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On Almost Pretopological Vector Spaces

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Abstract

In this paper, we introduce the notion of almost pretopological vector spaces and present some examples of almost pretopological vector spaces. Almost pretopological vector spaces are defined by using regular open sets and pre-open sets. The relationships of almost pretopological vector spaces with certain other types of spaces are provided. Along with some useful results, it is proved that in almost pretopological vector spaces, translation and scalar multiple of a regular open (resp. regular closed) set are pre-open (resp. pre-closed).

Subject Areas

Functional Analysis, Topology

Keywords

Pre-Open Sets, Regular Open Sets, δ-Open Sets, Almost Pretopological Vector Spaces

1. Introduction

The concept of topological vector spaces was first introduced and studied by Kolmogroff [1] in 1934. Due to nice properties, the notion earns a great importance in fixed point theory, operator theory and various other advanced branches of Mathematics. In 2015, M. Khan *et al.* [2] introduced the notion of s-topological vector spaces which is a generalization of topological vector spaces. Later, in 2016, M. Khan and M.A. Iqbal [3] introduced and studied another class of spaces called irresolute topological vector spaces which is contained in the class of s-topological vector spaces but independent of topological vector spaces. Ibrahim [4] initiated the study of α-topological vector spaces.

The purpose of the present paper is to introduce a new class of spaces, namely almost pretopological vector spaces. These spaces are defined by using regular open sets and pre-open sets. The relationships of almost pretopological vector spaces with certain other types of spaces are presented. Some basic properties of almost pretopological vector spaces are given.

2. Preliminaries

Throughout this paper, (X,τ) (or simply X) means a topological space. For a subset $A \subseteq X$, the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. The notation $\mathbb{R}$ (resp. $\mathbb{C}$) represents the set of real numbers (resp. the set of complex numbers) and ϵ (resp. η) represents negligibly small positive number.

Definition 2.1. Let X be a topological space. A subset A of X is called

1) regular open if $A = Int(Cl(A))$.

2) pre-open [5] if $A \subseteq Int(Cl(A))$.

3) semi-open [6] if $A \subseteq Cl(Int(A))$.

Definition 2.2. A subset A of a topological space X is said to be δ-open [7] if for each $\in A$, there exists a regular open set U in X such that $x \in U \subseteq A$.

The complement of a regular open (resp. pre-open, semi-open, δ-open) set is called regular closed (resp. pre-closed, semi-closed, δ-closed [7]). The intersection of all pre-closed (resp. δ-closed) sets containing a subset A of X is called the pre-closure (resp. δ-closure) of A and is denoted by $pCl(A)$ (resp. $Cl_\delta(A)$. It is known that a subset A of X is pre-closed (resp. δ-closed) if and only if $A = pCl(A)$ (resp. $A = Cl_\delta(A)$). A point $x \in pCl(A)$ (resp. $Cl_\delta(A)$) if and only if $A \cap U \neq \varnothing$ for each pre-open (resp. regularly open) set U in X containing x. The union of all pre-open (resp. δ-open) sets in X that are contained in $A \subseteq X$ is called the pre-interior (resp. δ-interior) of A and is denoted by $pInt(A)$ (resp. $Int_\delta(A)$). A point $x \in X$ is called a pre-interior point of $A \subseteq X$ if there exists a pre-open set U in X such that $x \in U \subseteq A$. The set of all pre-interior points of A is equal to $pInt(A)$. It is well-known that a subset $A \subseteq X$ is pre-open (resp. δ-open) if and only if $A = pInt(A)$ (resp. $A = Int_\delta(A)$).

The family of all regular open (resp. pre-open, pre-closed) sets in X is denoted by $RO(X)$ (resp. $PO(X)$, $PC(X)$). The family of all pre-open sets in X containing x is denoted by $PO(X, x)$. If $A \in PO(X), B \in PO(Y)$ (X and Y are topological spaces), then $A \times B \in PO(X \times Y)$ (with respect to product topology).

Definition 2.3. [8] A function $f: X \to Y$ from a topological space X to a topological space Y is called almost precontinous if for each $x \in X$ and each regular open set $V \subseteq Y$ containing $f(x)$, there exists a pre-open set $U \subseteq X$ containing x such that $f(U) \subseteq V$.

