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The Power Integrations of Trigonometric and Hyperbolic Functions

Ahmed M. Ali*, Omar M. Abd-Alkanee

The Surveying Department, Department of Mathematics, University of Mosul, Mosul, Iraq
Email: *ahmed_math79@yahoo.com, moammed.omar8875@yahoo.com

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Abstract

For importance of the trigonometric integrals, we have in this paper finding a series of power, some of trigonometric functions that did not exist before in the first section. As shown in the Section two, where, the integration of trigonometric function with power n has been achieved and approved, this result is considered as the first achievement. while in the third section we find integrals for multiplying of trigonometric functions with powers n and m. Finally, in Section 4, we find series of power of hyperbolic functions, integrals of hyperbolic functions with powers n and integrals for multiplying of hyperbolic functions with powers n and m.

Subject Areas

Financial Mathematics, Function Theory, Functional Analysis, Integral Equation

Keywords

Trigonometric Functions, Hyperbolic Functions, Integrations Method

1. Introduction

We follow the terminology of [1] [2] [3]. The Trigonometry had resulted from the continuous interaction between mathematics and astronomy, and this remained the special relationship between trigonometry and astronomy until the third century AD, when it began to disconnect Nasir al-Din al-Tusi (1201-1274) AD. In the middle of the seventeenth century when calculus has been developed by Issac Newton and that is done by inventing new form and relationship between mathematics and physical phenomena, moreover. Newton's work proved many functions as a series of infinity. with respect x and then Newton obtained series since x and similar series of cosine of x as well as tangent x with the invention of calculus, and re-considered the trigonometric functions where still play an important role in both pure and applied mathematics analysis [4] [5].

Theorem 1.1 [6]: For all $k \in N$, then:

1) $\sin^{2k-1}(x) = \frac{(-1)^{k-1}}{2^{2k-2}} \sum_{n=1}^{k} (-1)^{n-1} \binom{2k-1}{n-1} \sin(2k-2n+1)x$.

2) $\sin^{2k}(x) = \frac{1}{2^{2k}} \binom{2k}{k} + \frac{(-1)^k}{2^{2k-1}} \sum_{n=0}^{k-1} (-1)^n \binom{2k}{n} \cos(2k-2n)x$.

3) $\cos^{2k-1}(x) = \frac{1}{2^{2k-2}} \sum_{n=1}^{k} \binom{2k-1}{n-1} \cos(2k-2n+1)x$.

4) $\cos^{2k}(x) = \frac{1}{2^{2k}} \binom{2k}{k} + \frac{1}{2^{2k-1}} \sum_{n=0}^{k-1} \binom{2k}{n} \cos(2k-2n)x$.#

Theorem 1.2: For all $k \in N$, then:

$$\tan^{2k}(x) = (-1)^k + \sec^2(x) \sum_{n=1}^{k} (-1)^{n+1} \tan^{2k-2n}(x) \quad (1.2.1)$$

$$\tan^{2k+1}(x) = (-1)^k \tan(x) + \sec^2(x) \sum_{n=1}^{k} (-1)^{n+1} \tan^{2k-2n+1}(x) \quad (1.2.2)$$

$$\cot^{2k}(x) = (-1)^k + \csc^2(x) \sum_{n=1}^{k} (-1)^{n+1} \cot^{2k-2n}(x) \quad (1.2.3)$$

$$\cot^{2k+1}(x) = (-1)^k \cot(x) + \csc^2(x) \sum_{n=1}^{k} (-1)^{n+1} \cot^{2k-2n+1}(x) \quad (1.2.4)$$

Proof:

1) We use mathematical induction on k, $k \geq 1$.

For $k = 1$, we find by direct:

$$\tan^2(x) = -1 + \sec^2(x).$$

For $k = 2$, then

$$\begin{aligned}\tan^4(x) &= \tan^2(x)\left(-1+\sec^2(x)\right) \\ &= 1 - \sec^2(x) + \tan^2(x)\sec^2(x) \\ &= 1 + \sec^2(x)\left(\tan^2(x) - 1\right) \\ &= (-1)^2 + \sec^2(x) \sum_{n=1}^{2} (-1)^{n+1} \tan^{4-2n}(x)\end{aligned}$$

We assume that (1.2.1) is true for $k = m$, $m \geq 1$ and prove that is true for $k = m+1$.

$$\tan^{2(m+1)}(x) = \tan^{2m}(x) \cdot \tan^2(x) = -\tan^{2m}(x) + \tan^{2m}(x)\sec^2(x)$$

For hypotheses induction, we get:

$$\begin{aligned}&\tan^{2(m+1)}(x) \\ &= -\left[(-1)^m + \sec^2(x) \sum_{n=1}^{m} (-1)^{n+1} \tan^{2m-2n}(x)\right] + \sec^2(x)\tan^{2m}(x) \\ &= (-1)^{m+1} + \sec^2(x) \sum_{n=1}^{m+1} (-1)^{n+1} \tan^{2(m+1)-2n}(x)\end{aligned}$$

Therefore, the relation is true for all k, $k \geq 1$.

2) Also use mathematical induction on k, $k \geq 1$.

For $k = 1$, we find that:

$$\tan^3(x) = \tan(x)\left(-1 + \sec^2(x)\right) = -\tan(x) + \tan(x)\sec^2(x)$$

For $k = 2$, then

$$\begin{aligned}\tan^5(x) &= \tan^3(x)\left(-1 + \sec^2(x)\right) \\ &= -\tan^3(x) + \tan^3(x)\sec^2(x) \\ &= \tan(x) - \sec^2(x)\tan(x) + \sec^2(x)\tan^3(x) \\ &= \tan(x) + \sec^2(x)\left(\tan^3(x) - \tan(x)\right) \\ &= (-1)^2\tan(x) + \sec^2(x)\sum_{n=1}^{2}(-1)^{n+1}\tan^{5-2n}(x)\end{aligned}$$

We assume that (1.2.2) is true for $k = m$, $m \geq 1$ and prove that is true for $k = m + 1$.

$$\tan^{2(m+1)+1}(x) = \tan^{2m+1}(x)\cdot\tan^2(x) = -\tan^{2m+1}(x) + \tan^{2m+1}(x)\sec^2(x)$$

For hypotheses induction, we get:

$$\begin{aligned}&\tan^{2(m+1)+1}(x) \\ &= -\left[(-1)^m\tan(x) + \sec^2(x)\sum_{n=1}^{m}(-1)^{n+1}\tan^{2m-2n+1}(x)\right] + \sec^2(x)\tan^{2m+1}(x) \\ &= (-1)^{m+1}\tan(x) + \sec^2(x)\sum_{n=1}^{m+1}(-1)^{n+1}\tan^{2(m+1)+1-2n}(x)\end{aligned}$$

Therefore, the relation is true for all k, $k \geq 1$.

