cI_p\right)\right]-cI_pPM^2>0. \tag{110}$$

Equations (52), (109), and (110) demonstrate $G_7 > G_8 > 0$.

b) Lemma 1 implies that T_7^* and T_8^* exist.

c) Equations (44), (46), and lemma 1 imply that $TRC_7(T)$ and $TRC_8(T)$ are convex on $T>0$.

8. a) If $\Delta_{74} \leq 0$, then

$$2A \geq -2\left(c+\frac{h_m}{\theta}\right)\left[\frac{P}{\theta}\left(e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}-1\right)-D\left(\frac{W}{D\rho}\right)e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}\right]$$
$$+DM^2\left(sI_e-cI_p\right)-sI_eDN^2+D\left(h_r\rho+cI_p\right)\left(\frac{W}{D\rho}\right)^2. \tag{111}$$

Equation (111) implies

$$G_4 \geq D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+\left(h_o\rho+cI_p\right)\right]+\frac{W^2\left(h_r-h_o\right)}{D\rho}>0. \tag{112}$$

$$G_7 \geq D\left(\frac{W}{D\rho}\right)^2\left[\left(c+\frac{h_m}{\theta}\right)\theta\frac{D}{P}e^{\theta\frac{D}{P}\left(\frac{W}{D\rho}\right)}+\left(h_o\rho+cI_p\right)\right]>0. \tag{113}$$

Equations (52), (112), and (113) demonstrate $G_4 > G_7 > 0$.

b) Lemma 1 implies that T_4^* and T_7^* exist.

c) Equations (38), (44), and lemma 1 imply that $TRC_4(T)$ and $TRC_7(T)$ are convex on $T>0$.

Incorporate the above arguments, we have completed the proof of Lemma 2. □

6. The Determination of the Optimal Cycle Time T^* of $TRC(T)$

Theorem 1. *Suppose* $\frac{W}{D\rho} < N$.

1) *If* $0 < \Delta_{12}$, *then* $TRC(T^*) = TRC_1(T_1^*)$ *and* $T^* = T_1^*$.

2) *If* $\Delta_{12} \leq 0 < \Delta_{23}$, *then* $TRC(T^*) = TRC_2(T_2^*)$ *and* $T^* = T_2^*$.

3) *If* $\Delta_{23} \leq 0 < \Delta_{34}$, *then* $TRC(T^*) = TRC_3(T_3^*)$ *and* $T^* = T_3^*$.

4) *If* $\Delta_{34} \leq 0 < \Delta_{45}$, *then* $TRC(T^*) = TRC_4(T_4^*)$ *and* $T^* = T_4^*$.

5) *If* $\Delta_{45} \leq 0$, *then* $TRC(T^*) = TRC_5(T_5^*)$ *and* $T^* = T_5^*$.

Proof. 1) If $0<\Delta_{12}$, then $0<\Delta_{12}<\Delta_{23}<\Delta_{34}<\Delta_{45}$. So, lemmas 1, 2, and Equations (59a)-(59c) imply

a) $TRC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $\left[T_1^*,\frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is increasing on $\left[\frac{W}{D\rho},N\right]$.

c) $TRC_3(T)$ is increasing on $[N,M]$.

d) $TRC_4(T)$ is increasing on $\left[M,\frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D},\infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (19a)-(19e) and 1.1 - 1.5 reveal that $TRC(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,\infty)$. Hence, $T^*=T_1^*$ and $TRC(T^*)=TRC_1(T_1^*)$.

2) If $\Delta_{12}\le 0<\Delta_{23}$, then $\Delta_{12}\le 0<\Delta_{23}<\Delta_{34}<\Delta_{45}$. So, lemmas 1, 2, and Equations (59a)-(59c) imply

a) $TRC_1(T)$ is decreasing on $\left[0,\frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho},T_2^*\right]$ and increasing on $[T_2^*,N]$.

c) $TRC_3(T)$ is increasing on $[N,M]$.

d) $TRC_4(T)$ is increasing on $\left[M,\frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D},\infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (19a)-(19e) and 2.1-2.5 reveal that $TRC(T)$ is decreasing on $(0,T_2^*]$ and increasing on $[T_2^*,\infty)$. Hence, $T^*=T_2^*$ and $TRC(T^*)=TRC_2(T_2^*)$.