Also, we recall some definitions that will be used later.

Definition 2.4. Let L be a vector space over the field F ($\mathbb{R}$ or $\mathbb{C}$). Let T be a topology on L such that

1) For each $x, y \in L$ and each open neighbourhood W of $x + y$ in L, there

exist open neighborhoods U and V of x and y respectively, in L such that $U+V\subseteq W$ and

2) For each $\lambda\in F, x\in L$ and each open neighbourhood W of λx in L, there exist open neighborhoods U of λ in F and V of x in L such that $U\cdot V\subseteq W$.

Then the pair $\left(L_{(F)},T\right)$ is called topological vector space.

Definition 2.5. [2] Let L be a vector space over the field F ($\mathbb{R}$ or $\mathbb{C}$) and Let T be a topology on L such that

1) For each $x,y\in L$ and each open set W in L containing $x+y$, there exist semi-open sets U and V in L containing x and y respectively such that $U+V\subseteq W$ and

2) For each $\lambda\in F, x\in L$ and each open set W in L containing λx, there exist semi-open sets U in F containing λ and V in L containing x such that $U\cdot V\subseteq W$.

Then the pair $\left(L_{(F)},T\right)$ is called s-topological vector space.

Definition 2.6. [3] Let L be a vector space over the field F ($\mathbb{R}$ or $\mathbb{C}$) and Let T be a topology on L such that

3) For each $x,y\in L$ and each semi-open set W in L containing $x+y$, there exist semi-open sets U and V in L containing x and y respectively such that $U+V\subseteq W$ and

4) For each $\lambda\in F, x\in L$ and each semi-open set W in L containing λx, there exist semi-open sets U in F containing λ and V in L containing x such that $U\cdot V\subseteq W$.

Then the pair $\left(L_{(F)},T\right)$ is called irresolute topological vector space.

3. Almost Pretopological Vector Spaces

In this section, we define almost pretopological vector spaces and investigate their relationships with certain other types of spaces. Some general properties of almost pretopological vector spaces are also discussed.

Definition 3.1. Let E be a vector space over the field K, where $K=\mathbb{R}$ or $\mathbb{C}$ with standard topology. Let τ be a topology on E such that the following conditions are satisfied:

1) For each $x,y\in E$ and each regular open set $W\subseteq E$ containing $x+y$, there exist pre-open sets U and V in E containing x and y respectively, such that $U+V\subseteq W$, and

2) For each $x,y\in E$ and each regular open set $W\subseteq E$ containing λx, there exist pre-open sets U in K containing λ and V in E containing x such that $U\cdot V\subseteq W$.

Then the pair $\left(E_{(K)},\tau\right)$ is called an almost pretopological vector space (written in short, APTVS).

In order to catch the basic grasp of almost pretopological vector spaces, we first present some examples of almost pretopological vector spaces and then through these examples, we investigate their relationships with some other existing spaces.

Example 3.1. Consider the field $K=\mathbb{R}$ with standard topology. Let $E=\mathbb{R}$ be endowed with the topology τ on E generated by the base $B=\{(a,b):a,b\in\mathbb{R}\}$. We show that $(E_{(K)},\tau)$ is an almost pretopological vector space. For this, we have to check the following:

1) Let $x,y\in X$. Consider any regular open set $W=(x+y-\epsilon,x+y+\epsilon)$ in E containing $x+y$. Then we can choose pre-open sets $U=(x-\eta,x+\eta)$ and $V=(y-\eta,y+\eta)$ in E containing x and y respectively, such that $U+V\subseteq W$ for each $\eta<\frac{\epsilon}{2}$. This verifies the first condition of the definition of almost pretopological vector spaces.

2) Let $\lambda\in\mathbb{R}$ and $x\in E$. Consider a regular open set $W=(\lambda x-\epsilon,\lambda x+\epsilon)$ in E containing λx. We have following cases:

Case (I). If $\lambda>0$ and $x>0$, then for the choice of pre-open sets $U=(\lambda-\eta,\lambda+\eta)$ in $\mathbb{R}$ containing λ and $V=(x-\eta,x+\eta)$ in E containing x, we have $U\cdot V\subseteq W$ for each $\eta<\frac{\epsilon}{\lambda+x+1}$.