3) We can prove it with the same method.

4) We can prove it with the same method. #

Theorem 1.3: For all $k \in N$, then:

$$\sec^{2k}(x) = \sec^2(x)\sum_{n=0}^{k-1}\binom{k-1}{n}\tan^{2n}(x). \tag{1.3.1}$$

$$\csc^{2k}(x) = \csc^2(x)\sum_{n=0}^{k-1}\binom{k-1}{n}\cot^{2n}(x). \tag{1.3.2}$$

Proof:

1) $$\begin{aligned}\sec^{2k}(x) &= \sec^2(x)\cdot\left(\sec^2(x)\right)^{k-1} = \sec^2(x)\cdot\left(1 + \tan^2(x)\right)^{k-1} \\ &= \sec^2(x)\sum_{n=0}^{k-1}\binom{k-1}{n}\tan^{2n}(x)\end{aligned}$$ (by binomial theory).

2) $$\begin{aligned}\csc^{2k}(x) &= \csc^2(x)\cdot\left(\csc^2(x)\right)^{k-1} = \csc^2(x)\cdot\left(1 + \cot^2(x)\right)^{k-1} \\ &= \csc^2(x)\sum_{n=0}^{k-1}\binom{k-1}{n}\cot^{2n}(x)\end{aligned}$$ (by binomial theory).

2. The Integration of Trigonometric Function Power

In this section, we find the integrations of power of function trigonometric

Theorem 2.1: For all $k \in N$, then:

1) $$\int \sin^{2k-1}(x)\mathrm{d}x = \frac{(-1)^{k-1}}{2^{2k-2}}\sum_{n=1}^{k}\frac{(-1)^{n-1}\binom{2k-1}{n-1}}{2k-2n+1}\cos(2k-2n+1)x + c$$

2) $$\int \sin^{2k}(x)\mathrm{d}x = \frac{1}{2^{2k}}\binom{2k}{k}x + \frac{(-1)^{k}}{2^{2k-1}}\sum_{n=0}^{k-1}\frac{(-1)^{n}\binom{2k}{n}}{2k-2n}\sin(2k-2n)x + c$$

3) $$\int \cos^{2k-1}(x)\mathrm{d}x = \frac{1}{2^{2k-2}}\sum_{n=1}^{k}\frac{\binom{2k-1}{n-1}}{2k-2n+1}\sin(2k-2n+1)x + c .$$

4) $$\int \cos^{2k}(x)\mathrm{d}x = \frac{1}{2^{2k}}\binom{2k}{k}x + \frac{1}{2^{2k-1}}\sum_{n=0}^{k-1}\frac{\binom{2k}{n}}{2k-2n}\sin(2k-2n)x + c . \#$$

Proof: Directed from Theorem (1.1). #

Theorem 2.2: For all $k \in N$, then:

1) $$\int \tan^{2k}(x)\mathrm{d}x = (-1)^{k}x + \sum_{n=1}^{k}\frac{(-1)^{n+1}}{2k-2n+1}\tan^{2k-2n+1}(x) + c .$$

2) $$\int \tan^{2k+1}(x)\mathrm{d}x = (-1)^{k+1}\ln\left|\cos(x)\right| + \sum_{n=1}^{k}\frac{(-1)^{n+1}}{2k-2n+2}\tan^{2k-2n+2}(x) + c$$

3) $$\int \cot^{2k}(x)\mathrm{d}x = (-1)^{k}x + \sum_{n=1}^{k}\frac{(-1)^{n}}{2k-2n+1}\cot^{2k-2n+1}(x) + c .$$

4) $$\int \cot^{2k+1}(x)\mathrm{d}x = (-1)^{k}\ln\left|\sin(x)\right| + \sum_{n=1}^{k}\frac{(-1)^{n}}{2k-2n+2}\cot^{2k-2n+2}(x) + c$$

5) $$\int \sec^{2k}(x)\mathrm{d}x = \sum_{n=0}^{k-1}\frac{\binom{k-1}{n}}{2n+1}\tan^{2n+1}(x) + c .$$

6) $$\int \csc^{2k}(x)\mathrm{d}x = -\sum_{n=0}^{k-1}\frac{\binom{k-1}{n}}{2n+1}\cot^{2n+1}(x) + c .$$

Proof: Directed from Theorem (1.2) and Theorem (1.3). #

Lemma 2.3: For all $k \in N$, then:

$$\int \sec^{2k+1}(x)\mathrm{d}x = \frac{1}{2k}\left[\sec^{2k-1}(x)\tan(x) + (2k-1)\int \sec^{2k-1}(x)\mathrm{d}x\right]$$

Proof: $\int \sec^{2k+1}(x)\mathrm{d}x = \int \sec^{2k-1}(x)\sec^{2}(x)\mathrm{d}x$

Using integration by parts:

Let $u = \sec^{2k-1}(x)$, then $\mathrm{d}u = (2k-1)\sec^{2k-1}(x)\tan(x)\mathrm{d}x$

$\mathrm{d}v = \sec^{2}(x)\mathrm{d}x$, then $v = \tan(x)$

$$\begin{aligned}
&\therefore \int \sec^{2k+1}(x)\mathrm{d}x \\
&= \sec^{2k-1}(x)\tan(x) - (2k-1)\int \sec^{2k-1}(x)\tan^{2}(x)\mathrm{d}x \\
&= \sec^{2k-1}(x)\tan(x) - (2k-1)\int \sec^{2k-1}(x)\left(\sec^{2}(x)-1\right)\mathrm{d}x \\
&= \sec^{2k-1}(x)\tan(x) - (2k-1)\int \sec^{2k+1}(x)\mathrm{d}x + (2k-1)\int \sec^{2k-1}(x)\mathrm{d}x
\end{aligned}$$

$$\therefore 2k\int \sec^{2k+1}(x)\mathrm{d}x = \sec^{2k-1}(x)\tan(x)+(2k-1)\int \sec^{2k-1}(x)\mathrm{d}x .$$

