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Fixed Point Results for K-Iteration Using Non-Linear Type Mappings

Anju Panwar[1], Ravi Parkash Bhokal[2*]

[1]Department of Mathematics, M. D. U. Rohtak, Haryana, India
[2]Government College, Dujana, Jhajjar (Haryana), India
Email: anjupanwar15@gmail.com, *ravibhokal08@gmail.com

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Abstract

In this paper we establish convergence and stability results using general contractive condition, quasi-nonexpansive mapping and mean non expansive mapping for K-iteration process. We shall also generalize the K-iteration process for a pair of distinct mappings and with the help of example we claim that the generalized iteration process has better convergence rate than the K-iteration process for single mapping and some of the existing iteration processes. Suitable examples are given in the support of main results.

Subject Areas

Mathematical Analysis

Keywords

K-Iteration Process, Opial's Condition, Mean Non-Expansive Mapping, Quasi Non-Expansive Mapping

1. Introduction and Preliminary Definitions

Let (X,d) be a metric space and $T: X \to X$ be a self map defined on *X*. Let $F(T) = \{z \in X : Tz = z\}$ denote the set of fixed point of *T*. For $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n \geq 0, \tag{1.1}$$

is called the Picard iteration.

For $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the

*Corresponding author.

Mann iteration process [1].

In 2013, Khan [2] produced a new type of iteration process by introducing the concept of the following Picard-Mann hybrid iterative process for a single mapping *T*. For the initial value $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Ty_n,$$
$$y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, n \geq 0, \quad (1.3)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$.

Khan [2] showed that the rate of convergence of Picard-Mann hybrid iterative process is more than the Picard iteration scheme, Mann iteration scheme [1] and Ishikawa iterative schemes [3].

In this direction Gursoy and Karakaya [4], gave new iteration process as follows:

For the initial value $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n Tz_n, \\ x_{n+1} = Ty_n \end{cases} \quad (1.4)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ is known as Picard-S iterative process. By giving appropriate example, Gursoy and Karakaya [4] proved that their iterative process has better convergence rate than Picard, Mann, Ishikawa, Noor and Normal-S iterative processes.

Karakaya *et al.* in their paper [5], introduced a new hybrid iterative process as

$$\begin{cases} x_0 \in X, \\ y_n = T(1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = T(1-\alpha_n)y_n + \alpha_n Ty_n \end{cases} \quad (1.5)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$.

With the help of suitable example it was claimed by Karakaya *et al.* [5], that their iteration process converges faster than the iteration process of Gursoy and Karakaya [4].

In 2016, Thakur *et al.* [6] introduced a new iteration scheme called Thakur New Iteration Scheme as for the initial value $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ y_n = T(1-\alpha_n)x_n + \alpha_n z_n, \\ x_{n+1} = Ty_n \end{cases} \quad (1.6)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$.

In [6] it was claimed that the Thakur New Iteration Scheme has higher convergence rate than the iteration process of Karakaya *et al.* [7].

In the recent work of Hussain *et al.* [8], a new iteration scheme has been developed and it is claimed that it has better convergence rate than the iterative process Thakur *et al.* [6]. This iteration process is called K-iteration process and is given as:

For the initial value $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T(1-\alpha_n)Tx_n + \alpha_n T z_n, \\ x_{n+1} = T y_n \end{cases} \tag{1.7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$.

In the present work we shall generalize some convergence and stability results for K-iteration process. We shall also prove convergence and stability results for more general form of K-iteration process and K-iteration process for a pair of two distinct mappings.

Definition 1.1 [3]: Let X be a real Banach space. The mapping $T: X \to X$ is said to be asymptotically quasi-nonexpansive if $F(T) \neq \varnothing$ and there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - q\| \leq (1+\mu_n)\|x - q\| \tag{1.8}$$

for all $x \in X, q \in F(T)$ and $n \geq 0$.

Definition 1.2 [9]: Let X be a real Banach space. The mapping $T: X \to X$ is said to be mean non-expansive if there exists two non negative real numbers a, b such that $a + b \leq 1$ and for all $x, y \in X$,

$$\|Tx - Ty\| = a\|x - y\| + b\|x - Ty\|$$

Definition 1.3 [10]: Let $\{z_n\}_{n=0}^{\infty}$ be any sequence in *X*. Then the iterative process $x_{n+1} = f(T, x_n)$ which converges to a fixed point *q*, is said to be stable with respect to the mapping *T* if for $\varphi_n = \|z_{n+1} - f(T, z_n)\|, n = 0,1,2,\cdots$, we have $\lim_{n\to\infty} \varphi_n = 0$ if and only if $\lim_{n\to\infty} z_n = q$.

