Suzuki-Type Fixed Point Results in $b_2$-Metric Spaces

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Abstract
A common fixed point theorem for Suzuki-type contractions in the setting of $b_2$-metric space is established in this paper. Our result extends some known results from metric spaces to $b_2$-metric space. The research is meaningful and I recommend it to be published in the journal.

Subject Areas
Mathematical Analysis

Keywords
Common Fixed Point, Complete $b_2$-Metric Space, Suzuki Contraction

1. Introduction
Banach fixed point principle [1] is simple but forceful, which is a classical tool for many aspects. There are many generalizations of this principle, see [2] [3] [4] [5], from which, an interesting generalization is introduced by Suzuki [6] in 2008.

Many generalized spaces of Metric space have been established. Among them, $b$-metric [7] and 2-metric [8] have been extensively researched. Both of these metrics of those spaces are not continuous functions of its variables. In order to solve this problem, the author of [9] established the notion of $b_2$-metric space generalizing from both spaces above. And in this paper, we proved a common fixed point result for two maps in $b_2$-metric space [9]. Our purpose is to present a fixed point result of two maps under a newly Suzuki-type contractive condition in this space, and the fixed point theory in $b_2$-metric space is perfected.

2. Preliminaries
The following definitions will be presented before giving our results.

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Definition 2.1. [9] Let $X$ be a nonempty set, $s \geq 1$ be a real number and let $d : X \times X \times X \to \mathbb{R}$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.
2) If at least two of three points $x, y, z$ are the same, then $d(x, y, z) = 0$.
3) The symmetry:

$$d(x, y, z) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$$

for all $x, y, z \in X$.
4) The rectangle inequality:

$$d(x, y, z) \leq s \left[ d(x, y, a) + d(y, z, a) + d(z, x, a) \right]$$

for all $x, y, z, a \in X$.

Then $d$ is called a $b_2$ metric on $X$ and $(X, d)$ is called a $b_2$ metric space with parameter $s$. Obviously, for $s = 1$, $b_2$ metric reduces to 2-metric.

Definition 2.2. [9] Let \{ $x_n$ \} be a sequence in a $b_2$ metric space $(X, d)$.

1) A sequence \{ $x_n$ \} is said to be $b_2$-convergent to $x \in X$, written as $\lim_{n \to \infty} x_n = x$, if all $a \in X$ imply that $d(x, x_n, a) \to 0$, when $n \to \infty$.
2) \{ $x_n$ \} is Cauchy sequence if and only if $d(x, x_n, a) \to 0$, when $n \to \infty$.
3) $(X, d)$ is said to be complete if every $b_2$-Cauchy sequence is a $b_2$-convergent sequence.

Definition 2.3. [9] Let $(X, d)$ and $(X', d')$ be two $b_2$-metric spaces and let $f : X \to X'$ be a mapping. Then $f$ is said to be $b_2$-continuous at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(z, x, a) < \delta$ imply that $d'(fz, fx, a) < \varepsilon$. The mapping $f$ is $b_2$-continuous on $X$ if it is $b_2$-continuous at all $z \in X$.

Definition 2.4. [9] Let $(X, d)$ and $(X', d')$ be two $b_2$-metric spaces. Then a mapping $f : X \to X'$ is $b_2$-continuous at a point $x \in X'$ if and only if it is $b_2$-sequentially continuous at $x$; that is, whenever \{ $x_n$ \} is $b_2$-convergent to $x$, \{ $fx_n$ \} is $b_2$-convergent to $fx$.

Lemma 2.5. [10] Let $(X, d)$ be a $b_2$ metric space with $s \geq 1$ and let \{ $x_n$ \} be a sequence in $X$ such that

$$d(x_n, x_{n+1}, a) \leq \lambda d(x_{n+1}, x_n, a)$$

for all $n \in \mathbb{N}$ and all $a \in X$, where $\lambda \in [0, 1/s)$. Then \{ $x_n$ \} is a $b_2$-Cauchy sequence in $(X, d)$.