Hence

$$\int \sec^{2k+1}(x)\mathrm{d}x = \frac{1}{2k}\left[\sec^{2k-1}(x)\tan(x)+(2k-1)\int \sec^{2k-1}(x)\mathrm{d}x\right].\#$$

Note: From clearly that:

$$\int \sec^{3}(x)\mathrm{d}x = \frac{1}{2}\left[\sec(x)\tan(x)+\ln\left|\sec(x)+\tan(x)\right|\right]+c .$$

Theorem 2.4: For all $k \in N$, then:

$$\int \sec^{2k+1}(x)\mathrm{d}x = \frac{\prod_{i=1}^{k-1}(2i+1)}{\prod_{i=1}^{k}(2i)}\ln\left|\sec(x)+\tan(x)\right| + \frac{1}{2k}\sec^{2k-1}(x)\tan(x)$$

$$+\tan(x)\sum_{j=1}^{k-1}\frac{\prod_{i=j}^{k-1}(2i+1)}{\prod_{i=j}^{k}(2i)}\sec^{2i-1}(x)+c$$

Proof: Since

$$\int \sec^{2k+1}(x)\mathrm{d}x = \frac{1}{2k}\left[\sec^{2k-1}(x)\tan(x)+(2k-1)\int \sec^{2k-1}(x)\mathrm{d}x\right]$$

(by Lemma 2.3). Then

$$\int \sec^{2k-1}(x)\mathrm{d}x = \frac{1}{2k-2}\left[\sec^{2k-3}(x)\tan(x)+(2k-3)\int \sec^{2k-3}(x)\mathrm{d}x\right]$$

and $\int \sec^{2k-3}(x)\mathrm{d}x = \frac{1}{2k-4}\left[\sec^{2k-5}(x)\tan(x)+(2k-5)\int \sec^{2k-5}(x)\mathrm{d}x\right].$

…

and so on

$$\int \sec^{5}(x)\mathrm{d}x = \frac{1}{4}\left[\sec^{3}(x)\tan(x)+3\int \sec^{3}(x)\mathrm{d}x\right]$$

and $\int \sec^{3}(x)\mathrm{d}x = \frac{1}{2}\left[\sec(x)\tan(x)+\ln\left|\sec(x)+\tan(x)\right|\right]+c .$

$$\therefore \int \sec^{2k+1}(x)\mathrm{d}x$$

$$= \frac{1}{2k}\sec^{2k-1}(x)\tan(x)+\frac{2k-1}{2k(2k-2)}\sec^{2k-3}(x)\tan(x)$$

$$+\frac{(2k-1)(2k-3)}{2k(2k-2)(2k-4)}\sec^{2k-5}(x)\tan(x)$$

$$+\frac{(2k-1)(2k-3)(2k-5)}{2k(2k-2)(2k-4)(2k-6)}\sec^{2k-7}(x)\tan(x)$$

$$+\frac{(2k-1)(2k-3)(2k-5)(2k-7)}{2k(2k-2)(2k-4)(2k-6)(2k-8)}\sec^{2k-9}(x)\tan(x)$$

$$+\cdots$$
$$+\frac{(2k-1)(2k-3)(2k-5)(2k-7)\cdots(5)}{2k(2k-2)(2k-4)(2k-6)(2k-8)\cdots(4)}\sec^3(x)\tan(x)$$
$$+\frac{(2k-1)(2k-3)(2k-5)(2k-7)\cdots(5)(3)}{2k(2k-2)(2k-4)(2k-6)(2k-8)\cdots(4)(2)}\sec(x)\tan(x)$$
$$+\frac{(2k-1)(2k-3)(2k-5)(2k-7)\cdots(5)(3)}{2k(2k-2)(2k-4)(2k-6)(2k-8)\cdots(4)(2)}\ln|\sec(x)+\tan(x)|+c$$

$$\therefore \int \sec^{2k+1}(x)\,\mathrm{d}x = \frac{\prod_{i=1}^{k-1}(2i+1)}{\prod_{i=1}^{k}(2i)}\ln|\sec(x)+\tan(x)| + \frac{1}{2k}\sec^{2k-1}(x)\tan(x)$$
$$+\tan(x)\sum_{j=1}^{k-1}\frac{\prod_{i=j}^{k-1}(2i+1)}{\prod_{i=j}^{k}(2i)}\sec^{2i-1}(x)+c$$ #

Lemma 2.5: For all $k \in N$, then:

$$\int \csc^{2k+1}(x)\,\mathrm{d}x = \frac{1}{2k}\left[-\csc^{2k-1}(x)\cot(x)+(2k-1)\int \csc^{2k-1}(x)\,\mathrm{d}x\right]$$

Proof: $\int \csc^{2k+1}(x)\,\mathrm{d}x = \int \csc^{2k-1}(x)\csc^2(x)\,\mathrm{d}x$

Using integration by parts:

Let $u=\csc^{2k-1}(x)$, then $\mathrm{d}u = -(2k-1)\csc^{2k-1}(x)\cot(x)\,\mathrm{d}x$

$\mathrm{d}v = \csc^2(x)\,\mathrm{d}x$, then $v=-\cot(x)$

$$\therefore \int \csc^{2k+1}(x)\,\mathrm{d}x$$
$$= -\csc^{2k-1}(x)\cot(x)-(2k-1)\int \csc^{2k-1}(x)\cot^2(x)\,\mathrm{d}x$$
$$= -\csc^{2k-1}(x)\cot(x)-(2k-1)\int \csc^{2k-1}(x)\left(\csc^2(x)-1\right)\mathrm{d}x$$
$$= -\csc^{2k-1}(x)\cot(x)-(2k-1)\int \csc^{2k+1}(x)\,\mathrm{d}x+(2k-1)\int \csc^{2k-1}(x)\,\mathrm{d}x$$
$$\therefore 2k\int \csc^{2k+1}(x)\,\mathrm{d}x = -\csc^{2k-1}(x)\cot(x)+(2k-1)\int \csc^{2k-1}(x)\,\mathrm{d}x$$
$$\therefore \int \csc^{2k+1}(x)\,\mathrm{d}x = \frac{1}{2k}\left[-\csc^{2k-1}(x)\tan(x)+(2k-1)\int \csc^{2k-1}(x)\,\mathrm{d}x\right].$$ #