Definition 1.4 [7]: A space X is said to satisfy Opial's condition if for each sequence $\{x_n\}_{n=0}^{\infty}$ in X such that x_n converges weakly to *x* we have for all $y \in X$, $x \neq y$ following holds:

1) $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$,
2) $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$.

Lemma 1.5 [11]: Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be non-negative real sequences satisfying the inequality:

$$a_{n+1} \leq (1-b_n)a_n + b_n,$$

where $b_n \in (0,1)$, for all $n \in N$, $\sum_{n=1}^{\infty} b_n = \infty$ and $\frac{b_n}{a_n} \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.6 [12]: Let δ be a real number such that $0 \leq \delta < 1$, and $\{\epsilon_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then for any sequence of positive numbers $\{a_n\}_{n=0}^{\infty}$ satisfying $a_{n+1} \leq \delta a_n + \epsilon_n, n = 0,1,2,\cdots$, we have $\lim_{n\to\infty} a_n = 0$.

Lemma 1.7 [13]: Let X be a real Banach space and $\{g_n\}$ be any sequence in X such that $0 < g_n < 1$ for all $n \in N$. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be non-negative real sequences satisfying $\limsup_{n\to\infty}\|a_n\| \leq c$, $\limsup_{n\to\infty}\|b_n\| \leq c$ and $\limsup_{n\to\infty}\|g_n a_n + (1-g_n)b_n\| = c$ holds for some $c \geq 0$. Then $\limsup_{n\to\infty}\|a_n - b_n\| = 0$.

2. Main Results

Theorem 2.1: Let X be a Banach space and $T: X \to X$ be a mapping satisfying the condition

$$\|Tx - q\| \le \delta \|x - q\| \tag{2.1}$$

where $q \in F, x \in X$ and $0 \le \delta < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the K-iterative process given by (1.7). Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T)$.

Proof: From (1.7) and (2.1) we have,

$$\|x_{n+1} - q\| = \|Ty_n - q\| \le \delta \|Ty_n - q\| \tag{2.2}$$

And

$$\begin{aligned} \|y_n - q\| &= \|T((1-\alpha_n)Tx_n + \alpha_n Tz_n) - q\| \\ &\le \delta \|(1-\alpha_n)Tx_n + \alpha_n Tz_n - q\| \\ &\le \delta \|(1-\alpha_n)(Tx_n - q) + \alpha_n (Tz_n - q)\| \\ &\le \delta [(1-\alpha_n)\|Tx_n - q\| + \alpha_n \|Tz_n - q\|] \\ &\le \delta [(1-\alpha_n)\|Tx_n - q\| + \alpha_n \|Tz_n - q\|] \\ &\le \delta^2 [(1-\alpha_n)\|x_n - q\| + \alpha_n \|z_n - q\|] \end{aligned} \tag{2.3}$$

Again using (1.7) and (2.1) we get,

$$\begin{aligned} \|z_n - q\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - q\| \\ &\le (1-\beta_n)\|x_n - q\| + \beta_n \|Tx_n - q\| \\ &\le (1-\beta_n)\|x_n - q\| + \beta_n \delta \|x_n - q\| \end{aligned} \tag{2.4}$$

Using (2.4) in (2.3) we get,

$$\begin{aligned} \|y_n - q\| &\le \delta^2 [(1-\alpha_n)\|x_n - q\| + \alpha_n (1-\beta_n)\|x_n - q\| + \alpha_n \beta_n \delta \|x_n - q\|] \\ &\le \delta^2 (1 - \alpha_n + \alpha_n (1-\beta_n) + \alpha_n \beta_n \delta)\|x_n - q\| \\ &\le \delta^2 (1 - \alpha_n \beta_n (1-\delta))\|x_n - q\| \end{aligned} \tag{2.5}$$

Using (2.5) in (2.2) we get,

$$\|x_{n+1} - q\| \le \delta^3 (1 - \alpha_n \beta_n (1-\delta))\|x_n - q\|$$

Since $0 \le \delta < 1, \alpha_n \in [0,1)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Hence by using lemma (1.6), we have

$$\lim_{n \to \infty} \|x_{n+1} - q\| = 0.$$

Hence the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to *q*.