3. Main Results

Theorem 3.1. Let $(X, d)$ be a complete $b_2$ metric space and in each variable $d$ is continuous. Let $f : X \to X'$ be a selfmap and $\phi = \phi : [0, 1) \to (1/(s+1), 1]$ be defined by:
\[
\phi(\rho) = \begin{cases} 
1,0 \leq \rho \leq \frac{\sqrt{5} - 1}{2}, \\
\frac{1-\rho}{\rho^2}, \frac{\sqrt{5} - 1}{2} \leq \rho \leq b, \\
\frac{1}{s + \rho}, b \leq \rho < 1,
\end{cases} \tag{3.1}
\]

where \( b_s = \frac{1 - s + \sqrt{1 + 6s + s^2}}{4} \) is the positive solution of \( \frac{1 - \rho}{\rho^2} = \frac{1}{s + \rho} \). If there exists \( \rho \in (0,1) \) such that for each \( x, y \in X \),

\[
\phi(\rho)d(x, fx, a) \leq d(x, y, a) \Rightarrow d(fx, fy, a) \leq \frac{\rho}{s}N(x, y, a), \tag{3.2}
\]

where

\[
N(x, y, a) = \max \left\{ d(x, y, a), d(x, fx, a), d(y, fy, a) \right\}
\]

then \( f \) has a unique fixed point \( z \) in \( X \) and the sequence \( \{T^n x\} \) converges to \( z \).

**Proof** From (3.1) and take \( y = fx \), we get the inequality as follows:

\[
d(fx, f^2x, a) \leq \frac{\rho}{s} \max \left\{ d(x, fx, a), d(x, fx, a), d(fx, f^2x, a) \right\} = \frac{\rho}{s} \max \left\{ d(x, fx, a), d(fx, f^2x, a) \right\}
\]

from the above relation, we get

\[
d(fx, f^2x, a) \leq \frac{\rho}{s} d(x, fx, a), \text{ for each } x \in X \tag{3.3}
\]

Given \( v_0 \in X \) and construct a sequence \( \{v_n\} \) letting \( v_{n+1} = f v_n = f^{a+1}v_0 \), for all \( n \in N \). Then by taking \( x = v_{n+1} \) in (3.3) we get

\[
d(v_n, v_{n+1}, a) \leq \frac{\rho}{s} d(v_{n+1}, v_n) \tag{3.4}
\]

since \( \rho \in (0,1) \), we have \( \frac{\rho}{s} < \frac{1}{s} \), by Lemma 2.6, we get the conclusion that \( \{v_n\} \) is a Cauchy sequence, so there exists \( z \) in \( X \), such that \( f v_n \to z \) as \( n \to \infty \).

Since \( v_n \to z \) and \( f v_n \to z \), that is \( d(v_n, f v_n, a) \to 0 \) and by the continuity of \( d \), we have \( d(v_n, x, a) \to d(x, z, a) \) as \( n \to \infty \), for every \( x \neq z \), so there exists \( n_0 \in N \) such that \( \phi(\rho)d(v_n, f v_n, a) < d(v_n, x, a) \), for each \( n \geq n_0 \), now for such above \( n \) and from the assumption (3.2) we get

\[
d(f v_n, f x, a) \leq \frac{\rho}{s} \max \left\{ d(v_n, x, a), d(v_n, v_{n+1}, a), d(x, fx, a) \right\}, \text{ for } x \neq z \tag{3.5}
\]

taking \( n \to \infty \) we have

\[
d(fx, z, a) \leq \frac{\rho}{s} \max \left\{ d(x, z, a), d(x, fx, a) \right\} \tag{3.6}
\]

In (3.3), take \( x = f^{a+1}z \), we have

\[
d(f^{a+1}z, f z, a) \leq \frac{\rho}{s} d(f^{a+1}z, f^a z, a), \text{ for } n \in N \tag{3.7}
\]
by induction, we have
\[ d\left(f^nz, f^{n+1}z, a\right) \leq \frac{\rho^n}{s}d(z, f^z, a) \] (3.8)

Now we claim that
\[ d\left(f^nz, z, a\right) \leq d\left(f^z, z, a\right), \text{ for every } n \in \mathbb{N} \] (3.9)

this inequality is true for \( n = 1 \), assume (3.9) holds for some \( n \in \mathbb{N} \), if \( f^nz = z \), then we have \( f^{n+1}z = f^z \) and
\[ d\left(f^{n+1}z, z, a\right) = d\left(f^z, z, a\right) \leq d\left(f^z, z, a\right) \] (3.9.1)

if \( f^nz \neq z \), then we can obtain the following inequality from (3.6), and that is:
\[ d\left(f^{n+1}z, z, a\right) \leq \frac{\rho}{s} \max \{d\left(f^nz, z, a\right), d\left(f^nx, f^{n+1}x, a\right)\} \] (3.9.2)

By the induction hypothesis (3.9) for some \( n \in \mathbb{N} \) and (3.8), we have
\[ d\left(f^{n+1}z, z, a\right) \leq \frac{\rho}{s} \max \left\{d\left(f^nz, z, a\right), \frac{\rho}{s}d\left(f^x, z, a\right)\right\} \]
\[ = \frac{\rho}{s}d\left(f^z, z, a\right) \leq d\left(f^z, z, a\right) \]

Therefore, (3.9) is true for every \( n \in \mathbb{N} \).