Note: From clearly that:

$$\int \csc^3(x)\,\mathrm{d}x = -\frac{1}{2}\left[\csc(x)\cot(x)+\ln|\csc(x)+\cot(x)|\right]+c.$$

Theorem 2.6: For all $k \in N$, then:

$$\int \csc^{2k+1}(x)\,\mathrm{d}x = -\frac{\prod_{i=1}^{k-1}(2i+1)}{\prod_{i=1}^{k}(2i)}\ln|\csc(x)+\cot(x)| - \frac{1}{2k}\csc^{2k-1}(x)\cot(x)$$
$$-\cot(x)\sum_{j=1}^{k-1}\frac{\prod_{i=j}^{k-1}(2i+1)}{\prod_{i=j}^{k}(2i)}\csc^{2i-1}(x)+c$$

Proof: Since

$\int \csc^{2k+1}(x)\mathrm{d}x = \frac{1}{2k}\left[-\csc^{2k-1}(x)\cot(x) + (2k-1)\int \csc^{2k-1}(x)\mathrm{d}x\right]$ (by Lemma 2.3) Then

$$\int \csc^{2k-1}(x)\mathrm{d}x = \frac{1}{2k-2}\left[-\csc^{2k-3}(x)\cot(x) + (2k-3)\int \csc^{2k-3}(x)\mathrm{d}x\right]$$

and $\int \csc^{2k-3}(x)\mathrm{d}x = \frac{1}{2k-4}\left[-\csc^{2k-5}(x)\cot(x) + (2k-5)\int \csc^{2k-5}(x)\mathrm{d}x\right]$.

...

and so on

$$\int \csc^{5}(x)\mathrm{d}x = \frac{1}{4}\left[-\csc^{3}(x)\cot(x) + 3\int \csc^{3}(x)\mathrm{d}x\right]$$

and $\int \csc^{3}(x)\mathrm{d}x = \frac{1}{2}\left[-\csc(x)\cot(x) - \ln\left|\csc(x)+\cot(x)\right|\right] + c$.

$$\begin{aligned}
&\therefore \int \csc^{2k+1}(x)\mathrm{d}x \\
&= \frac{-1}{2k}\csc^{2k-1}(x)\cot(x) - \frac{2k-1}{2k(2k-2)}\csc^{2k-3}(x)\cot(x) \\
&\quad - \frac{(2k-1)(2k-3)}{2k(2k-2)(2k-4)}\csc^{2k-5}(x)\cot(x) \\
&\quad - \frac{(2k-1)(2k-3)(2k-5)}{2k(2k-2)(2k-4)(2k-6)}\csc^{2k-7}(x)\cot(x) \\
&\quad - \frac{(2k-1)(2k-3)(2k-5)(2k-7)}{2k(2k-2)(2k-4)(2k-6)(2k-8)}\csc^{2k-9}(x)\cot(x) \\
&\quad - \cdots \\
&\quad - \frac{(2k-1)(2k-3)(2k-5)(2k-7)\cdots(5)}{2k(2k-2)(2k-4)(2k-6)(2k-8)\cdots(4)}\csc^{3}(x)\cot(x) \\
&\quad - \frac{(2k-1)(2k-3)(2k-5)(2k-7)\cdots(5)(3)}{2k(2k-2)(2k-4)(2k-6)(2k-8)\cdots(4)(2)}\csc(x)\cot(x) \\
&\quad - \frac{(2k-1)(2k-3)(2k-5)(2k-7)\cdots(5)(3)}{2k(2k-2)(2k-4)(2k-6)(2k-8)\cdots(4)(2)}\ln\left|\csc(x)+\cot(x)\right| + c
\end{aligned}$$

$$\begin{aligned}
\therefore \int \csc^{2k+1}(x)\mathrm{d}x &= -\frac{\prod_{i=1}^{k-1}(2i+1)}{\prod_{i=1}^{k}(2i)}\ln\left|\csc(x)+\cot(x)\right| - \frac{1}{2k}\csc^{2k-1}(x)\cot(x) \\
&\quad - \cot(x)\sum_{j=1}^{k-1}\frac{\prod_{i=j}^{k-1}(2i+1)}{\prod_{i=j}^{k}(2i)}\csc^{2i-1}(x) + c
\end{aligned}$$

#

3. The Integration of Multiply Trigonometric Function Power

Theorem 3.1: For all $n \in N$, then:

1) $\int \sin^{n}(x)\cos(x)\mathrm{d}x = \begin{cases} \frac{1}{n+1}\sin^{n+1}(x) + c, & n \neq -1 \\ \ln\left|\sin(x)\right| + c, & n = -1 \end{cases}$.

2) $\int \cos^n(x)\sin(x)\,\mathrm{d}x = \begin{cases} \dfrac{-1}{n+1}\cos^{n+1}(x)+c, & n \neq -1 \\ -\ln|\cos(x)|+c, & n=-1 \end{cases}.$

3) $\int \tan^n(x)\sec^2(x)\,\mathrm{d}x = \begin{cases} \dfrac{1}{n+1}\tan^{n+1}(x)+c, & n \neq -1 \\ \ln|\tan(x)|+c, & n=-1 \end{cases}.$

4) $\int \cot^n(x)\csc^2(x)\,\mathrm{d}x = \begin{cases} \dfrac{-1}{n+1}\cot^{n+1}(x)+c, & n \neq -1 \\ -\ln|\cot(x)|+c, & n=-1 \end{cases}.$

5) $\int \sec^n(x)\tan(x)\,\mathrm{d}x = \begin{cases} \dfrac{1}{n}\sec^{n}(x)+c, & n \neq -1 \\ -\cos(x)+c, & n=-1 \end{cases}.$

6) $\int \csc^n(x)\cot(x)\,\mathrm{d}x = \begin{cases} \dfrac{-1}{n}\csc^{n}(x)+c, & n \neq -1 \\ \sin(x)+c, & n=-1 \end{cases}.$

where c is integral constant.