3) If $\Delta_{23}\le 0<\Delta_{34}$, then $\Delta_{12}<\Delta_{23}\le 0<\Delta_{34}<\Delta_{45}$. So, lemmas 1, 2, and Equations (59a)-(59c) imply

a) $TRC_1(T)$ is decreasing on $\left[0,\frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho},N\right]$.

c) $TRC_3(T)$ is decreasing on $[N,T_3^*]$ and increasing on $[T_3^*,M]$.

d) $TRC_4(T)$ is increasing on $\left[M,\frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D},\infty\right)$.

Since $TRC(T)$ is continuous on $T>0$, Equations (19a)-(19e) and 3.1 - 3.5 reveal that $TRC(T)$ is decreasing on $(0,T_3^*]$ and increasing on $[T_3^*,\infty)$. Hence, $T^*=T_3^*$ and $TRC(T^*)=TRC_3(T_3^*)$.

4) If $\Delta_{34}\le 0<\Delta_{45}$, then $\Delta_{12}<\Delta_{23}<\Delta_{34}\le 0<\Delta_{45}$. So, lemmas 1, 2, and Equations (59a)-(59c) imply

a) $TRC_1(T)$ is decreasing on $\left[0, \frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, N\right]$.

c) $TRC_3(T)$ is decreasing on $[N, M]$.

d) $TRC_4(T)$ is decreasing on $\left[M, T_4^*\right]$ and increasing on $\left[T_4^*, \frac{PM}{D}\right]$.

e) $TRC_5(T)$ is increasing on $\left[\frac{PM}{D}, \infty\right)$.

Since $TRC(T)$ is continuous on $T > 0$, Equations (19a)-(19e) and 4.1-4.5 reveal that $TRC(T)$ is decreasing on $\left(0, T_4^*\right]$ and increasing on $\left[T_4^*, \infty\right)$. Hence, $T^* = T_4^*$ and $TRC\left(T^*\right) = TRC_4\left(T_4^*\right)$.

5) If $\Delta_{45} \le 0$, then $\Delta_{12} < \Delta_{23} < \Delta_{34} < \Delta_{45} \le 0$. So, lemmas 1, 2, and Equations (59a)-(59c) imply

a) $TRC_1(T)$ is decreasing on $\left[0, \frac{W}{D\rho}\right]$.

b) $TRC_2(T)$ is decreasing on $\left[\frac{W}{D\rho}, N\right]$.

c) $TRC_3(T)$ is decreasing on $[N, M]$.

d) $TRC_4(T)$ is decreasing on $\left[M, T_4^*\right]$.

e) $TRC_5(T)$ is decreasing on $\left[\frac{PM}{D}, T_5^*\right]$ and increasing on $\left[T_5^*, \infty\right)$.

Since $TRC(T)$ is continuous on $T > 0$, Equations (19a)-(19e) and 5.1 - 5.5 reveal that $TRC(T)$ is decreasing on $\left(0, T_5^*\right]$ and increasing on $\left[T_5^*, \infty\right)$. Hence, $T^* = T_5^*$ and $TRC\left(T^*\right) = TRC_5\left(T_5^*\right)$.

Incorporating all argument above arguments, we have completed the proof of theorem 1. □

Applying lemmas 1, 2, and Equations (25a)-(25e), the following results hold.

Theorem 2. *Suppose* $N \le \frac{W}{D\rho} < M$.

1) *If* $0 < \Delta_{16}$, *then* $TRC\left(T^*\right) = TRC_1\left(T_1^*\right)$ *and* $T^* = T_1^*$.

