



The Phase-Space Noncommutativity Effect on the Large and Small Wave-Function Components Approach at Dirac Equation

Ilyas Haouam

Département de Physique, Faculté des Sciences, University des Frères Mentouri Constantine 1, Constantine, Algeria
Email: ilyashaouam@live.fr

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Abstract

By the large and small wave-function components approach we achieved the nonrelativistic limit of the Dirac equation in interaction with an electromagnetic potential in noncommutative phase-space, and we tested the effect of the phase-space noncommutativity on it, knowing that the nonrelativistic limit of the Dirac equation gives the Schrödinger-Pauli equation.

Subject Areas

Applied Physics, Mathematical Analysis, Modern Physics, Quantum Mechanics, Theoretical Physics

Keywords

Schrödinger-Pauli Equation, Noncommutative Nonrelativistic Limit, Noncommutativity of Phase-Space, Noncommutative Dirac Equation, Moyal Product, Bopp-Shift Transformation

1. Introduction

In the last few years there has been much interests in the study of physics in non-commutative space, knowing that the study of noncommutative geometry has a long history [1] [2], the studies of noncommutativity in phase-space and their involvement for quantum field theories play an important role in various fields of physics especially in the theory of strings, and in the matrix model of M-theory [3], also in the description of quantum gravity.

The common method for studying the noncommutativity of quantum mechanics (NCQM) is the correspondence between commutative space and the non-commutative space using the method of translation which known as Bopp-shift,

or using the Moyal star product [4] [5] [6] [7].

In the quantum-mechanical description of particles, there are various relativistic or non-relativistic wave equations as the usual Schrödinger equation applies to the spin-0 particles in the non-relativistic domain, and the Klein-Gordon equation is the relativistic equation appropriate for spin-0 particles [8] [9] [10], and in regards to the spin-1/2 particles are governed by the relativistic Dirac.

To move from the relativistic quantum mechanics toward the nonrelativistic one it is very necessary to pass by the nonrelativistic limit, which is to transform the physical information under the condition $v \ll c$, we speak of the nonrelativistic limit for low speeds in front of the speed of the light or for the regime of weak or low energy in front of the mass energy $\frac{v}{c} \approx \frac{pC}{mc^2} \ll 1$, the nonrelativistic limit can be achieved through various ways, the most important ways are the Foldy-Wouthuysen transformation (it is only applicable to weak fields) [11] [12], and the Douglas-Kroll-Hell transformation [13] [14] [15], this canonical transformation is an unitary transformation allows separating (block-diagonalize) Dirac hamiltonian into two parts, one part describes electrons, while the other gives rise to negative energy states, which are the so-called positronic states, and the classical approach which is the large and small wave function components approach [16] [17], in this work we investigate the nonrelativistic limit of the Dirac equation according to the large and small wave function components approach to derive directly the Schrödinger-Pauli equation [18] [19] [20], but in the case of particles with spin 1 or higher, only relativistic equations are usually considered [21], noting that the uses of the Schrödinger-Pauli equation represented on the study of the fine structure of the hydrogen atom, and various scattering problems knowing that it does not consider the spin of the particles in the studies, but it can be introduced by assuming the presence of an electromagnetic field in the Dirac equation before the extraction of the nonrelativistic equation, which describes the interaction of a spin 1/2 particle with the external electromagnetic field. It correctly predicts the spin of the particle and the gyromagnetic ratio, in fact the examples of using and applying Schrödinger-Pauli equation are many and we can not all mention them.