Case (II). If $\lambda<0$ and $x<0$, then $\lambda x>0$. We can choose pre-open neighborhoods $U=(\lambda-\eta,\lambda+\eta)$ of λ in $\mathbb{R}$ and $V=(x-\eta,x+\eta)$ of x in E such that $U\cdot V\subseteq W$ for sufficiently appropriate $\eta\leq\frac{-\epsilon}{\lambda+x-1}$.

Case (III). If $\lambda=0$ and $x>0$, (resp. $\lambda>0$ and $x=0$). Then, for the selection of pre-open neighborhoods $U=(-\eta,\eta)$ (resp. $U=(\lambda-\eta,\lambda+\eta)$) of λ in $\mathbb{R}$ and $V=(x-\eta,x+\eta)$ (resp. $V=(-\eta,\eta)$) of x in E, we see that $U\cdot V\subseteq W$ for each $\eta<\frac{\epsilon}{x+1}$ (resp. $\eta<\frac{\epsilon}{\lambda+1}$).

Case (IV). If $\lambda=0$ and $x<0$, (resp. $\lambda<0$ and $x=0$). Then, for the selection of pre-open sets $U=(-\eta,\eta)$ (resp. $U=(\lambda-\eta,\lambda+\eta)$) in $\mathbb{R}$ containing λ and $V=(x-\eta,x+\eta)$ (resp. $V=(-\eta,\eta)$) in E containing x, we have $U\cdot V\subseteq W$ for every $\eta<\frac{\epsilon}{1-x}$ (resp. $\eta<\frac{\epsilon}{1-\lambda}$).

Case (V). If $\lambda=0$ and $x=0$. Then, for pre-open neighborhoods $U=(-\eta,\eta)$) of λ in $\mathbb{R}$ and $U=(-\eta,\eta)$ of x in E, we have $U\cdot V\subseteq W$ for each $\eta<\sqrt{\epsilon}$.

This proves that the pair $(E_{(R)},\tau)$ is an almost pretopological vector space.

After tasting this example, an immediate question that comes into mind: is there any other topology on $\mathbb{R}$ which turns it out an almost pretopological vector space. The answer is in affirmative. In fact, there are many other topologies on $\mathbb{R}$ which turn it out an almost pretopological vector space. Let us present some of them.

Example 3.2. Consider the field $K=\mathbb{R}$ with standard topology. Let $E=\mathbb{R}$ be the vector space over the field K, where E is endowed with the topology $\tau=\{\varnothing,\{0\},\mathbb{R}\}$. Then $(E_{(K)},\tau)$ is an almost pretopological vector space.

Example 3.3. Let $E=\mathbb{R}$ be the vector space of real numbers with the topology τ generated by the base $B=\{(a,b):a,b\in\mathbb{R}\}\cup\{(c,d)\cap D:c,d\in\mathbb{R}$ and D

denotes the set of irrational numbers}. Then $\left(E_{(R)},\tau\right)$ is an almost pretopological vector space.

So far we have presented examples of almost pretopological vector spaces, we now present an example which lies beyond the class of almost pretopological vector spaces.

Example 3.4. Consider the field $K=\mathbb{R}$ with standard topology. Let $E=\mathbb{R}$ with the topology τ generated by the base $\mathcal{B}=\{(a,b),[1,c):a,b,c\in\mathbb{R}\}$. Then $\left(E_{(R)},\tau\right)$ is not an almost pretopological vector space because [1, 2) is a regular open set in E containing 1 = 1.1 but there do not exist pre-open sets U in K containing 1 and V in E containing 1 such that $U\cdot V\subseteq[1,2)$.

Remark 3.1. Clearly, by definition, every topological vector space is almost pretopological vector space but the converse is not true in general because, Example 3.2 and Example 3.3 are almost pretopological vector spaces which are not topological vector spaces.

Remark 3.2. Almost pretopological vector spaces are independent of s-topological vector spaces. It is easy to check that Example 3.2 is not an s-topological vector space. Furthermore, Example 3.4 is not an almost pretopological vector space which is shown an s-topological vector spaces in [2].

Theorem 3.1. *Let A be any δ-open subset of an almost pretoplogical vector space E. then the following are true:*

(i) $x+A\in PO(E)$ *for each* $x\in E$

(ii) $\lambda A\in PO(E)$ *for each non-zero scalar* λ.