Corollary 2.2: (Akewe and Okeke [14]) Let X be a Banach space and $T: X \to X$ be a mapping satisfying the condition

$$\|Tx - q\| \le \delta \|x - q\|$$

where $q \in F, x \in X$ and $0 \le \delta < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the Picard-Mann hybrid iterative process given by (1.3). Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to *q*.

Remark 2.3: Theorem 2.1 gives generalization to many results in the literature by considering a wider class of contractive type operators and more general iterative process, including the results of Chidume [15], Bosede and Rhoades [16] and Akewe and Okeke [14].

Theorem 2.4: Let X be a Banach space and $T: X \to X$ be a mapping satisfying the condition

$$\|Tx - q\| \le \delta \|x - q\|$$

where $q \in F, x \in X$ and $0 \le \delta < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the K-iterative process given by (1.7). Then the iteration process (1.7) is T-stable.

Proof: By theorem 2.1, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to *q*. Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ be real sequences in *X*.

Let $\varphi_n = \|u_{n+1} - Tv_n\|, n = 0,1,2,\cdots$, where

$$w_n = (1-\beta_n)u_n + \beta_n T u_n,$$

$$v_n = T\left((1-\alpha_n)Tu_n + \alpha_n T w_n\right),$$

$$u_{n+1} = Tv_n,$$

and let $\lim_{n\to\infty} \varphi_n = 0$.

We shall prove that $\lim_{n\to\infty} u_n = q$.

Now,

$$\begin{aligned} \|u_{n+1} - q\| &= \|u_{n+1} - Tv_n\| + \|Tv_n - q\| \\ &\le \varphi_n + \delta\|v_n - q\| \end{aligned} \tag{2.6}$$

$$\begin{aligned} \|v_n - q\| &= \left\|T\left((1-\alpha_n)Tu_n + \alpha_n Tw_n\right) - q\right\| \\ &\le \delta\left\|(1-\alpha_n)Tu_n + \alpha_n Tw_n - q\right\| \\ &\le \delta\left\|(1-\alpha_n)(Tu_n - q) + \alpha_n (Tw_n - q)\right\| \\ &\le \delta\left[(1-\alpha_n)\|Tu_n - q\| + \alpha_n\|Tw_n - q\|\right] \\ &\le \delta\left[(1-\alpha_n)\|Tu_n - q\| + \alpha_n\|Tw_n - q\|\right] \\ &\le \delta^2\left[(1-\alpha_n)\|u_n - q\| + \alpha_n\|w_n - q\|\right] \end{aligned} \tag{2.7}$$

Again using (1.7) and (2.1) we get,

$$\begin{aligned} \|w_n - q\| &= \|(1-\beta_n)u_n + \beta_n Tu_n - q\| \\ &\le (1-\beta_n)\|u_n - q\| + \beta_n\|Tu_n - q\| \\ &\le (1-\beta_n)\|u_n - q\| + \beta_n\delta\|u_n - q\| \\ &\le \left(1-\beta_n(1-\delta)\right)\|u_n - q\| \end{aligned} \tag{2.8}$$

Using (2.8) in (2.7) we get,

$$\begin{aligned} \|v_n - q\| &\le \delta^2\left[(1-\alpha_n)\|u_n - q\| + \alpha_n\left(1-\beta_n(1-\delta)\right)\|u_n - q\|\right] \\ &\le \delta^2\left(1-\alpha_n\beta_n(1-\delta)\right)\|u_n - q\| \end{aligned} \tag{2.9}$$

Using (2.9) in (2.6) we get,

$$\|u_{n+1} - q\| \le \varphi_n + \delta^3\left(1-\alpha_n\beta_n(1-\delta)\right)\|u_n - q\| \tag{2.10}$$

Since $0 \le \delta < 1$ and since $0 \le \alpha_n, \beta_n \le 1$ we have by lemma (1.6)

$$\lim_{n\to\infty} u_n = q.$$

Conversely let $\lim_{n\to\infty} u_n = q$. We shall show that $\lim_{n\to\infty} \varphi_n = 0$.