2. Length-Momentum Noncommutativity

At string scales (very small scales) the space does not commute anymore, so that we consider the operators of coordinates and momentum in the noncommutative phase-space x_i^{nc} and p_j^{nc} respectively, then considering a noncommutative algebra satisfying the commutation relations

$$\begin{aligned} [x_i^{nc}, x_j^{nc}] &= i\Theta_{ij} \\ [p_i^{nc}, p_j^{nc}] &= i\bar{\Theta}_{ij} \quad (i, j = 1, 2), \\ [x_i^{nc}, p_j^{nc}] &= i\hbar^{eff} \delta_{ij} \end{aligned} \quad (1)$$

taking into account the effective Plank constant

$$\hbar^{eff} = \hbar \left(1 + \frac{\Theta \cdot \bar{\Theta}}{4\hbar^2} \right). \quad (2)$$

where $\Theta_{ij} = \epsilon_{ijk} \Theta_k$, $\Theta_k = (0, 0, \Theta)$, $\bar{\Theta}_{ij} = \epsilon_{ijk} \bar{\Theta}_k$, $\bar{\Theta}_k = (0, 0, \bar{\Theta})$, $\Theta, \bar{\Theta}$ are antisymmetric constant matrices (noncommutative parameters) with the dimension of (length)² and (momentum)² respectively.

The mapping between the noncommutative phase space and the commutative one doing through the *Bopp-Shift* linear transformations [22] [23].

$$\begin{aligned} x^{nc} &= x - \frac{1}{2\hbar} \Theta p_y, & y^{nc} &= y + \frac{1}{2\hbar} \Theta p_x \\ p_x^{nc} &= p_x + \frac{1}{2\hbar} \bar{\Theta} y, & p_y^{nc} &= p_y - \frac{1}{2\hbar} \bar{\Theta} x \end{aligned} \quad (3)$$

vanishing the noncommutative parameters the system will reduce to the commutative one.

With another method the noncommutativity in space can be realized using the Moyal product (\star -product) [24] [25] [26].

$$\begin{aligned} (f \star g)(x) &= \exp \left[\frac{i}{2} \Theta_{ab} \partial_{x_a} \partial_{x_b} \right] f(x_a) g(x_b) \\ &= f(x) g(x) + \sum_{n=1} \left(\frac{1}{n!} \right) \left(\frac{i}{2} \right)^n \Theta^{a_1 b_1} \dots \Theta^{a_n b_n} \partial_{a_1} \dots \partial_{a_n} f(x) \partial_{b_1} \dots \partial_{b_n} g(x), \end{aligned} \quad (4)$$

in other term the noncommutativity information is encoded in the Star product

$$(A, \star) = (A^{nc}, \cdot) \quad (5)$$

3. Nonrelativistic Limit of the Noncommutative Dirac Equation

3.1. Noncommutative Dirac Equation

Starting with the Dirac equation in the noncommutative phase space [27] [28]

$$H(x, p) \star \psi(x) = H(x^{nc}, p^{nc}) \cdot \psi(x^{nc}) = E\psi, \quad (6)$$

where the Dirac equation in interaction with electromagnetic four-potential $\{A_\mu = (A_0, A_i(x))\}$ in commutative phase space is

$$\left\{ c\hat{\alpha}_i \left(\hat{p}_i - \frac{e}{c} A_i(x) \right) + eA_0(x) + \hat{\beta}mc^2 \right\} \psi = E\psi, \quad (7)$$

where the momentum \hat{p}_i is giving by $\hat{p}_i = i\hbar\nabla_i$, and the matrices $\hat{\alpha}_i$ and $\hat{\beta}$ satisfy the anticommutation relations

$$\{\hat{\alpha}_i, \hat{\alpha}_j\} = 2\delta_{ij}, \quad \{\hat{\alpha}_i, \hat{\beta}\} = 0, \quad \hat{\alpha}_i^2 = \hat{\beta}^2 = 1. \quad (8)$$

Using the Equation (4), we achieve the noncommutativity in space $\{x \rightarrow x^{nc}\}$, then the Dirac Equation (7) transforms to Equation (6) as

$$\left\{ c\hat{\alpha}_i \left(\hat{p}_i - \frac{e}{c} A_i(x) \right) + eA_0(x) + \hat{\beta}mc^2 \right\} \star \psi(x) = E\psi_{nc}, \quad (9)$$

as $A(x) = \hbar x$ for that the derivation in Equation (4) will automatically stop in

the first order

$$(f \star g)(x) = \frac{i}{2} \Theta^{ab} \partial_a f \partial_b g + O(\Theta^2), \quad (10)$$

it means that terms upper than the first order will vanish, the Equation (9) transforms to