Proof: Directed proof. #

Theorem 3.2: For all $n,m \in N$ and m is an odd number, then:

$$\int \sin^m(x)\cos^n(x)\,\mathrm{d}x = \sum_{i=0}^{k} \frac{(-1)^{i+1}}{n+1+2i}\binom{k}{i}\cos^{n+1+2i}(x)+c.$$

Proof: Since m is an odd number, then $\exists k \in N$ such that $m = 2k+1$.

$$\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$$

$$= \int \sin^{2k+1}(x)\cos^n(x)\,\mathrm{d}x = \int \left(\sin^2(x)\right)^k \sin(x)\cos^n(x)\,\mathrm{d}x$$

$$= \int \left(1-\cos^2(x)\right)^k \sin(x)\cos^n(x)\,\mathrm{d}x = \int \sum_{i=0}^{k}(-1)^i\binom{k}{i}\cos^{2i}(x)\sin(x)\cos^n(x)\,\mathrm{d}x$$

$$= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\int \cos^{n+2i}(x)\sin(x)\,dx = \sum_{i=0}^{k}\frac{(-1)^{i+1}}{n+1+2i}\binom{k}{i}\cos^{n+1+2i}(x)+c \quad \#$$

Theorem 3.3: For all $n,m \in N$ and m is an even number, then:

$$\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$$

$$= \begin{cases} \displaystyle\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left[\frac{1}{2^\alpha}\binom{\alpha}{\frac{\alpha}{2}}x + \frac{1}{2^{\alpha-1}}\sum_{j=0}^{\frac{\alpha}{2}-1}\frac{\binom{\alpha}{j}}{\alpha-2j}\sin(\alpha-2j)x\right]+c \\ \qquad \text{if } \alpha = 2i+n,\ n \text{ is even} \\ \displaystyle\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left[\frac{1}{2^{\alpha-1}}\sum_{j=1}^{\frac{\alpha+1}{2}}\frac{\binom{\alpha-1}{j-1}}{\alpha-2j+1}\sin(\alpha-2j+1)x\right]+c \\ \qquad \text{if } \alpha = 2i+n,\ n \text{ is odd} \end{cases} \tag{3.3}$$

Proof: Since m is an even number, then $\exists k \in N$ such that $m = 2k$.

$$\begin{aligned}&\int \sin^m(x)\cos^n(x)\mathrm{d}x \\ &= \int \sin^{2k}(x)\cos^n(x)\mathrm{d}x \\ &= \int \left(\sin^2(x)\right)^k \cos^n(x)\mathrm{d}x \\ &= \int \left(1-\cos^2(x)\right)^k \cos^n(x)\mathrm{d}x \\ &= \int \sum_{i=0}^{k}(-1)^i \binom{k}{i}\cos^{2i+n}(x)\mathrm{d}x\end{aligned}$$

By Theorem 2.1, we have (3.3).

Theorem 3.4: For all $n, m \in N$ and m is an odd number, then:

$$\int \tan^m(x)\sec^n(x)\mathrm{d}x = \sum_{i=0}^{k}\frac{(-1)^{k+i}}{n+2i}\binom{k}{i}\sec^{n+2i}(x)+c.$$

Proof: Since m is an odd number, then $\exists k \in N$ such that $m = 2k+1$.

$$\begin{aligned}&\int \tan^m(x)\sec^n(x)\mathrm{d}x \\ &= \int \tan^{2k+1}(x)\sec^n(x)\mathrm{d}x = \int \left(\tan^2(x)\right)^k \tan(x)\sec^n(x)\mathrm{d}x \\ &= \int \left(\sec^2(x)-1\right)^k \tan(x)\sec^n(x)\mathrm{d}x \\ &= \int (-1)^k\left(1-\sec^2(x)\right)^k \tan(x)\sec^n(x)\mathrm{d}x \\ &= \int (-1)^k \sum_{i=0}^{k}(-1)^i\binom{k}{i}\sec^{2i}(x)\tan(x)\sec^n(x)\mathrm{d}x \\ &= \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\int \sec^{n+2i-1}(x)\sec(x)\tan(x)\mathrm{d}x \qquad \# \\ &= \sum_{i=0}^{k}\frac{(-1)^{k+i}}{n+2i}\binom{k}{i}\sec^{n+2i}(x)+c\end{aligned}$$

Theorem 3.5: For all $n, m \in N$ and m is an even number, then:

$$\int \tan^m(x)\sec^n(x)dx =$$

$$\int \tan^m(x)\sec^n(x)\mathrm{d}x$$

$$= \begin{cases} \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\left[\sum_{j=0}^{\frac{\alpha}{2}-1}\frac{\binom{\frac{\alpha}{2}-1}{j}}{2j+1}\tan^{2j+1}(x)\right]+c, & \text{if } \alpha = 2i+n,\ n \text{ is even} \\ \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\left[\frac{1}{\alpha-1}\sec^{\alpha-2}(x)\tan(x)+\frac{\prod_{m=1}^{\frac{\alpha-3}{2}}(2m+1)}{\prod_{m=1}^{\frac{\alpha-1}{2}}(2m)}\ln\left|\sec(x)+\tan(x)\right|\right. \\ \left. +\tan(x)\sum_{j=1}^{\frac{\alpha-3}{2}}\frac{\prod_{m=j}^{\frac{\alpha-3}{2}}(2m+1)}{\prod_{m=j}^{\frac{\alpha-1}{2}}(2m)}\sec^{2m-1}(x)+c\right], & \text{if } \alpha = 2i+n,\ n \text{ is odd}\end{cases} \tag{3.5}$$

Proof: Since m is an even number, then $\exists k \in N$ such that $m = 2k$.