Now

$$\begin{aligned}\varphi_n &= \|u_{n+1} - Tv_n\| \\ &\le \|u_{n+1} - q\| + \|Tq - Tv_n\| \\ &\le \|u_{n+1} - q\| + \delta\|v_n - q\|\end{aligned} \tag{2.11}$$

Substituting (2.9) in (2.11),

$$\varphi_n \le \|u_{n+1} - q\| + \delta^3\left(1 - \alpha_n\beta_n(1-\delta)\right)\|u_n - q\| \tag{2.12}$$

Since $\lim_{n\to\infty} u_n = q$, we have from (2.12) $\lim_{n\to\infty} \varphi_n = 0$. Hence the K-iteration scheme is T-stable.

From theorem 2.4, we have the following corollary.

Corollary 2.5: Let X be a Banach space and $T : X \to X$ be a mapping satisfying the condition

$$\|Tx - q\| \le \delta\|x - q\|,$$

where $q \in F, x \in X$ and $0 \le \delta < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the Picard-Mann hybrid iterative process given by (1.3). Then the iteration process (1.3) is T-stable.

Example 2.6: Let $X = [0,1]$ and consider the mapping $Tx = \frac{x}{2}$. The clearly the mapping T satisfies the inequality (2.1). Now $F(T) = 0$. Now we claim that the K-iteration scheme (1.7) is T-stable. Let us take $\alpha_n = \beta_n = \frac{1}{2}$ and consider the sequences $x_n = y_n = z_n = \frac{1}{n}$. Then clearly $\lim_{n\to\infty} x_n = 0$.

Now

$$\begin{aligned}\varphi_n &= \|x_{n+1} - Ty_n\| = \left\|x_{n+1} - \frac{y_n}{2}\right\| \\ &= \left\|x_{n+1} - \frac{T\left((1-\alpha_n)Tx_n + \alpha_n Tz_n\right)}{2}\right\| \\ &= \left\|x_{n+1} - \frac{(1-\alpha_n)Tx_n + \alpha_n Tz_n}{4}\right\| \\ &= \left\|x_{n+1} - \left(\frac{(1-\alpha_n)x_n}{8} + \frac{\alpha_n z_n}{8}\right)\right\| \\ &= \left\|x_{n+1} - \left(\frac{(1-\alpha_n)x_n}{8} + \frac{\alpha_n(1-\beta_n)x_n}{8} + \frac{\alpha_n\beta_n Tx_n}{8}\right)\right\| \\ &= \left\|x_{n+1} - \left(\frac{(1-\alpha_n)x_n}{8} + \frac{\alpha_n(1-\beta_n)x_n}{8} + \frac{\alpha_n\beta_n x_n}{16}\right)\right\| \\ &= \left\|\frac{1}{n+1} - \left(\frac{1}{16n} + \frac{1}{32n} + \frac{1}{64n}\right)\right\| = \left\|\frac{1}{n+1} - \frac{1}{8n}\right\|\end{aligned}$$

$$=\left\|\frac{7n-1}{8n(n+1)}\right\|=\left\|\frac{7-\frac{1}{n}}{8(n+1)}\right\| \qquad (2.13)$$

Taking limit $n\to\infty$ in (2.13), we have $\lim_{n\to\infty}\varphi_n=0$. Hence the K-iteration process is T-stable.

Now we shall prove the convergence and stability results for asymptotically quasi-nonexpansive mapping by considering the more general form of K-iteration process as:

$$z_n=(1-\beta_n)x_n+\beta_n T^n x_n,$$
$$y_n=T^n\left((1-\alpha_n)T^n x_n+\alpha_n T^n z_n\right),$$
$$x_{n+1}=T^n y_n, \text{ where } n=0,1,2,\cdots, \qquad (2.14)$$

Theorem 2.7: Let H be a non-empty closed convex subset of a Banach space X and $T:H\to H$ be asymptotically quasi-nonexpansive mapping with real sequence $\mu_n\subseteq[0,\infty)$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the K-iterative process given by (2.14) and satisfies the assumption that $\sum_{n=0}^{\infty}\alpha_n\beta_n\mu_n=\infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some fixed point q of the mapping T.