$$\begin{aligned} & (H \star \psi)(x) \\ &= H(x, p^{nc}) \psi(x) + \frac{i}{2} \Theta_{ab} \partial_a \left(c \hat{\alpha}_i \left(\hat{p}_i - \frac{e}{c} A_i(x) \right) + e A_0(x) + \hat{\beta} m c^2 \right) \partial_b \psi(x) \\ &= E \psi_{nc}, \end{aligned} \quad (11)$$

as $\partial_a (c \hat{\alpha}_i \hat{p}_i) = \partial_a (\hat{\beta} m c^2) = 0$, Equation (20) becomes

$$H(x, p^{nc}) \psi(x) - \frac{ie}{2} \Theta_{ab} \partial_a (c \hat{\alpha}_i A_i(x) + e A_0(x)) \partial_b \psi(x) = E \psi_{nc}, \quad (12)$$

after there we achieve the noncommutativity in phase $\{p \rightarrow p^{nc}\}$ using the Equation (3) for finding the entire noncommutative phase space Dirac equation,

$$\begin{aligned} & H(x, p^{nc}) \star \psi(x) \\ &= \left\{ c \hat{\alpha}_i \left(p_i + \frac{1}{2\hbar} \bar{\Theta}_{ij} x_j - \frac{e}{c} A_i(x) \right) + e A_0(x) + \hat{\beta} m c^2 \right. \\ & \quad \left. - \frac{ie}{2} \Theta_{ab} \partial_a (c \hat{\alpha}_i A_i(x) + e A_0(x)) \partial_b \right\} \psi(x) \\ &= E \psi_{nc}, \end{aligned} \quad (13)$$

we rewrite, the Equation (13) in more compact forme (see **Appendix**):

$$\begin{aligned} & H_{nc} \star \psi_{nc} \\ &= \left\{ c \bar{\alpha} \left(\bar{p} - \frac{e}{c} \bar{A} \right) + e A_0 + \hat{\beta} m c^2 + \frac{c}{\hbar} (\bar{\alpha} \times \bar{x}) \cdot \bar{\Theta} + \frac{e}{\hbar} (\bar{\nabla} (\bar{\alpha} \bar{A} - A_0) \times \bar{p}) \cdot \bar{\Theta} \right\} \psi_{nc} \\ &= E \psi_{nc}. \end{aligned} \quad (14)$$

3.2. Large and Small Wave-Function Components Approach

It is possible to define the nonrelativistic limit of the Dirac equation, using several ways, including that there is the Douglas-Kroll-Hell approach, it used mostly as part of relativistic quantum chemistry, and the Foldy-Wouthuysen transformation, which are both canonical transformation, and the method of development in power of \hbar [29], and the classical approach, the latter one depends on the upper two components of the Dirac wave-function ψ in the standard representation are much larger than the lower two components, using this property we can derive simply the Schrödinger-Pauli equation.

To define the nonrelativistic limit of the phase-space noncommutative Dirac equation we should firstly study the case of an electron at rest, so that without the electromagnetic interaction $\{p\psi = 0, A^\mu = 0\}$, Equation (14) becomes

$$H_{nc} \psi_{nc} = \left\{ \hat{\beta} m_0 c^2 + \frac{c}{\hbar} (\bar{\alpha} \times \bar{x}) \cdot \bar{\Theta} \right\} \psi_{nc} = E \psi_{nc}. \quad (15)$$

This system of equations is simply solved, and leads to the following

four-solutions

$$\begin{aligned} \psi_{nc}^1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar} \left(m_0 c^2 + \frac{c}{\hbar} (\vec{\alpha} \times \vec{x}) \cdot \vec{\Theta} \right) t} & \psi_{nc}^2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar} \left(m_0 c^2 + \frac{c}{\hbar} (\vec{\alpha} \times \vec{x}) \cdot \vec{\Theta} \right) t} \\ \psi_{nc}^3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+\frac{i}{\hbar} \left(m_0 c^2 + \frac{c}{\hbar} (\vec{\alpha} \times \vec{x}) \cdot \vec{\Theta} \right) t} & \psi_{nc}^4 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+\frac{i}{\hbar} \left(m_0 c^2 + \frac{c}{\hbar} (\vec{\alpha} \times \vec{x}) \cdot \vec{\Theta} \right) t} \end{aligned}, \tag{16}$$

ψ_{nc}^1 and ψ_{nc}^2 correspond to the positive energy value and ψ_{nc}^3 , ψ_{nc}^4 to the negative one.