2) *If* $\Delta_{16} \le 0 < \Delta_{63}$, *then* $TRC\left(T^*\right) = TRC_6\left(T_6^*\right)$ *and* $T^* = T_6^*$.

3) *If* $\Delta_{63} \le 0 < \Delta_{34}$, *then* $TRC\left(T^*\right) = TRC_3\left(T_3^*\right)$ *and* $T^* = T_3^*$.

4) *If* $\Delta_{34} \le 0 < \Delta_{45}$, *then* $TRC\left(T^*\right) = TRC_4\left(T_4^*\right)$ *and* $T^* = T_4^*$.

5) *If* $\Delta_{45} \le 0$, *then* $TRC\left(T^*\right) = TRC_5\left(T_5^*\right)$ *and* $T^* = T_5^*$.

Applying lemmas 1, 2, and Equations (27a)-(27e), the following results hold.

Theorem 3. *Suppose* $M \le \frac{W}{D\rho} < \frac{PM}{D}$.

1) *If* $0 < \Delta_{16}$, *then* $TRC\left(T^*\right) = TRC_1\left(T_1^*\right)$ *and* $T^* = T_1^*$.

2) *If* $\Delta_{16} \le 0 < \Delta_{67}$, *then* $TRC\left(T^*\right) = TRC_6\left(T_6^*\right)$ *and* $T^* = T_6^*$.

3) *If* $\Delta_{67} \le 0 < \Delta_{74}$, *then* $TRC\left(T^*\right) = TRC_7\left(T_7^*\right)$ *and* $T^* = T_7^*$.

4) *If* $\Delta_{74} \le 0 < \Delta_{45}$, *then* $TRC\left(T^*\right) = TRC_4\left(T_4^*\right)$ *and* $T^* = T_4^*$.

5) *If* $\Delta_{45} \le 0$, *then* $TRC\left(T^*\right) = TRC_5\left(T_5^*\right)$ *and* $T^* = T_5^*$.

Applying lemmas 1, 2, and Equations (29a)-(29e), the following results hold.

Theorem 4. *Suppose* $\frac{PM}{D} \le \frac{W}{D\rho}$.

1) *If* $0 < \Delta_{16}$, *then* $TRC\left(T^*\right) = TRC_1\left(T_1^*\right)$ *and* $T^* = T_1^*$.

2) *If* $\Delta_{16} \le 0 < \Delta_{67}$, *then* $TRC\left(T^*\right) = TRC_6\left(T_6^*\right)$ *and* $T^* = T_6^*$.

3) *If* $\Delta_{67} \le 0 < \Delta_{78}$, *then* $TRC\left(T^*\right) = TRC_7\left(T_7^*\right)$ *and* $T^* = T_7^*$.

4) *If* $\Delta_{78} \le 0 < \Delta_{85}$, *then* $TRC\left(T^*\right) = TRC_8\left(T_8^*\right)$ *and* $T^* = T_8^*$.

5) *If* $\Delta_{85} \le 0$, *then* $TRC\left(T^*\right) = TRC_5\left(T_5^*\right)$ *and* $T^* = T_5^*$.

7. Sensitivity Analyses

We executes the sensitivity analyses for this research, [1], and [3] models to determine the unique solution T_i^* when $TRC_i'\left(T^*\right) = 0, i = 1 \sim 8$ by Maple 18.00. Given $P = 9000$ units/year, $D = 5500$ units/year, $W = 800$ units, $A = \$1000/\text{order}$, $\theta = 0.1$, $h_m = \$0.7/\text{unit/year}$, $h_o = \$1.5/\text{unit/year}$, $h_r = \$4.5/\text{unit/year}$, $s = \$14/\text{unit}$, $c = \$6/\text{unit}$, $I_p = \$0.4/\text{year}$, $I_e = \$0.21/\text{year}$, $M = 120 \text{ days} = 120/365 \text{ year}$, and $N = 65 \text{ days} = 65/365 \text{ year}$. The results of the sensitivity analyses show the critical parameters by both increasing and decreasing 25% and 50% of the values.