Proof: From the iterative process (2.14) we have,

$$\begin{aligned}\|z_n-q\|&=\left\|(1-\beta_n)x_n+\beta_n T^n x_n-q\right\|\\&\le(1-\beta_n)\|x_n-q\|+\beta_n\left\|T^n x_n-q\right\|\\&\le(1-\beta_n)\|x_n-q\|+\beta_n(1+\mu_n)\|x_n-q\|\\&\le(1+\beta_n\mu_n)\|x_n-q\|\end{aligned} \qquad (2.15)$$

and

$$\begin{aligned}\|y_n-q\|&=\left\|T^n\left((1-\alpha_n)T^n x_n+\alpha_n T^n z_n\right)-q\right\|\\&\le(1+\mu_n)\left\|(1-\alpha_n)T^n x_n+\alpha_n T^n z_n-q\right\|\\&\le(1+\mu_n)\left\|(1-\alpha_n)(T^n x_n-q)+\alpha_n(T^n z_n-q)\right\|\\&\le(1+\mu_n)\left[(1-\alpha_n)\left\|T^n x_n-q\right\|+\alpha_n\left\|T^n z_n-q\right\|\right]\\&\le(1+\mu_n)\left[(1-\alpha_n)(1+\mu_n)\|x_n-q\|+\alpha_n(1+\mu_n)\|z_n-q\|\right]\\&\le(1+\mu_n)^2\left[(1-\alpha_n)\|x_n-q\|+\alpha_n\|z_n-q\|\right]\\&\le(1+\mu_n)^2\left[(1-\alpha_n)\|x_n-q\|+\alpha_n(1+\beta_n\mu_n)\|x_n-q\|\right]\\&\le(1+\mu_n)^2(1-\alpha_n\beta_n\mu_n)\|x_n-q\|\end{aligned} \qquad (2.16)$$

Again using (2.14) we have,

$$\begin{aligned}\|x_{n+1}-q\|&\le\left\|T^n y_n-q\right\|\\&\le(1+\mu_n)\|y_n-q\|\\&\le(1+\mu_n)^3(1-\alpha_n\beta_n\mu_n)\|x_n-q\|\end{aligned} \qquad (2.17)$$

By repeating the above process, we have the following inequalities

$$\|x_{n+1}-q\| \le (1+\mu_n)^3 (1-\alpha_n\beta_n\mu_n)\|x_n-q\|$$
$$\|x_n-q\| \le (1+\mu_{n-1})^3 (1-\alpha_{n-1}\beta_{n-1}\mu_{n-1})\|x_{n-1}-q\|$$
$$\|x_{n-1}-q\| \le (1+\mu_{n-2})^3 (1-\alpha_{n-2}\beta_{n-2}\mu_{n-2})\|x_{n-2}-q\|$$
$$\cdots$$
$$\|x_1-q\| \le (1+\mu_0)^3 (1-\alpha_0\beta_0\mu_0)\|x_0-q\|$$

So we can write,

$$\|x_{n+1}-q\| \le (1+\mu_0)^{3(n+1)}\|x_0-q\|\prod_{j=0}^{n}(1-\alpha_j\beta_j\mu_j)$$

Since $1-x \le \mathrm{e}^{-x}$ for all $x\in[0,1]$. Now $1-\alpha_j\beta_j\mu_j < 1$, so we can write,

$$\begin{aligned}\|x_{n+1}-q\| &\le (1+\mu_0)^{3(n+1)}\|x_0-q\|\mathrm{e}^{-(1-\alpha_j\beta_j\mu_j)} \\ &\le (1+\mu_0)^{3(n+1)}\|x_0-q\|\mathrm{e}^{-\sum_{j=0}^{n}\alpha_j\beta_j\mu_j}\end{aligned} \tag{2.18}$$

Taking limit $n\to\infty$ in (2.18), we have $\lim_{n\to\infty}\|x_n-q\|=0$, that is the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to fixed point q of the mapping T.

Theorem 2.8: Let H be a non-empty closed convex subset of a Banach space X and $T:H\to H$ be asymptotically quasi-nonexpansive mapping with real sequence $\mu_n\subseteq[0,\infty)$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the K-iterative process given by (2.14) and satisfies the assumption that $\sum_{n=0}^{\infty}\alpha_n\beta_n\mu_n=\infty$. Then the iterative process (2.14) is T-stable.

Proof: Let $\{u_n\}_{n=0}^{\infty}\subset X$ be any arbitrary sequence. Let the sequence generated by the iterative process (2.14) is $x_{n+1}=f(T,x_n)$ converging to the fixed point q.

Let $\varphi_n=\|u_{n+1}-f(T,x_n)\|$.

We shall prove that $\lim_{n\to\infty}\varphi_n=0$ if and only if $\lim_{n\to\infty}u_n=q$.