Proof. (i) Let $y\in x+A$. Then $-x+y\in A$. Since A is δ-open set in E, there exists $W\in RO(E)$ such that $-x+y\in W\subseteq A$. Since E is an APTVS, there exist pre-open sets U in E containing $-x$ and V in E containing y such that $U+V\subseteq W$ and therein, $y\in V\subseteq x+A$. Consequently, $y\in pInt(x+A)$ and therefore $pInt(x+A)=x+A$. Thus, $x+A\in PO(E)$.

(ii) Let $x\in\lambda A$. Since A is δ-open, there exists a regularly open set W in E such that $\frac{1}{\lambda}x\in W\subseteq A$. By definition of almost pretoplological vector spaces, there exist pre-open sets $U\in PO\left(K,\frac{1}{\lambda}\right)$, $V\in PO(E,x)$ such that $U\cdot V\subseteq W\subseteq A$. This gives that $V\subseteq\lambda A$. Consequently, $\lambda A\in PO(E)$.

Corollary 3.1.1. *Let A and B be any δ-open subsets of an almost pretopological vector spaces E. Then* $A+B\in PO(E)$.

Corollary 3.1.2. *Let A be any δ-open subset of an almost pretopological vector space E. Then the following are true*:

(i) $x+A\subseteq Int(Cl(x+A))$ *for each* $x\in E$.

(ii) $\lambda A\in PC(E)$ *for each non-zero scalar* λ.

Theorem 3.2. *Let F be any δ-closed subset of an almost pretopological vector space E. Then the following are valid*:

(i) $x+F\in PC(E)$ *for each* $x\in E$.

(ii) $\lambda F\in PC(E)$ *for each non-zero scalar* λ.

Proof. (i) Suppose that $y \in pCl(x+F)$. If we consider $z=-x+y$ and let W be any open set in E containing z, then $Int(Cl(W))$ is regularly open set in E such that $z \in W \subseteq Int(Cl(W))$. The definition of almost pretopological vector spaces yields pre-open sets U and V in E containing $-x$ and y respectively, such that $U+V \subseteq Int(Cl(W))$. Since $y \in pCl(x+F)$, there is $r \in V \cap (x+F)$. Now $-x+r \in F \cap (U+V) \subseteq F \cap Int(Cl(W)) \Rightarrow F \cap Int(Cl(W)) \neq \varnothing$. Therefore, $z \in Cl_\delta(F)=F$ and, as a result, $y \in x+F$. This proves that $x+F=pCl(x+F)$ and hence $x+F \in PC(E)$.

(ii) Suppose that $x \in pCl(\lambda F)$. Let W be any open set in E containing $\frac{1}{\lambda}x$. By definition of almost pretopological vector spaces, there exist pre-open sets U in K containing $\frac{1}{\lambda}$ and V in E containing x such that $U \cdot V \subseteq Int(Cl(W))$. Since $x \in pCl(\lambda F)$, $V \cap (\lambda F) \neq \varnothing$. So, there is $r \in E$ such that $r \in V$ and $r \in \lambda F$. Now, $\frac{1}{\lambda} r \in F \cap (U \cdot V) \subseteq F \cap Int(Cl(W)) \Rightarrow F \cap Int(Cl(W)) \neq \varnothing$. Therefore, $x \in \lambda Cl_\delta(F)=\lambda F$ and thereby, $pCl(\lambda F)=\lambda F$. Hence the assertion follows.

Corollary 3.2.1. *Let F be any δ-closed subset of an almost pretopological vector space E. Then the following are valid*:

(i) $Cl(Int(x+F)) \subseteq x+F$ *for each* $x \in E$.

(ii) $Cl(Int(\lambda F)) \subseteq \lambda F$ *for each non-zero scalar* λ.

Now, we investigate further properties of almost pretopological vector spaces by using their basic idea.

Theorem 3.3. *For any subset A of an almost pretopological vector space E,* $x+pCl(A) \subseteq Cl_\delta(x+A)$ *for each* $x \in E$.