At first, therefore we restrict ourselves to solutions of positive energy. In order to show that the Dirac equation reproduces the two component Pauli equation in the nonrelativistic limit.

The nonrelativistic limit of the Equation (14) can be most efficiently studied in the representation

$$\psi_{nc} = \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix}, \tag{17}$$

where the four-component spinor ψ_{nc} is decomposed into two-two component spinors $\tilde{\phi}_{nc}$ and $\tilde{\chi}_{nc}$, with $\left\{ \vec{p} - \frac{e}{c} \vec{A} \rightarrow \vec{\Pi} \right\}$, the Dirac Equation (14) becomes

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} &= c \vec{\alpha} \vec{\Pi} \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} + e A_0 \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} + \hat{\beta} m_0 c^2 \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} \\ &+ \frac{c}{\hbar} (\vec{\alpha} \times \vec{x}) \cdot \vec{\Theta} \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} + \frac{e}{\hbar} (\vec{\nabla} (\vec{\alpha} \vec{A} - A_0) \times \vec{p}) \cdot \vec{\Theta} \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix}, \end{aligned} \tag{18}$$

according to the Dirac matrices

$$\vec{\alpha} = \begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{19}$$

and setting $\Xi_{\bar{\Theta}} = (\vec{\alpha} \times \vec{x}) \cdot \vec{\Theta}$, and $\Xi_{\Theta} = (\vec{\nabla} (\vec{\alpha} \vec{A} - A_0) \times \vec{p}) \cdot \vec{\Theta}$ it comes

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} &= \begin{pmatrix} c \hat{\sigma} \vec{\Pi} \tilde{\phi}_{nc} \\ c \hat{\sigma} \vec{\Pi} \tilde{\chi}_{nc} \end{pmatrix} + e A_0 \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} + m_0 c^2 \begin{pmatrix} \tilde{\phi}_{nc} \\ -\tilde{\chi}_{nc} \end{pmatrix} \\ &+ \frac{c}{\hbar} \Xi_{\bar{\Theta}} \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} + \frac{e}{\hbar} \Xi_{\Theta} \begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix}, \end{aligned} \tag{20}$$

if the rest energy $m_0 c^2$, as the largest occurring energy, is additionally separated

by $\begin{pmatrix} \tilde{\phi}_{nc} \\ \tilde{\chi}_{nc} \end{pmatrix} = \begin{pmatrix} \phi_{nc} \\ \chi_{nc} \end{pmatrix} e^{-\frac{i}{\hbar} \left(m_0 c^2 + \frac{c}{\hbar} \Xi_{\bar{\Theta}} \right) t}$, the Equation (18) takes the form

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \phi_{nc} \\ \chi_{nc} \end{pmatrix} = \begin{pmatrix} c \hat{\sigma} \vec{\Pi} \chi_{nc} \\ c \hat{\sigma} \vec{\Pi} \phi_{nc} \end{pmatrix} + e A_0 \begin{pmatrix} \phi_{nc} \\ \chi_{nc} \end{pmatrix} - 2m_0 c^2 \begin{pmatrix} 0 \\ \chi_{nc} \end{pmatrix} + \frac{e}{\hbar} \Xi_{\Theta} \begin{pmatrix} \phi_{nc} \\ \chi_{nc} \end{pmatrix}, \tag{21}$$

firstly considering the lower of the above equation. Using the slow-time dependence $E_0 \gg i\hbar \frac{\partial}{\partial t}$, and the weak coupling of the electromagnetic potential $E_0 \gg eA_0$ approach, which means that the kinetic energy as well as the potential energy are small compared to the rest energy, by another term the transition to the nonrelativistic limit is realized by assuming that the momentum is small compared to the characteristic quantity mc and that the Coulomb interaction energy is weak compared to the mass energy, so that the Equation (21) goes to