First suppose $\lim_{n\to\infty}\varphi_n=0$. Now we have

$$\begin{aligned}\|u_{n+1}-q\| &= \|u_{n+1}-f(T,u_n)\|+\|f(T,u_n)-q\| \\ &= \varphi_n+\left\|T^n\left(T^n(1-\beta_n)T^nu_n+\beta_nT^n\left((1-\alpha_n)u_n+\alpha_nT^nu_n\right)\right)-q\right\| \\ &\le \varphi_n+(1+\mu_n)^3(1-\alpha_n\beta_n\mu_n)\|x_n-q\|\end{aligned} \tag{2.19}$$

where $\alpha_n,\beta_n\in[0,1]$, $\lim_{n\to\infty}\varphi_n=0$ and $\lim_{n\to\infty}\mu_n=0$.

Now using (2.19) together with lemma (1.5), we have $\lim_{n\to\infty}\|u_n-q\|=0$ that is $\lim_{n\to\infty}u_n=q$.

Conversely let $\lim_{n\to\infty}u_n=q$. we have

$$\begin{aligned}\varphi_n &= \|u_{n+1}-f(T,u_n)\| \\ &\le \|u_{n+1}-q\|+\|f(T,u_n)-q\| \\ &\le \|u_{n+1}-q\|+(1+\mu_n)^3(1-\alpha_n\beta_n\mu_n)\|u_n-q\|\end{aligned}$$

Taking limit $n\to\infty$ both sides of (6) we have $\lim_{n\to\infty}\varphi_n=0$. Hence (2.14) is T-stable.

Now we shall prove the convergence results for mean non-expansive mapping by modifying the K-iteration process for two mappings as:

$$z_n = (1-\beta_n)x_n + \beta_n S x_n,$$
$$y_n = T\left((1-\alpha_n)Sx_n + \alpha_n T z_n\right),$$
$$x_{n+1} = Ty_n, \text{ where } n = 0,1,2,\cdots, \tag{2.20}$$

Lemma 2.9: Let H be a non-empty closed convex subset of a Banach space X and $S,T : H \to H$ be two mean non-expansive mapping such that $F = F(T) \cap F(S) \neq \phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the K-iterative process given by (2.20). Then $\lim_{n\to\infty}\|x_n - q\|$ exists for some $q \in F$.

Proof: We have

$$\begin{aligned}
\|z_n - q\| &= \|(1-\beta_n)x_n + \beta_n Sx_n - q\| \\
&\le (1-\beta_n)\|x_n - q\| + \beta_n\|Sx_n - q\| \\
&\le (1-\beta_n)\|x_n - q\| + \beta_n\left(a_1\|x_n - q\| + b_1\|x_n - q\|\right) \\
&\le (1-\beta_n)\|x_n - q\| + \beta_n(a_1 + b_1)\|x_n - q\| \\
&\le \|x_n - q\|
\end{aligned} \tag{2.21}$$

Again using (2.20) and (2.21)

$$\begin{aligned}
\|y_n - q\| &= \left\|T\left((1-\alpha_n)Sx_n + \alpha_n Tz_n\right) - q\right\| \\
&\le a_2\left\|\left((1-\alpha_n)Sx_n + \alpha_n Tz_n\right) - q\right\| + b_2\left\|\left((1-\alpha_n)Sx_n + \alpha_n Tz_n\right) - q\right\| \\
&\le (a_2 + b_2)\left\|\left((1-\alpha_n)Sx_n + \alpha_n Tz_n\right) - q\right\| \\
&\le (1-\alpha_n)\|Sx_n - q\| + \alpha_n\|Tz_n - q\| \\
&\le (1-\alpha_n)\left(a_1\|x_n - q\| + b_1\|x_n - q\|\right) + \alpha_n\left(a_2\|z_n - q\| + b_2\|z_n - q\|\right) \\
&\le (1-\alpha_n)(a_1 + b_1)\|x_n - q\| + \alpha_n(a_2 + b_2)\|z_n - q\| \\
&\le (1-\alpha_n)\|x_n - q\| + \alpha_n\|z_n - q\| \\
&\le \|x_n - q\|
\end{aligned} \tag{2.22}$$

Again using (2.20) and (2.22)

$$\begin{aligned}
\|x_{n+1} - q\| &\le \|Ty_n - q\| \\
&\le a_2\|y_n - q\| + b_2\|y_n - q\| \\
&\le (a_2 + b_2)\|y_n - q\| \\
&\le \|y_n - q\| \\
&\le \|x_n - q\|
\end{aligned} \tag{2.23}$$

This shows that $\{\|x_n - q\|\}$ is non-increasing and bounded sequence for $q \in F$. Hence $\lim_{n\to\infty}\|x_n - q\|$ exists.