With the computational outcomes, we can compare for T^* and $TRC\left(T^*\right)$ for this research, [1], and [3] models as shown in **Table 2** and **Table 3**, we also derive a relative comparison of the impact of the parameters on T^* and $TRC\left(T^*\right)$ in the sensitivity analyses as shown in **Figures 9-14**.

According to **Table 2** and **Table 3** and **Figures 9-14**, it can be seen the variables impact order cycle time T^* for this research, [1], and [3] models:

1) this research model

a) Positive & Major: the ordering cost A.

b) Positive & Minor: the unit holding cost per item for product in a rented warehouse h_r.

c) Negative & Minor: the unit holding cost per item for raw materials in a raw materials warehouse h_m, the unit holding cost per item for product in an owned warehouse h_o, the interest rate payable I_p, and the deterioration rate θ.

d) Negative & Major: the unit selling price per item s, the unit purchasing price per item c, and the interest rate earned I_e.

2) [1]'s model

a) Positive & Major: the ordering cost A, the unit holding cost per item for product in a rented warehouse h_r, and the interest rate earned I_e.

b) Positive & Minor: none.

c) Negative & Minor: the unit purchasing price per item c and the unit holding cost per item for raw materials in a raw materials warehouse h_m.

d) Negative & Major: the unit selling price per item s, the unit holding cost per item for product in an owned warehouse h_o, and the interest rate payable I_p.

3) [3] model

a) Positive & Major: the ordering cost A and the interest rate earned I_e.

b) Positive & Minor: the interest rate payable I_p.

c) Negative & Minor: the unit purchasing price per item c, the unit holding

Table 2. The sensitivity analyses for T^* of this research, [1], and [3] models.

Parameters	+/−	this research	[1]	[3]
	−50%	0.270233815	0.101786596	0.327853393
	−25%	0.308263707	0.205770647	0.365752394
A	0%	0.336680821	0.262664107	0.394563833
	+25%	0.362872495	0.309262520	0.421410042
	+50%	0.387290961	0.349705848	0.446645529
	−50%	0.320429903	0.36875616	0.415415749
	−25%	0.353084192	0.316762278	0.405123971
s	0%	0.336680821	0.262664107	0.394563833
	+25%	0.319433195	0.194027125	0.383713181
	+50%	0.301196267	0.079376429	0.372546631
	−50%	0.313929929	0.34787516	0.418378299
	−25%	0.341413098	0.267593486	0.404628808
c	0%	0.336680821	0.262664107	0.394563833
	+25%	0.329239493	0.259153569	0.386870646
	+50%	0.330211234	0.256525704	0.380796006
	−50%	0.297988701	0.344291552	
	−25%	0.340421870	0.266881021	
h_m	0%	0.336680821	0.262664107	
	+25%	0.333061397	0.258640968	
	+50%	0.329557115	0.254797204	
	−50%	0.348680773	0.297528747	0.406832431
	−25%	0.342733499	0.279860234	0.400745084
h_o	0%	0.336680821	0.262664107	0.394563833
	+25%	0.330516996	0.245821301	0.388284193
	+50%	0.324235566	0.229212566	0.381901310
	−50%	0.328130944	0.181953853	0.399872524
	−25%	0.332846661	0.225942070	0.396914129
h_r	0%	0.336680821	0.262664107	0.394563833
	+25%	0.339860159	0.294847498	0.392651524
	+50%	0.342539502	0.323848207	0.391065153
	−50%	0.339166974	0.450722838	0.380709435
	−25%	0.337755094	0.348933273	0.388867371
I_p	0%	0.336680821	0.262664107	0.394563833
	+25%	0.335835953	0.178927888	0.398768059
	+50%	0.335154093	0.068271068	0.401999000
	−50%	0.368756160	0.221426362	0.440417015
	−25%	0.353084192	0.242921867	0.418119460
I_e	0%	0.336680821	0.262664107	0.394563833
	+25%	0.319433195	0.281022833	0.369509613
	+50%	0.301196267	0.298253637	0.342628209
	−50%	0.343327907		
	−25%	0.339965886		
θ	0%	0.336680821		
	+25%	0.333470273		
	+50%	0.330331881		

Table 3. The sensitivity analyses for $TRC\left(T^{*}\right)$ of this research, [1], and [3] models.