$$\int \tan^m(x)\sec^n(x)\,dx$$
$$= \int \tan^{2k}(x)\sec^n(x)\,dx = \int \left(\tan^2(x)\right)^k \sec^n(x)\,dx$$
$$= \int \left(\sec^2(x)-1\right)^k \sec^n(x)\,dx = \int (-1)^k \left(1-\sec^2(x)\right)^k \sec^n(x)\,dx$$
$$= \int \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} \sec^{2i+n}(x)\,dx$$

By Theorem 2.2, we have (3.5). #

Theorem 3.6: For all $n, m \in N$ and m is an odd number, then:

$$\int \cot^m(x)\csc^n(x)\,dx = \sum_{i=0}^{k} \frac{(-1)^{k+i+1}}{n+2i} \binom{k}{i} \csc^{n+2i}(x) + c.$$

Proof: Since m is an odd number, then $\exists k \in N$ such that $m = 2k+1$.

$$\int \cot^m(x)\csc^n(x)\,dx$$
$$= \int \cot^{2k+1}(x)\csc^n(x)\,dx = \int \cot^{2k}(x)\cot(x)\csc^n(x)\,dx$$
$$= \int \left(\cot^2(x)\right)^k \cot(x)\csc^n(x)\,dx = \int \left(\csc^2(x)-1\right)^k \cot(x)\csc^n(x)\,dx$$
$$= \int (-1)^k \left(1-\csc^2(x)\right)^k \cot(x)\csc^n(x)\,dx$$
$$= \int (-1)^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \csc^{2i}(x)\cot(x)\csc^n(x)\,dx$$
$$= \sum_{i=0}^{k} -(-1)^{k+i} \binom{k}{i} \int \csc^{n+2i-1}(x)\left(-\csc(x)\right)\cot(x)\,dx \qquad \#$$
$$= \sum_{i=0}^{k} \frac{(-1)^{k+i+1}}{n+2i} \binom{k}{i} \csc^{n+2i}(x) + c$$

Theorem 3.7: For all $n, m \in N$ and m is an even number, then: $\int \cot^m(x)\csc^n(x)\,dx =$

$$\int \cot^m(x)\csc^n(x)\,dx$$
$$= \begin{cases} -\sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} \left[\sum_{j=0}^{\frac{\alpha}{2}-1} \frac{\binom{\frac{\alpha}{2}-1}{j}}{2j+1} \cot^{2j+1}(x) \right] + c, & \text{if } \alpha = 2i+n,\ n \text{ is even} \\ -\sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} \left[\frac{1}{\alpha-1} \csc^{\alpha-2}(x)\cot(x) + \frac{\prod_{m=1}^{\frac{\alpha-3}{2}} (2m+1)}{\prod_{m=1}^{\frac{\alpha-1}{2}} (2m)} \ln\left|\csc(x)+\cot(x)\right| \right. \\ \left. + \cot(x) \sum_{j=1}^{\frac{\alpha-3}{2}} \frac{\prod_{m=j}^{\frac{\alpha-3}{2}} (2m+1)}{\prod_{m=j}^{\frac{\alpha-1}{2}} (2m)} \csc^{2m-1}(x) \right] + c, & \text{if } \alpha = 2i+n,\ n \text{ is odd} \end{cases} \quad (3.7)$$

Proof: Since m is an even number, then $\exists k \in N$ such that $m = 2k$

$$\begin{aligned}
&\int \cot^m(x)\csc^n(x)\,dx \\
&= \int \cot^{2k}(x)\csc^n(x)\,dx \\
&= \int \left(\cot^2(x)\right)^k \csc^n(x)\,dx \\
&= \int \left(\csc^2(x)-1\right)^k \csc^n(x)\,dx \\
&= \int (-1)^k \sum_{i=0}^{k}(-1)^i \binom{k}{i}\csc^{2i+n}(x)\,dx \\
&= \int \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\csc^{2i+n}(x)\,dx \\
&= \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\int \csc^{2i+n}(x)\,dx
\end{aligned}$$

Using Theorems 2.2 and 2.5, we have (3.7). #

Remark: There are many integration formulas, we can be finding it by previous results which obtained it from this paper. For example:

1) $\int \sin^m(x) f(x)\,dx$, $\int \cos^m(x) f(x)\,dx$,

where $f(x) \in \left\{\tan^n(x), \cot^n(x), \sec^n(x), \csc^n(x)\right\}$, $\forall n, m \in N$.

2) $\int \tan^m(x) g(x)\,dx$, where $g(x) \in \left\{\cot^n(x), \sec^n(x), \csc^n(x)\right\}$, $\forall n, m \in N$.

3) $\int \cot^m(x)\sec^n(x)\,dx$.

4. The Hyperbolic Functions

The combinations of the exponential functions $\exp(x)$ and $\exp(-x)$ called hyperbolic functions. These functions which arise in various engineering applications, have many properties in common with the trigonometric functions. The hyperbolic functions have resulted from vibratory motions inside elastic solid and more general in many problems where mechanical energy is gradually absorbed by a surrounding medium. There are importance applications on hyperbolic functions and power integration of hyperbolic functions in physics, mathematics transformations and numerical analysis.

Remark: The proofs for this following result are similarly of the results for previous sections.

4.1. The Hyperbolic Functions Power

Theorem 4.1.1: For all $k \in N$, then:

1) $\sinh^{2k-1}(x) = \dfrac{1}{2^{2k-2}}\sum_{n=0}^{k-1}(-1)^n\binom{2k-1}{n}\sinh(2k-1-2n)x$.

2) $\sinh^{2k}(x) = \dfrac{(-1)^k}{2^{2k}}\binom{2k}{k} + \dfrac{1}{2^{2k-1}}\sum_{n=0}^{k-1}(-1)^n\binom{2k}{n}\cosh(2k-2n)x$.

3) $\cosh^{2k-1}(x) = \dfrac{1}{2^{2k-2}}\sum_{n=0}^{k-1}\binom{2k-1}{n}\cosh(2k-1-2n)x$.

4) $\cosh^{2k}(x)=\frac{1}{2^{2k}}\binom{2k}{k}+\frac{1}{2^{2k-1}}\sum_{n=0}^{k-1}\binom{2k}{n}\cosh(2k-2n)x$.