Lemma 2.10: Let H be a non-empty closed convex subset of a Banach space X and $S,T : H \to H$ be two mean non-expansive mapping such that $F = F(T) \cap F(S) \neq \phi$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the K-iterative process given by (2.20). Also consider that

$\lim_{n\to\infty}\|Sx_n - q\| = \lim_{n\to\infty}\|Tx_n - q\| = 0$ for some $q \in F$. Then $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Proof: Let $q \in F$. In lemma (2.9) we have proved the existence of

$$\lim_{n\to\infty} \|x_n - q\|. \text{ Let } \lim_{n\to\infty} \|x_n - q\| = c. \tag{2.24}$$

W.L.O.G. let $c > 0$.

Now from (2.20) and (2.24) we have,

$$\limsup_{n\to\infty} \|z_n - q\| \le \limsup_{n\to\infty} \|x_n - q\| = c \tag{2.25}$$

Now

$$\begin{aligned} \|Sx_n - q\| &\le a_1 \|x_n - q\| + b_1 \|x_n - q\| \\ &\le (a_1 + b_1)\|x_n - q\| \le \|x_n - q\| \end{aligned}$$

$$\text{Implies that } \limsup_{n\to\infty} \|Sx_n - q\| \le \limsup_{n\to\infty} \|x_n - q\| = c \tag{2.26}$$

Now

$$\begin{aligned} \|x_{n+1} - q\| &\le \|Ty_n - q\| \le a_2 \|y_n - q\| + b_2 \|y_n - q\| \\ &\le (a_2 + b_2)\|y_n - q\| \le \|y_n - q\| \\ &\le \|T((1-\alpha_n)Sx_n + \alpha_n Tz_n) - q\| \\ &\le a_2 \|((1-\alpha_n)Sx_n + \alpha_n Tz_n) - q\| + b_2 \|((1-\alpha_n)Sx_n + \alpha_n Tz_n) - q\| \\ &\le (a_2 + b_2)\|((1-\alpha_n)Sx_n + \alpha_n Tz_n) - q\| \end{aligned}$$

$$\begin{aligned} &\le (1-\alpha_n)\|Sx_n - q\| + \alpha_n \|Tz_n - q\| \\ &\le (1-\alpha_n)(a_1 \|x_n - q\| + b_1 \|x_n - q\|) + \alpha_n (a_2 \|z_n - q\| + b_2 \|z_n - q\|) \\ &\le (1-\alpha_n)(a_1 + b_1)\|x_n - q\| + \alpha_n (a_2 + b_2)\|z_n - q\| \\ &\le (1-\alpha_n)\|x_n - q\| + \alpha_n \|z_n - q\| \\ &\le \|x_n - q\| - \alpha_n \|x_n - q\| + \alpha_n \|z_n - q\| \end{aligned}$$

$$\Rightarrow \frac{\|x_{n+1} - q\| - \|x_n - q\|}{\alpha_n} = \|z_n - q\| - \|x_n - q\|$$

and hence

$$\|x_{n+1} - q\| - \|x_n - q\| \le \frac{\|x_{n+1} - q\| - \|x_n - q\|}{\alpha_n} = \|z_n - q\| - \|x_n - q\|$$

$$\text{which implies that } \|x_{n+1} - q\| \le \|z_n - q\| \tag{2.27}$$

Taking limit inferior in (2.27) we obtain

$$c \le \liminf_{n\to\infty} \|z_n - q\| \tag{2.28}$$

From (2.20) and (2.28) we have

$$\begin{aligned} c &= \lim_{n\to\infty} \|z_n - q\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n Sx_n - q\| \\ &= \lim_{n\to\infty} \|\beta_n (Sx_n - q) + (1-\beta_n)(x_n - q)\| \end{aligned} \tag{2.29}$$

Now from (2.24), (2.26), (2.29) and lemma (1.7), we have $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.