Now we assume that \( f^z \neq z \) and consider the two following possible cases to prove that \( f^z = z \).

Case 1. Take \( 0 \leq \rho < b_1 \), therefore \( \phi(\rho) \leq \frac{1-\rho}{\rho^2} \). Firstly we claim that
\[ d\left(f^nz, f^z, a\right) \leq \frac{\rho}{s}d\left(f^z, z, a\right), \text{ for all } n \in \mathbb{N} \] (3.10)

It is obvious for \( n = 1 \) and this follows from (3.8) for \( n = 2 \).

From (3.9) we have \( d\left(z, f^nz, f^z\right) \leq d\left(f^z, z, f^z\right) = 0 \), that is, \( d\left(z, f^nz, f^z\right) = 0 \) (3.11)

Now assume that (3.10) holds for some \( n \geq 2 \), then from part 4 of Definition 2.1 and (3.11) we have
\[ d\left(z, f^z, a\right) \leq s\left(d\left(z, f^nz, a\right) + d\left(f^nz, f^z, a\right) + d\left(z, f^nz, f^z\right)\right) \]
\[ \leq s\left(d\left(z, f^nz, a\right) + d\left(f^nz, f^z, a\right)\right) \]
\[ \leq s\left(d\left(z, f^nz, a\right) + \frac{\rho}{s}d\left(f^z, z, a\right)\right) \] (3.10.1)

and that is \( d\left(z, f^z, a\right) \leq \frac{s}{1-\rho}d\left(z, f^nz, a\right) \), using (3.8), it follows that
\[ \phi(\rho)d\left(f^nz, f^{n+1}z, a\right) \]
\[ \leq \frac{1-\rho}{\rho^2}d\left(f^nz, f^{n+1}z, a\right) \leq \frac{1-\rho}{\rho^2}d\left(f^nz, f^{n+1}z, a\right) \]
\[ \leq \frac{1-\rho}{\rho^2}d\left(z, f^z, a\right) \leq \frac{1-\rho}{\rho^2}d\left(z, f^z, a\right) \]
\[ \leq \frac{1}{s}d\left(z, f^z, a\right) \leq d\left(f^nz, z, a\right) \] (3.10.2)
from (3.2)
\[
d\left(f^{n+1}z, fz, a\right) \leq \frac{\rho}{s} \max\left\{d\left(f^n z, z, a\right), d\left(f^n z, f^{n+1}z, a\right), d\left(z, fz, a\right)\right\}
\leq \frac{\rho}{s} d\left(z, fz, a\right)
\]
(3.10.3)

By induction with using (3.8) and (3.9), it is easy for us to get the relation (3.10).

Now from \(fz \neq z\) and (3.10), we get for each \(n \in \mathbb{N}\) \(f^n z \neq z\), therefore, (3.6) and (3.8) show that
\[
d\left(f^{n+1}z, fz, a\right) \leq \frac{\rho}{s} \max\left\{d\left(f^n z, z, a\right), d\left(f^n z, f^{n+1}z, a\right)\right\}
\leq \frac{\rho}{s} \max\left\{d\left(f^n z, z, a\right), \frac{\rho^n}{s} d\left(z, fz, a\right)\right\}
\]
(3.12)

From part 4 of Definition 2.1 and (3.11), we get
\[
d\left(f^n x, z, a\right) \leq s \left(d\left(fz, f^n z, a\right) + d\left(f^n z, z, a\right) + d\left(fz, f^n z\right)\right)
\leq s \left(d\left(fz, f^n z, a\right) + d\left(f^n z, z, a\right)\right)
\]
(3.12.1)

It follows from (3.10) that
\[
d\left(f^n z, z, a\right) \geq \frac{1}{s} d\left(fz, z, a\right) - d\left(fz, f^n z, a\right)
\geq \frac{1}{s} d\left(fz, z, a\right) - \frac{\rho}{s} d\left(fz, z, a\right) \geq \frac{1 - \rho}{s} d\left(fz, z, a\right)
\]
(3.12.2)

There exists \(n_1 \in \mathbb{N}\), for \(n \geq n_1\) and \(0 < \rho < b_1\) such that \(1 - \rho \geq \rho^n\) for such \(n\), we get
\[
d\left(f^n z, z, a\right) \geq \frac{\rho^n}{s} d\left(fz, z, a\right) \geq \frac{\rho^n}{s} d\left(fz, z, a\right)
\]
(3.12.3)

Then taking \(n \to \infty\) from (3.12) we have
\[
d\left(f^{n+1}z, z, a\right) \leq \frac{\rho}{s} d\left(f^n z, z, a\right) \leq \cdots \leq \left(\frac{\rho}{s}\right)^{n_1} d\left(f^{n_1}z, z, a\right) \to 0
\]
(3.12.4)

That is, \(f^n z \to z\), and from (3.10), we get
\[
\lim_{s \to \infty} d\left(fz, z, a\right) \leq \frac{\rho}{s} \lim_{s \to \infty} d\left(fz, z, a\right)
\]
(3.12.5)

which is impossible except \(fz = z\).