Proof. Let $y \in pCl(A)$ and let W be any open neighborhood of $z=x+y$ in E. Then, there exist pre-open sets $U \in PO(E,x)$ and $V \in PO(E,y)$ such that $U+V \subseteq Int(Cl(W))$. Since $y \in pCl(A)$, there is $a \in A \cap V$. Now $x+a \in (x+A) \cap (U+V) \subseteq (x+A) \cap Int(Cl(W))$ implies that $z \in Cl_\delta(x+A)$; that is, $x+pCl(A) \subseteq Cl_\delta(x+A)$. This completes the proof.

If the set A in Theorem 3.3 is replaced by the δ-open set, then the following inclusion holds in almost pretopological vector spaces.

Theorem 3.4. *For any δ-open subset of an almost pretopological vector space E,* $x+pCl(A) \subseteq Cl(x+A)$ *for each* $x \in E$.

Proof. Suppose that $y \in pCl(A)$ and let W be any open neighborhood of $z=x+y$ in E. Since $Int(Cl(W))$ is regularly open, by definition of almost pretopological vector spaces, we get pre-open sets U and V in E containing x and y respectively, such that $U+V \subseteq Int(Cl(W))$. Since $y \in pCl(A)$, $A \cap V \neq \varnothing$. So, there is $a \in A \cap V$.

Consequently, $x+a \in (x+A) \cap (U+V) \subseteq (x+A) \cap Int(Cl(W))$

$$\Rightarrow (x+A) \cap Int(Cl(W)) \neq \varnothing$$
$$\Rightarrow (x+A) \cap Cl(W) \neq \varnothing.$$

Since A is δ-open, $Int(Cl(x+A)) \cap Cl(W) \neq \varnothing$

$$\Rightarrow Int(Cl(x+A))\cap W \neq \varnothing$$
$$\Rightarrow Cl(x+A)\cap W \neq \varnothing .$$

Since W is open, $(x+A)\cap W \neq \varnothing$. This results in $z \in Cl(x+A)$; that is, $x + pCl(A) \subseteq Cl(x+A)$. The proof is finished.

The analogs of Theorem 3.3 and 3.4 respectively, are the following results:

Theorem 3.5. *For any subset A of an almost pretopological vector space E, $\lambda \cdot pCl(A) \subseteq Cl_\delta(\lambda A)$ for each non-zero scalar λ.*

Proof. The proof follows along the lines of the proof of Theorem 3.3.

Theorem 3.6. *For any δ-open subset A of an almost pretopological vector space E, $\lambda \cdot pCl(A) \subseteq Cl(\lambda A)$ for each non-zero scalar λ.*

Proof. Follows along the lines of the proof of Theorem 3.4.

Theorem 3.7. *Let A be any subset of an almost pretopological vector space E. Then the following assertions are true:*

(i) $pCl(x+A) \subseteq x + Cl_\delta(A)$ *for each* $x \in E$.

(ii) $pCl(\lambda A) \subseteq \lambda Cl_\delta(A)$ *for each non-zero scalar* λ.

Proof. (i) Let $y \in pCl(x+A)$. Consider $z = -x + y$ and let W be any open set in E containing *z*. Then, there exist pre-open sets $U, V \in PO(E)$ such that $-x \in U$, $y \in V$ and $U + V \subseteq Int(Cl(W))$. Now, $y \in pCl(x+A) \Rightarrow (x+A)\cap V \neq \varnothing \Rightarrow$ there is $r \in (x+A)\cap V$. This gives, $-x + r \in A\cap(U+V) \subseteq A\cap Int(Cl(W))$. This proves that $z \in Cl_\delta(A)$; that is, $y \in x + Cl_\delta(A)$. Hence the assertion follows.

(ii) Let $x \in pCl(\lambda A)$. Consider $y = \frac{1}{\lambda}x$. Let W be any open set in E containing *y*. By definition of almost pretopological vector space, we get pre-open sets $U \in PO\left(K, \frac{1}{\lambda}\right)$ and $V \in PO(E,x)$ such that $U \cdot V \subseteq Int(Cl(W))$. Since $x \in pCl(\lambda A)$, $(\lambda A)\cap V \neq \varnothing$. So, there is $a \in (\lambda A)\cap V$. Now $\frac{1}{\lambda}a \in A\cap(U \cdot V) \subseteq A\cap Int(Cl(W))$ showing that $A\cap Int(Cl(W)) \neq \varnothing$. Therefore, $y \in Cl_\delta(A)$; that is, $x \in \lambda Cl_\delta(A)$. Thereby the assertion follows.