Now,

$$\|Tx_n - q\| \le a_2 \|x_n - q\| + b_2 \|x_n - q\| \le \|x_n - q\|$$
$$\Rightarrow \limsup_{n\to\infty} \|Tx_n - q\| \le \limsup_{n\to\infty} \|x_n - q\| \le c \tag{2.30}$$

Using the conditions of the lemma in (2.30), we can write

$$C = \lim_{n\to\infty} \|\beta_n (Tx_n - q) + (1-\beta_n)(x_n - q)\| \tag{2.31}$$

Using (2.24), (2.30), (2.31) along with the lemma (1.7), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Theorem 2.11: Let H be a non-empty closed convex subset of a Banach space X satisfying Opial's condition and S, T and $\{x_n\}_{n=0}^{\infty}$ be same as defined in the lemma (2.10) .Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to some $q \in F$.

Proof: From lemma (2.10) we have, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Since X is uniformly convex and hence it is reflexive so there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $\{x_{n_m}\}$ converges weakly to some $q_1 \in F$. Since H is closed so $q_1 \in H$. Now we claim the weak convergence of $\{x_n\}$ to q_1 . Let it is not true, then there exists a subsequence of $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to q_2 and let $q_1 \ne q_2$. Also $q_2 \in F$. Now from lemma (2.9) $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ both exist. Using Opial's condition we have,

$$\lim_{n\to\infty} \|x_n - q_1\| \le \lim_{n\to\infty} \|x_{n_m} - q_1\| < \lim_{n\to\infty} \|x_{n_m} - q_2\|$$
$$= \lim_{n\to\infty} \|x_n - q_2\| = \lim_{n\to\infty} \|x_{n_i} - q_2\|$$
$$< \lim_{n\to\infty} \|x_{n_i} - q_1\| \le \lim_{n\to\infty} \|x_n - q_1\|$$

This is a contradiction, so we must have $q_1 = q_2$. Thus the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to some $q \in F$.

Theorem 2.12: Let H be a non-empty closed compact subset of a Banach space X and S, T and $\{x_n\}_{n=0}^{\infty}$ be same as defined in the lemma (2.10). Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $q \in F$.

Proof: Since H is compact and hence it is sequentially compact. So there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to $q \in H$.

Now

$$\|x_{n_i} - Tq\| = \|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tq\|$$
$$\le \|x_{n_i} - Tx_{n_i}\| + a_2 \|x_{n_i} - q\| + b_2 \|x_{n_i} - q\| \tag{2.32}$$
$$\le \|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - q\|$$

Taking limit $n \to \infty$ in (2.32) we have, $Tq = q$ that is $q \in F$. We have earlier proved that $\lim_{n\to\infty} \|x_n - q\|$ exists for $q \in F$. Hence the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $q \in F$.

In [8] it is proves that the K-iteration process converges faster than Picard-S, Thakur-New and Vatan two-step iterative process. Now we shall compare the rate of convergence the K-iteration process defined in [8] and our new modified K-iteration process for two mappings.

Table 1. Iterative values of K-iteration process and Modified K-iteration process.

	K-iteration	Modified K-iteration
x_0	2.25	2.25
x_1	2.030273437500000	2.013360362386860
x_2	2.003665924072266	2.000717402289730
x_3	2.000443920493126	2.000038531785984
x_4	2.000053755997215	2.000002069576723
x_5	2.000006509515288	2.000000111158901
x_6	2.000000788261617	2.000000005970444
x_7	2.000000095453555	2.000000000320678
x_8	2.000000011558829	2.000000000017224
x_9	2.000000001399702	2.000000000000925
x_{10}	2.000000000169495	2.000000000000050
x_{11}	2.000000000020525	2.000000000000003
x_{12}	2.000000000002486	**2.000000000000000**
x_{13}	2.000000000000301	**2.000000000000000**
x_{14}	2.000000000000036	**2.000000000000000**
x_{15}	2.000000000000004	**2.000000000000000**
x_{16}	**2.000000000000000**	**2.000000000000000**

Example 2.13: Let $S,T:[0,3]\to[0,3]$ be two mappings defined by $T(x)=\frac{x+2}{2}$ and $s(x)=(x+2)^{\frac{1}{2}}$. Let α_n,β_n be the sequences defined by $\alpha_n=\beta_n=\frac{1}{4}$. Let the initial approximation be $x_0=2.25$. Clearly S, T has unique common fixed point 2. The convergence pattern of K-iteration process and modified K-iteration process is shown in Table 1.

Clearly we can conclude from Table 1, that the modified K-iteration process has better rate of convergence than the k-iteration process.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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