5) $\tanh^{2k}(x)=1-\operatorname{sech}^2(x)\sum_{n=1}^{k}\tanh^{2k-2n}(x)$.

6) $\tanh^{2k+1}(x)=\tanh(x)-\operatorname{sech}^2(x)\sum_{n=1}^{k}\tanh^{2k-2n+1}(x)$.

7) $\coth^{2k}(x)=1+\operatorname{csch}^2(x)\sum_{n=1}^{k}\coth^{2k-2n}(x)$.

8) $\coth^{2k+1}(x)=\coth(x)+\operatorname{csch}^2(x)\sum_{n=1}^{k}\coth^{2k-2n+1}(x)$.

9) $\operatorname{sech}^{2k}(x)=\operatorname{sech}^2(x)\sum_{n=0}^{k-1}(-1)^n\binom{k-1}{n}\tanh^{2n}(x)$.

10) $\operatorname{csch}^{2k}(x)=\operatorname{csch}^2(x)\sum_{n=0}^{k-1}(-1)^{k+n-1}\binom{k-1}{n}\coth^{2n}(x)$.

4.2. The Integration of Hyperbolic Functions Power

Theorem 4.2.1: For all $k\in N$, then:

1) $\int\sinh^{2k-1}(x)=\frac{1}{2^{2k-2}}\sum_{n=0}^{k-1}\frac{(-1)^n}{2k-1-2n}\binom{2k-1}{n}\cosh(2k-1-2n)x+c$

2) $\int\sinh^{2k}(x)=\frac{(-1)^k}{2^{2k}}\binom{2k}{n}x+\frac{1}{2^{2k-1}}\sum_{n=0}^{k-1}\frac{(-1)^n}{2k-2n}\binom{2k}{n}\sinh(2k-2n)x+c$

3) $\int\cosh^{2k-1}(x)=\frac{1}{2^{2k-2}}\sum_{n=0}^{k-1}\frac{1}{2k-1-2n}\binom{2k-1}{n}\sinh(2k-1-2n)x+c$

4) $\int\cosh^{2k}(x)=\frac{1}{2^{2k}}\binom{2k}{k}x+\frac{1}{2^{2k-1}}\sum_{n=0}^{k-1}\frac{1}{2k-2n}\binom{2k}{n}\sinh(2k-2n)x+c$

5) $\int\tanh^{2k}(x)=x-\sum_{n=1}^{k}\frac{1}{2k+1-2n}\tanh^{2k+1-2n}(x)+c$.

6) $\int\tanh^{2k+1}(x)=\ln|\cosh(x)|-\sum_{n=1}^{k}\frac{1}{2k+2-2n}\tanh^{2k+2-2n}(x)+c$.

7) $\int\coth^{2k}(x)=x-\sum_{n=1}^{k}\frac{1}{2k+1-2n}\coth^{2k+1-2n}(x)+c$.

8) $\int\coth^{2k+1}(x)=\ln|\sinh(x)|-\sum_{n=1}^{k}\frac{1}{2k+2-2n}\coth^{2k-2n+2}(x)+c$.

9) $\int\operatorname{sech}^{2k}(x)=\sum_{n=0}^{k-1}\frac{(-1)^n}{2n+1}\binom{k-1}{n}\tanh^{2n+1}(x)+c$.

10) $\int\operatorname{csch}^{2k}(x)=\sum_{n=0}^{k-1}\frac{(-1)^{k+n}}{2n+1}\binom{k-1}{n}\coth^{2n+1}(x)+c$.

Lemma 4.2.2: For all $k\in N$, then:

1) $\int\operatorname{sech}^{2k+1}(x)\mathrm{d}x=\frac{1}{2k}\left[\operatorname{sech}^{2k-1}(x)\tanh(x)+(2k-1)\int\sec^{2k-1}(x)\mathrm{d}x\right]$

2) $\int\operatorname{csch}^{2k+1}(x)\mathrm{d}x=-\frac{1}{2k}\left[\operatorname{csch}^{2k-1}(x)\coth(x)+(2k-1)\int\operatorname{csch}^{2k-1}(x)\mathrm{d}x\right]$

Theorem 4.2.3: For all $k\in N$, then:

1) $$\int \operatorname{sec}^{2k+1}(x)\mathrm{d}x = \frac{\prod_{i=1}^{k-1}(2i+1)}{\prod_{i=1}^{k}(2i)}\tan^{-1}\mathrm{e}^{x} + \frac{1}{2k}\operatorname{sech}^{2k-1}(x)\tanh(x) + \tanh(x)\sum_{j=1}^{k-1}\frac{\prod_{i=j}^{k-1}(2i+1)}{\prod_{i=j}^{k}(2i)}\operatorname{sech}^{2i-1}(x)+c$$

2) $$\int \operatorname{csch}^{2k+1}(x)\mathrm{d}x = (-1)^{n+1}\frac{\prod_{i=1}^{k-1}(2i+1)}{\prod_{i=1}^{k}(2i)}\ln\left|\operatorname{csch}(x)+\coth(x)\right| - \frac{1}{2k}\operatorname{csch}^{2k-1}(x)\coth(x) + \coth(x)\sum_{j=1}^{k-1}\frac{\prod_{i=j}^{k-1}(2i+1)}{\prod_{i=j}^{k}(2i)}\operatorname{csch}^{2i-1}(x)+c$$

4.3. The Integration of Multiply Trigonometric Function Power

Theorem 4.3.1: For all $n \in N$, then:

1) $$\int \sinh^{n}(x)\cosh(x)\mathrm{d}x = \begin{cases}\frac{1}{n+1}\sinh^{n+1}(x)+c, & n \neq -1\\ \ln\left|\sinh(x)\right|+c, & n=-1\end{cases}.$$

2) $$\int \cosh^{n}(x)\sinh(x)\mathrm{d}x = \begin{cases}\frac{1}{n+1}\cosh^{n+1}(x)+c, & n \neq -1\\ \ln\left(\cosh(x)\right)+c, & n=-1\end{cases}.$$

3) $$\int \tanh^{n}(x)\operatorname{sech}^{2}(x)\mathrm{d}x = \begin{cases}\frac{1}{n+1}\tanh^{n+1}(x)+c, & n \neq -1\\ \ln\left|\tanh(x)\right|+c, & n=-1\end{cases}.$$