Theorem 3.8. *For any open subset A of an almost pretopological vector space E, the following are true:*

(i) $pCl(x+A) \subseteq x + Cl(A)$ *for each* $x \in E$.

(ii) $pCl(\lambda A) \subseteq \lambda Cl(A)$ *for each non-zero scalar* λ.

Proof. (i) Assume that $y \in pCl(x+A)$. Let $z = -x + y$ and W be any open neighborhood of *z*. Then, there exist pre-open sets U containing $-x$ and V containing *y* in E such that $U + V \subseteq Int(Cl(W))$. Since V is pre-open, $(x+A)\cap V \neq \varnothing$. So, there is $a \in E$ such that $a \in x + A$ and $a \in V$. This implies $-x + a \in A\cap(U+V) \subseteq A\cap Int(Cl(W)) \Rightarrow A\cap Int(Cl(W)) \neq \varnothing$. Since A is open, $A\cap W \neq \varnothing$. This results in $y \in x + Cl(A)$. Thus, $pCl(x+A) \subseteq x + Cl(A)$.

(ii) Assume that $x \in pCl(\lambda A)$. Let W be any open set in E containing $\frac{1}{\lambda}x$. By definition 3.1, there exist pre-open sets U in the topological field K containing

$\frac{1}{\lambda}$ and V in E containing x such that $U \cdot V \subseteq Int(Cl(W))$. Since V is pre-open, there is $a \in (\lambda A) \cap V$. Thereby we get, $\frac{1}{\lambda} a \in \frac{1}{\lambda}$.
$\lambda A \cap (U \cdot V) \subseteq A \cap Int(Cl(W)) \Rightarrow A \cap Int(Cl(W)) \neq \varnothing \Rightarrow A \cap Cl(W) \neq \varnothing$.
Since A is open, $A \cap W \neq \varnothing$. This shows that $\frac{1}{\lambda} x \in Cl(A)$ and hence $x \in \lambda Cl(A)$. Thereby the assertion follows.

Theorem 3.9. *For any subset A of an almost pretopological vector space E, the following assertions are valid*:

(i) $Int_\delta(x+A) \subseteq x + pInt(A)$ *for each* $x \in E$.

(ii) $x + Int_\delta(A) \subseteq pInt(x+A)$ *for each* $x \in E$.

Proof. (i) Let $y \in Int_\delta(x+A)$. Then $y = x+a$ for some $a \in A$. Since $Int_\delta(x+A)$ is δ-open, there exists $W \in RO(E)$ such that $y \in W \subseteq Int_\delta(x+A)$. By definition of almost pretopological vector spaces, there exist pre-open sets $U, V \in PO(X)$ such that $x \in U, a \in V$ and $U+V \subseteq W \subseteq Int_\delta(x+A) \subseteq x+A \Rightarrow y = x+a \in x+V \subseteq U+V \subseteq x+A \Rightarrow y = x+a \in x+pInt(A)$. Hence the assertion follows.

(ii) Let $y \in Int_\delta(A)$. Then there exists $U \in RO(E)$ such that $y \in U \subseteq Int_\delta(A)$. This gives that $x+y \in x+U \subseteq x+A$. In view of Theorem 3.1, $x+U \in PO(E)$ and consequently, $x+y \in pInt(x+A)$. Thus, $x+Int_\delta(A) \subseteq pInt(x+A)$.

The analog of Theorem 3.9 is the following:

Theorem 3.10. *For any subset A of an almost pretopological vector space E, the following are true*:

(i) $Int_\delta(\lambda A) \subseteq \lambda pInt(A)$ *for each non-zero scalar* λ.

(ii) $\lambda Int_\delta(A) \subseteq pInt(\lambda \cdot A)$ *for each non-zero scalar* λ.

Theorem 3.11. *Let A be any δ-open subset of an almost pretopological vector space E. Then* $Int(x+A) \subseteq x + Int_\delta(A)$ *for each* $x \in E$.