Parameters	+/−	this research	[1]	[3]
	−50%	33,763.13964	33,437.31541	534.510096
	−25%	34,674.33045	34,339.88184	1218.313736
A	0%	35,433.65613	35,092.02053	1788.150103
	+25%	36,192.98180	35,844.15922	2357.986469
	+50%	36,952.30747	36,596.29791	2927.822836
	−50%	36,371.40551	36,020.89415	2491.884636
	−25%	35,902.53082	35,556.45734	2140.017370
s	0%	35,433.65613	35,092.02053	1788.150103
	+25%	34,964.78145	34,627.58372	1436.282838
	+50%	34,495.90676	34,163.14691	1084.415571
	−50%	18,766.54316	18,591.89054	1697.209763
	−25%	27,100.09964	26,841.95553	1742.679933
c	0%	35,433.65613	35,092.02053	1788.150103
	+25%	43,767.21261	43,342.08552	1833.620272
	+50%	52,100.76910	51,592.15051	1879.090442
	−50%	35,238.69360	34,896.51322	
	−25%	35,336.17486	34,994.26687	
h_m	0%	35,433.65613	35,092.02053	
	+25%	35,531.13739	35,189.77418	
	+50%	35,628.61866	35,287.52784	
	−50%	35,174.46516	34,829.60381	1443.910425
	−25%	35,304.06065	34,960.81217	1616.030264
h_o	0%	35,433.65613	35,092.02053	1788.150103
	+25%	35,563.25162	35,223.22888	1960.269941
	+50%	35,692.84710	35,354.43724	2132.389781
	−50%	35,418.99651	35,079.46804	1765.193270
	−25%	35,426.32632	35,085.74428	1776.671687
h_r	0%	35,433.65613	35,092.02053	1788.150103
	+25%	35,440.98594	35,098.29677	1799.628520
	+50%	35,448.31575	35,104.57302	1811.106937
	−50%	35,433.65390	35,091.89054	1697.209763
	−25%	35,433.65501	35,091.95553	1742.679933
I_p	0%	35,433.65613	35,092.02053	1788.150103
	+25%	35,433.65725	35,092.08552	1833.620272
	+50%	35,433.65837	35,092.15051	1879.090442
	−50%	36,371.40551	36,020.89415	2491.884636
	−25%	35,902.53082	35,556.45734	2140.017370
I_e	0%	35,433.65613	35,092.02053	1788.150103
	+25%	34,964.78145	34,627.58372	1436.282838
	+50%	34,495.90676	34,163.14691	1084.415571
	−50%	35,264.66951		
	−25%	35,349.02050		
θ	0%	35,433.65613		
	+25%	35,518.56949		
	+50%	35,603.77148		