4) $$\int \coth^{n}(x)\operatorname{csch}^{2}(x)\mathrm{d}x = \begin{cases}\frac{-1}{n+1}\coth^{n+1}(x)+c, & n \neq -1\\ -\ln\left|\coth(x)\right|+c, & n=-1\end{cases}.$$

5) $$\int \operatorname{sech}^{n}(x)\tanh(x)\mathrm{d}x = \begin{cases}\frac{1}{n}\operatorname{sech}^{n}(x)+c, & n \neq -1\\ \cosh(x)+c, & n=-1\end{cases}.$$

6) $$\int \operatorname{csch}^{n}(x)\coth(x)\mathrm{d}x = \begin{cases}\frac{-1}{n}\operatorname{csch}^{n}(x)+c, & n \neq -1\\ \sinh(x)+c, & n=-1\end{cases}.$$

where c is integral constant.

Theorem 4.3.2: For all $n, m \in N$ and m is an odd number, then:

1) $$\int \sinh^m(x)\cosh^n(x)\mathrm{d}x = \sum_{i=0}^{k}\frac{(-1)^{k+i}}{n+2i+1}\binom{k}{i}\cosh^{n+2i+1}(x)+c.$$

2) $$\int \tanh^m(x)\operatorname{sech}^n(x)\mathrm{d}x = \sum_{i=0}^{k}\frac{(-1)^{i+1}}{n+2i}\binom{k}{i}\operatorname{sech}^{n+2i}(x)+c.$$

3) $$\int \coth^m(x)\operatorname{csch}^n(x)\mathrm{d}x = -\sum_{i=0}^{k}\frac{1}{n+2i}\binom{k}{i}\operatorname{csch}^{n+2i}(x)+c.$$

Theorem 4.3.3: For all $n,m\in N$ and m is an even number, then:

1) $$\int \sinh^m(x)\cosh^n(x)\mathrm{d}x = \begin{cases} \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\left[\frac{1}{2^{\alpha}}\binom{\alpha}{\frac{\alpha}{2}}x+\frac{1}{2^{\alpha-1}}\sum_{j=0}^{\frac{\alpha}{2}-1}\frac{\binom{\alpha}{j}}{\alpha-2j}\sinh(\alpha-2j)x\right]+c \\ \qquad \text{if } \alpha=2i+n,\ n \text{ is even} \\ \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\left[\frac{1}{2^{\alpha-1}}\sum_{j=0}^{\frac{\alpha-1}{2}}\frac{\binom{\alpha}{j-1}}{\alpha-2j}\sinh(\alpha-2j)x\right]+c \\ \qquad \text{if } \alpha=2i+n,\ n \text{ is odd} \end{cases}$$

2) $$\int \tanh^m(x)\operatorname{sech}^n(x)\mathrm{d}x = \begin{cases} \sum_{i=0}^{k}(-1)^{i}\binom{k}{i}\left[\sum_{j=0}^{\frac{\alpha}{2}-1}\frac{\binom{\frac{\alpha}{2}-1}{j}}{2j+1}\tanh^{2j+1}(x)\right]+c, \quad \text{if } \alpha=2i+n,\ n \text{ is even} \\ \sum_{i=0}^{k}(-1)^{i}\binom{k}{i}\left[\frac{1}{\alpha-1}\operatorname{sech}^{\alpha-1}(x)\tanh(x)+\frac{\prod_{m=1}^{\frac{\alpha-3}{2}}(2m+1)}{\prod_{m=1}^{\frac{\alpha-1}{2}}(2m)}\tan^{-1}\mathrm{e}^{x}\right. \\ \left.+\tanh(x)\sum_{j=1}^{\frac{\alpha-3}{2}}\frac{\prod_{m=j}^{\frac{\alpha-3}{2}}(2m+1)}{\prod_{m=j}^{\frac{\alpha-1}{2}}(2m)}\operatorname{sech}^{2m-1}(x)+c\right], \quad \text{if } \alpha=2i+n,\ n \text{ is odd} \end{cases}$$

3)
$$\int \coth^m(x)\operatorname{csch}^n(x)\,dx = \begin{cases} \sum_{i=0}^{k}\binom{k}{i}\left[\sum_{j=0}^{\frac{\alpha}{2}-1}\frac{\binom{\frac{\alpha}{2}-1}{j}}{2j+1}\coth^{2j+1}(x)\right]+c, & \text{if } \alpha=2i+n,\ n \text{ is even} \\ \sum_{i=0}^{k}\binom{k}{i}\left[\frac{-1}{\alpha-1}\operatorname{csch}^{\alpha-1}(x)\coth(x)+(-1)^{\frac{\alpha+1}{2}}\frac{\prod_{m=1}^{\frac{\alpha-3}{2}}(2m+1)}{\prod_{m=1}^{\frac{\alpha-1}{2}}(2m)}\ln\left|\operatorname{csch}(x)+\coth(x)\right|\right. \\ \left.+\coth(x)\sum_{j=1}^{\frac{\alpha-3}{2}}(-1)^{\frac{\alpha+1}{2}+j}\frac{\prod_{m=j}^{\frac{\alpha-3}{2}}(2m+1)}{\prod_{m=j}^{\frac{\alpha-1}{2}}(2m)}\operatorname{csch}^{2m-1}(x)\right]+c, & \text{if } \alpha=2i+n,\ n \text{ is odd} \end{cases}$$

Remark

There are many integration formulas, we can find it by previous results which obtained it from this paper. For example:

1) $\int \sinh^m(x) f(x)\,dx$, $\int \cosh^m(x) f(x)\,dx$,

where $f(x)\in\{\tanh^n(x),\coth^n(x),\operatorname{sech}^n(x),\operatorname{csch}^n(x)\}$, $\forall n,m\in N$.

2) $\int \tanh^m(x) g(x)\,dx$, where

$$g(x)\in\{\coth^n(x),\operatorname{sech}^n(x),\operatorname{csch}^n(x)\},\quad \forall n,m\in N.$$

3) $\int \coth^m(x)\operatorname{sech}^n(x)\,dx$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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