$$\begin{pmatrix} c\hat{\sigma}\vec{\Pi}\chi_{nc} \\ c\hat{\sigma}\vec{\Pi}\phi_{nc} \end{pmatrix} - 2m_0c^2 \begin{pmatrix} 0 \\ \chi_{nc} \end{pmatrix} + \frac{e}{\hbar}\Xi_{\Theta} \begin{pmatrix} \phi_{nc} \\ \chi_{nc} \end{pmatrix} = 0, \quad (22)$$

using the second equation of the above system Equation (22) then we obtain

$$\chi_{nc} = \frac{c\hat{\sigma}\vec{\Pi}}{2m_0c^2 - \frac{e}{\hbar}\Xi_{\Theta}}\phi_{nc}, \quad (23)$$

where χ_{nc} represent the small component of the wave function ψ_{nc} . Insertion of Equation (23) into the first equation of Equation (21) results in a nonrelativistic wave function for ϕ_{nc}

$$i\hbar \frac{\partial}{\partial t}\phi_{nc} = \frac{(\hat{\sigma}\vec{\Pi})(\hat{\sigma}\vec{\Pi})}{2m_0 - \frac{e}{\hbar c^2}\Xi_{\Theta}}\phi_{nc} + eA_0\phi_{nc} + \frac{e}{\hbar}\Xi_{\Theta}\phi_{nc}, \quad (24)$$

with the help of

$$(\hat{\sigma}\hat{A})(\hat{\sigma}\hat{B}) = \hat{A}\hat{B} + i\hat{\sigma}\hat{A} \times \hat{B}. \quad (25)$$

Finally, the Equation (24) becomes

$$i\hbar \frac{\partial}{\partial t}\phi_{nc} = \left\{ \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m_0 - \frac{e}{\hbar c^2}\Xi_{\Theta}} - \frac{e\hbar(\hat{\sigma}\vec{B})}{c\left(2m_0 - \frac{e}{\hbar c^2}\Xi_{\Theta}\right)} + eA_0 + \frac{e}{\hbar}\Xi_{\Theta} \right\} \phi_{nc}. \quad (26)$$

This is as it should be the noncommutative phase-space Schrödinger-Pauli equation.

For $\Theta = 0 \Rightarrow \Xi_{\Theta} = 0$, the Equation (26) returns to that of usual Schrödinger-Pauli equation [30] [31].

3.3. Gyromagnetic Factor of the Electron ($g = 2$)

According to $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$, we have

$$\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \left(\vec{p} - \frac{e}{2c}\vec{B} \times \vec{x}\right)^2 \approx \vec{p}^2 - \frac{e}{c}\vec{B} \cdot \vec{L}, \quad (27)$$

where $\vec{L} = \vec{x} \times \vec{p}$ and $\hat{S} = \frac{1}{2}\hbar\hat{\sigma}$ are the operator of orbital angular momentum and the spin operator respectively.

So that the Equation (26) finally takes the form of

$$i\hbar \frac{\partial}{\partial t} \phi_{nc} = \left\{ \frac{\vec{p}^2}{2m_0 - \frac{e}{\hbar c^2} \Xi_\Theta} - \frac{e}{c \left(2m_0 - \frac{e}{\hbar c^2} \Xi_\Theta \right)} \left(\vec{L} + 2\hat{S} \right) \vec{B} + eA_0 + \frac{e}{\hbar} \Xi_\Theta \right\} \phi_{nc}. \quad (28)$$

While we are in very tiny space scales, so the NC term $\Xi_\Theta \ll 1$, it is possible to use the Maclaurin series, by changing the variable

$$\frac{e}{2m_0 \hbar c^2} \Xi_\Theta = \Omega_\Theta, \quad (29)$$

$$\frac{e}{2m_0 \left(1 - \frac{e}{2m_0 \hbar c^2} \Xi_\Theta \right)} \cong \frac{1}{2m_0} \sum_{j=0}^n \Omega_\Theta^j, \quad (30)$$

we find that the Equation (28) goes to

$$i\hbar \frac{\partial}{\partial t} \phi_{nc} = \left\{ \frac{1}{2m_0} \sum_{j=0}^n \Omega_\Theta^j \left(\vec{p}^2 - \frac{e}{c} \left(\vec{L} + 2\hat{S} \right) \vec{B} \right) + eA_0 + 2m_0 c^2 \Omega_\Theta \right\} \phi_{nc}. \quad (31)$$