Case 2. Take \(b_i \leq \rho < 1\) and that is when \(\phi(\rho) = \frac{1}{s + \rho}\), we will prove that we can find a subsequence \(\{v_n\}\) of \(\{v_n\}\) such that for each \(i \in \mathbb{N}\),
\[
\phi(\rho) d\left(v_n, f v_n, a\right) = \phi(\rho) d\left(v_n, v_{n+1}, a\right) \leq d\left(v_n, z, a\right),
\]
(13.1)

we know for each \(n \in \mathbb{N}\) \(d\left(v_n, v_{n+1}, a\right) \leq \frac{\rho}{s} d\left(v_{n-1}, v_n, a\right)\) from (3.4), assume that for some \(n \in \mathbb{N}\).
\[
\frac{1}{s + \rho} d(v_n, v_{n+1}, a) > d(v_{n+1}, z, a), \quad (3.13.1)
\]

and
\[
\frac{1}{s + \rho} d(v_n, v_{n+1}, a) > d(v_n, z, a) \quad (3.13.2)
\]

then
\[
d(v_{n+1}, v_n, a) \leq s \left( d(v_{n+1}, z, a) + d(v_n, z, a) + d(v_{n+1}, v_n, z) \right) < \frac{s}{s + \rho} \left( d(v_{n+1}, v_n, a) + d(v_n, v_{n+1}, a) + sd(v_n, v_{n+1}, z) \right) \quad (3.13.3)
\]

taking \( n \to \infty \), we get a relation which is impossible. Therefore we have
\[
\phi(\rho) d(v_n, v_{n+1}, a) \leq d(v_{n+1}, z, a) \quad \text{or} \quad \phi(\rho) d(v_n, v_{n+1}, a) \leq d(v_{n+1}, z, a)
\]

for each \( n \in N \). \((3.13.4)\)

In other words, there is a subsequence \( \{v_{n_i}\} \) for \( \{v_n\} \) such that \((3.13)\) is true for every \( i \in N \), but from \((3.2)\) we have
\[
d(fv_{n_i}, fz, a) \leq \frac{\rho}{s} \max \left\{ d(v_{n_i}, z, a), d(fv_{n_i}, a), d(z, fz, a) \right\} \quad (3.13.5)
\]

Taking \( i \to \infty \), we have
\[
d(z, fz, a) \leq \frac{\rho}{s} d(z, fz, a) \quad (3.13.6)
\]

which is possible only if \( fz = z \).

Therefore, \( z \) is a fixed point of \( f \). Let \( w \) be another fixed point of \( f \), from \((3.6)\), we have
\[
d(w, z, a) = d(fw, z, a) \leq \frac{\rho}{s} \max \left\{ d(w, z, a), d(w, fz, z) \right\} = \frac{\rho}{s} d(w, z, a) \quad (3.14)
\]

which is a contraction unless \( d(w, z, a) = 0 \), and that is \( w = z \), \( f \) has a unique common fixed point \( z \in X \).

**Corollary** Let \((X, d)\) be a complete \( b_2 \)-metric space and \( d \) is continuous in every variable. Let \( f : X \to X \) be a selfmap and \( \phi : [0,1) \to \left( 1/(s+1), 1 \right] \) be defined by \((3.1)\). If there exists \( \rho \in [0,1) \) such that for each \( x, y \) of \( X \),
\[
\phi(\rho) d(x, fx, a) \leq d(x, y, a) \Rightarrow d(fx, fy, a) \leq \frac{\rho}{s} d(x, y, a) \quad (3.15)
\]

then \( f \) has a unique fixed point \( z \) in \( X \) and the sequence \( \{f^n x\} \) converges to \( z \), for each \( x \in X \).

**4. Conclusion**

A known existence theorems of common fixed points for two maps was proved for the generalized Suzuki-type contractions in \( b_2 \)-metric space. The results generalized and improved the field of fixed point theory for metric spaces and perfected the realization of the fixed point theory in this generalized space.
Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References