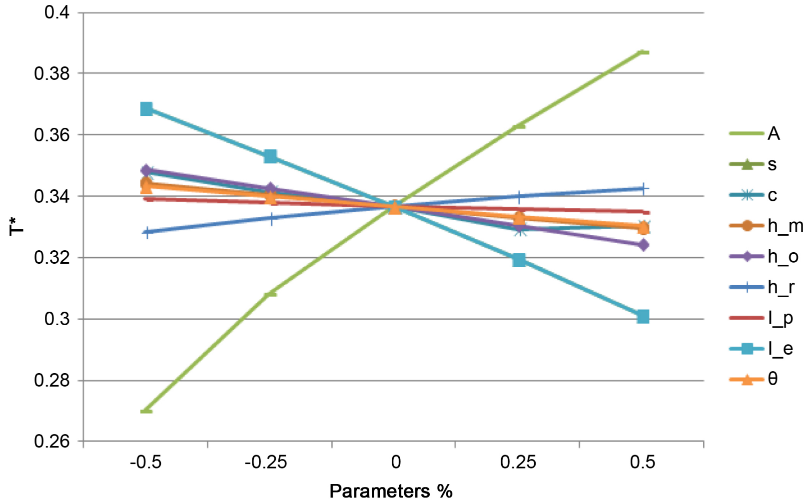

Figure 9. The sensitivity analyses for T^* of this research model.

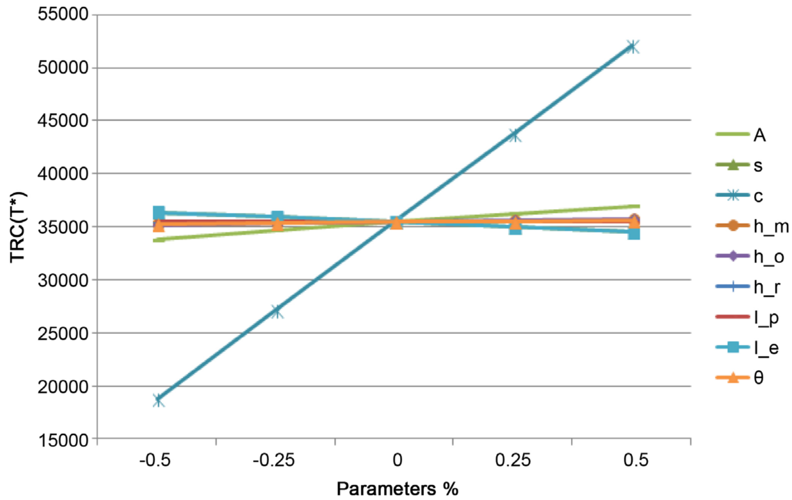

Figure 10. The sensitivity analyses for $TRC(T^*)$ of this research model.

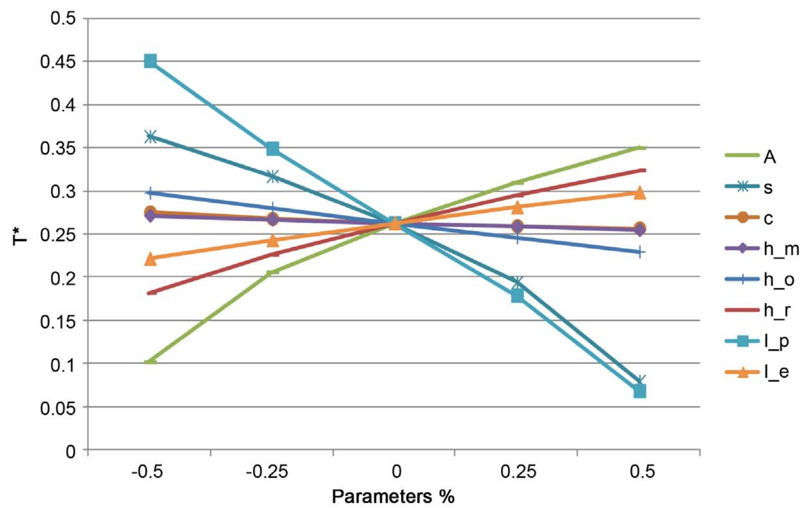

Figure 11. The sensitivity analyses for T^* of [1] model.