The Equation (28) represents the phase-space noncommutative Schrödinger-Pauli equation, and it contains the NC kinetic energy operator and the NC Zeeman coupling term (which had been added by hand by Pauli when we talk about the commutative term), and the term that associated with the NC diamagnetism, in the absence of magnetic field ($A = B = 0$), the Equation (31) takes its original form without the information about the spin, which is the noncommutative Schrödinger equation as follows

$$i\hbar \frac{\partial}{\partial t} \phi_{nc} = \left\{ \frac{1}{2m_0} \sum_{j=0}^n \Omega_\Theta^j \left(\vec{p}^2 \right) + 2m_0 c^2 \Omega_\Theta \right\} \phi_{nc}. \quad (32)$$

The Equation (31) is a first order equation of $\frac{1}{m}$, the nonrelativistic expansion of this equation allows to add potentials such the electrical potential, but also to find corrections terms if one realize the development in the second and third order of $\frac{1}{m}$, precisely we predict that in the seconde order we find the Darwin interaction and Spin-Orbit coupling NC terms knowing that Darwin's NC term is interpreted as a correction of the potential energy due to the Zitterbewegung phenomenon (the tremor movement) [32] [33], in the third order we find corrections of the kinetic energy and temporal dependence of the electric field NC terms.

4. Conclusion

In conclusion, the nonrelativistic limit of the Dirac equation with electromagnetic potential has been studied in noncommutative phase-space using the large and small wave-function components approach. We find that the effect of the noncommutativity in phase on the nonrelativistic limit vanished, but the effect of the

noncommutativity in space appeared widely and it reduced in the NC term Ξ_{Θ} (at least to the order of approximation we have considered). Under the condition that space-space and momentum-momentum are all commutative (namely, $\Theta = \bar{\Theta} = 0$), the results return to that of the usual quantum mechanics.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

Calculations between moving from the Equation (13) to the Equation (14):

Using $\bar{\Theta}_{ij} = \bar{\Theta}\epsilon_{ij}$ and $\bar{\Theta}_k = \frac{1}{2}\epsilon_{kij}\bar{\Theta}_{ij}$ then

$$c\alpha_i \frac{1}{2\hbar} \bar{\Theta}_{ij} x_j = 2c \frac{1}{2\hbar} \bar{\Theta}_k \epsilon_{kij}^{-1} \alpha_i x_j = c \frac{1}{2} \bar{\Theta}_k \epsilon_{kij} \alpha_i x_j, \text{ knowing that } \epsilon_{kij} = \epsilon_{ijk},$$

we know that $(u \times v)_\mu = \epsilon_{\mu\nu\lambda} u_\nu \cdot v_\lambda$, so that

$$c \frac{1}{2} \bar{\Theta}_k \epsilon_{kij} \alpha_i x_j = \frac{c}{\hbar} (\vec{\alpha} \times \vec{x})_k \cdot \bar{\Theta}_k = \frac{c}{\hbar} (\vec{\alpha} \times \vec{x}) \cdot \vec{\bar{\Theta}},$$

with the same manner we prove that

$$\begin{aligned} -ie\bar{\Theta}_k \epsilon_{abk} \partial_a (\alpha_i A_i - A_0) \partial_b &= \frac{-ie\hbar}{\hbar} \bar{\Theta}_k \epsilon_{abk} \partial_a (\vec{\alpha} \vec{A} - A_0) \partial_b \\ &= \frac{e}{\hbar} (\vec{\nabla} (\vec{\alpha} \vec{A} - A_0) \times \vec{p}) \cdot \vec{\bar{\Theta}} \end{aligned}$$