Proof. Let $y \in Int(x+A)$. Then $-x+y \in A$. Since A is δ-open, there exists $W \in RO(E)$ such that $-x+y \in W \subseteq A = Int_\delta(A) \Rightarrow y \in x+Int_\delta(A)$. This proves that $Int(x+A) \subseteq x+Int_\delta(A)$.

Theorem 3.12. *Let A be any semi-closed set in an almost pretopological vector space E. Then*

(i) $x + Int(A) \subseteq pInt(x+A)$ *for each* $x \in E$.

(ii) $\lambda Int(A) \subseteq pInt(\lambda \cdot A)$ *for each non-zero scalar* λ.

Proof. (i) Assume that $y \in x+Int(A)$. Since A is semi-closed, $Int(A) \in RO(E)$ and as a result, there exist pre-open sets U and V in E such that $-x \in U, y \in V$ and $U+V \subseteq Int(A)$. Now, $-x+V \subseteq U+V \subseteq Int(A) \subseteq A \Rightarrow y \in V \subseteq x+A$. Since V is pre-open, $y \in pInt(x+A)$ and hence $x+Int(A) \subseteq pInt(x+A)$.

(ii) Assume that $x \in \lambda \cdot Int(A)$. Then $\frac{1}{\lambda} x \in Int(A)$. Since $Int(A)$ is regularly open, there exist pre-open sets U in the topological field K containing $\frac{1}{\lambda}$ and V in E containing x such that $U \cdot V \subseteq Int(A)$. Whence we get $x \in V \subseteq \lambda A$ and hence $y \in pInt(\lambda A)$. Hence the assertion follows.

Presenting properties of some special functions on almost pretopological vector spaces.

Theorem 3.13. *For an almost pretopological vector space E, the following are always true*:

(1) *the translation mapping* $T_x : E \to E$ *defined by* $T_x(y) = x + y, \forall x, y \in E$, *is almost precontinuous.*

(2) *the multiplication mapping* $M_\lambda : E \to E$ *defined by* $M_\lambda(x) = \lambda x$, $x \in E$, *(λ is non-zero fixed scalar), is almost precontinous.*

Proof. (1) Let $y \in E$ be an arbitrary. Let W be any regular open set in E containing $T_x(y)$. Then, by the definition of almost pretopological vector space, there exist pre-open sets U in E containing x and V in E containing y such that $U + V \subseteq W$. This results in $x + V \subseteq W \Rightarrow T_x(V) \subseteq W$. This indicates that T_x is almost precontinuous at y and hence T_x is almost precontinuous.

(2) Let $x \in E$ and W be any regular open set in E containing λx. Then, there exist pre-open sets U in the topological field K containing λ and V in E containing x such that $U \cdot V \subseteq W$. This gives that $\lambda V \subseteq W$. This means that $M_\lambda(V) \subseteq W$ showing that M_λ is almost precontinous at x. since $x \in E$ was an arbitrary, it follows that M_λ is almost precontinous.

Theorem 3.14. *For an almost pretopological vector space E, the mapping* $\phi : E \times E \to E$ *defined by* $\phi(x, y) = x + y, \forall (x, y) \in E \times E$, *is almost precontinuous.*

Proof. Let $(x, y) \in E \times E$ and let W be any regular open set in E such that $\phi(x, y) \in W$. Since E is an APTVS, there exist pre-open sets U and V in E such that $x \in U, y \in V$ and $U + V \subseteq W$. Since $U \times V$ is pre-open in $E \times E$ (with respect to the product topology) such that $(x, y) \in U \times V$ and $\phi(U \times V) = U + V \subseteq W$, it follows that ϕ is almost precontinuous at (x, y) and consequently, ϕ is almost precontinuous.

Theorem 3.15. *For an almost pretopological vector space E, the mapping* $\psi : K \times E \to E$ *defined by* $\psi(\lambda, x) = \lambda x, \forall (\lambda, x) \in K \times E$, *is almost precontinuous.*

Proof. Let $\lambda \in K, x \in E$. Let W be any regular open set in E containing λx. Then, there exist pre-open sets U in the topological field K containing λ and V in E containing x such that $U \cdot V \subseteq W$. Since $U \times V$ is pre-open in $K \times E$ containing (λ, x) and $\psi(U \times V) = U \cdot V \subseteq W$, it follows that ψ is almost precontinuous at (λ, x) and hence ψ is almost precontinuous.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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