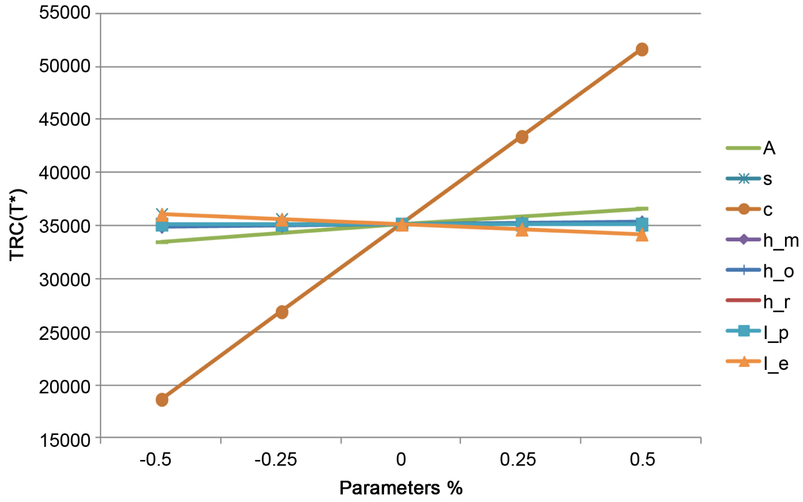

Figure 12. The sensitivity analyses for $TRC\left(T^*\right)$ of [1] model.

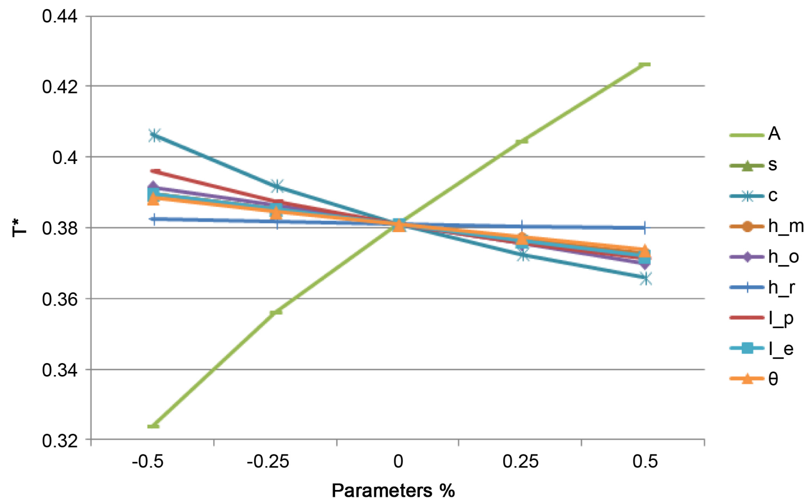

Figure 13. The sensitivity analyses for T^* of [3] model.

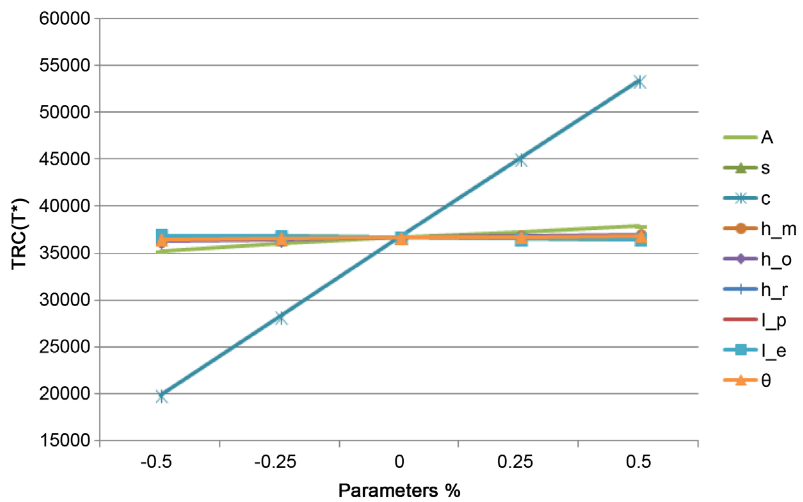

Figure 14. The sensitivity analyses for $TRC\left(T^*\right)$ of [3] model.

cost per item for product in an owned warehouse h_o, and the unit holding cost per item for product in a rented warehouse h_r.

d) Negative & Major: the unit selling price per item s.

Therefore, when making decisions on the order cycle time, variables with a relatively large influence must be considered as priority, while those with a small influence can be processed later.

On the other hand, it is seen that the variables impact the annual total relevant cost $TRC\left(T^*\right)$ for this research, [1], and [3] models:

1) this research model

a) Positive & Major: the unit purchasing price per item c.

b) Positive & Minor: the ordering cost A, the unit holding cost per item for raw materials in a raw materials warehouse h_m, the unit holding cost per item for product in an owned warehouse h_o, the unit holding cost per item for product in a rented warehouse h_r, the interest rate payable I_p, and the deterioration rate θ.

c) Negative & Minor: the unit selling price per item s and the interest rate earned I_e.

d) Negative & Major: none.

2) [1]'s model

a) Positive & Major: the unit purchasing price per item c.

b) Positive & Minor: the ordering cost A, the unit holding cost per item for raw materials in a raw materials warehouse h_m, the unit holding cost per item for product in an owned warehouse h_o, the unit holding cost per item for product in a rented warehouse h_r, and the interest rate payable I_p.

c) Negative & Minor: the unit selling price per item s and the interest rate earned I_e.

d) Negative & Major: none.

3) [3]'s model

a) Positive & Major: the ordering cost A, the unit purchasing price per item c, the unit holding cost per item for product in an owned warehouse h_o, and the interest rate payable I_p.

b) Positive & Minor: the unit holding cost per item for product in a rented warehouse h_r.

c) Negative & Minor: none.

d) Negative & Major: the unit selling price per item s and the interest rate earned I_e.

Therefore, when making decisions on the annual total relevant cost, variables with a relatively large influence can be considered as priority, while those with a small influence can be processed later.

We can organize the relative parameters impact to T^* and $TRC\left(T^*\right)$ for this research, [1], and [3] models, as shown in **Table 4** and **Table 5**.

8. Conclusions

From the basic EPQ model introduced by [4] to [3]'s complete models for the

Table 4. Comparison of the relative parameters impact to T^* of [1], and [3] models in the sensitivity analyses.

Impact	this research	[1]	[3]
Positive & Major	A	A, h_r, I_e	A, I_e
Positive & Minor	h_r		I_p
Negative & Minor	h_m, h_o, I_p, θ	c, h_m	c, h_o, h_r
Negative & Major	s, c, I_e	s, h_o, I_p	s

Table 5. Comparison of the relative parameters impact to $TRC(T^*)$ of [1], and [3] modelsin the sensitivity analyses.

Impact	this research	[1]	[3]
Positive & Major	c	c	A, c, h_o, I_p
Positive & Minor	$A, h_m, h_o, h_r, I_p, \theta$	A, h_m, h_o, h_r, I_p	h_r
Negative & Minor	s, I_e	s, I_e	
Negative & Major			s, I_e

two-level trade credit and limited storage capacity, it's all assumed that the raw materials required for production are timely. And [2] pointed out the holding cost of raw materials will change due to the impact of other factors, and it should also be included in the total relevant cost. [1] combined [2]'s the concept of holding cost of raw materials and [3]'s two-level trade credit and limited storage capacity model to present an inventory model with the holding cost of non-deteriorating raw materials. And this research further development with the holding cost of deteriorating raw materials.

After the sensitivity analyses, we reach the following conclusions in practical management:

1) When making decisions on the order cycle time T^* under limited resources, it gives priority order to the ordering cost A, the unit selling price per item s, the interest rate earned I_e, and the unit purchasing price per item c.

2) When making decisions on the annual total relevant cost $TRC(T^*)$ under limited resources, it only considers the unit purchasing price per item c.

Even though adding the holding cost of raw materials increases the complexity of the model, but it approximates real-world situations and provides more precise decisions for practical business management.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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