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Unique Common Fixed Points for Mappings Satisfying φ-Contractions on b_2 Metric Spaces

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Abstract

In this paper, we construct the convergence sequences on b_2 metric spaces and prove that mappings satisfying the φ contractions have the unique common fixed point, and the conclusion we obtained generalized many results on 2 metric spaces.

Subject Areas

Mathematical Analysis

Keywords

b_2-Metric Space, Common Fixed Point, Contractive Condition, Comparison Function

1. Introduction

As is known to all, the notion of metric spaces has been generalized by many scholars, of which the most essential work is the 2-metric spaces. In 1963, the notion of 2-metric spaces was first introduced by Gähler in [1], from then on, many scholars had proved the common fixed points theorems in this spaces [1] [2] [3]. In 1993, Czerwik introduced the notion of b-metric spaces [4], and proved theorems of common fixed points in this space. The two metric spaces above have obviously generalized the traditional metric spaces. Therefore, the fixed point theory has been developed a lot.

On the other hand, many scholars generalized the fixed point theorems by improving the contraction or expansive conditions,such as the φ-contraction, quasi-contraction [5] [6] [7]. Recently, Zead Mustafa has introduced the notion of b_2 metric spaces [8] which is a generalization of both 2 and b metric spaces. Some fixed point theorems were then obtained under various contractive

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conditions in this spaces [9] [10].

The purpose of this paper is to consider the common fixed points of a self-mappings family in the b_2-metric spaces satisfying the φ contractions to generalize the fixed points theorems in 2-metric spaces and improve the theorems in b_2 metric spaces.

2. Preliminary Notes

Before stating our main results, we introduce some necessary definitions as follows.

Definition 2.1 [1] *Let X be a non-empty set and let* $d: X \times X \times X \to \mathbb{R}$ *be a map satisfying the following conditions*:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

2) If at least two of three points x, y, z are the same, then $d(x,y,z)=0$.

3) The symmetry:
$d(x,y,z)=d(x,z,y)=d(y,x,z)=d(y,z,x)=d(z,x,y)=d(z,y,x)$ for all $x,y,z \in X$.

4) The rectangle inequality: $d(x,y,z) \leq d(x,y,a)+d(y,z,a)+d(z,x,a)$ for all $x,y,z,a \in X$.

Then *d* is called a 2-metric on *X* and (X,d) is called a 2-metric space.

Definition 2.2 [4] *Let X be a non-empty set and* $s \geq 1$ *be a given real number. A function* $d: X \times X \to \mathbb{R}^+$ *is a b-metric on X if for all* $x,y,z \in X$, *the following conditions hold*:

1) $d(x,y)=0$ if and only if $x=y$.

2) $d(x,y)=d(y,x)$.

3) $d(x,z) \leq s\left[d(x,y)+d(y,z)\right]$.

In this case, the pair (X,d) is called a *b*-metric space.

Definition 2.3 [6] *we call the function as the comparison function if it satisfies the following conditions.*

1) $\varphi:[0,+\infty] \to [0,+\infty)$.

2) φ is nondecreasing, sequentially continuous from the right.

3) for each $t \in R^+$, $\varphi(t) < t$.

Without loss of generality, we mark $\varphi(0)=0$.

Definition 2.4 [8] *Let X be a non-empty set,* $s \geq 1$ *be a real number and let d:* $X \times X \times X \to \mathbb{R}$ *be a map satisfying the following conditions*:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

2) If at least two of three points x, y, z are the same, then $d(x,y,z)=0$.

3) The symmetry:
$d(x,y,z)=d(x,z,y)=d(y,x,z)=d(y,z,x)=d(z,x,y)=d(z,y,x)$ for all $x,y,z \in X$.

4) The rectangle inequality:
$d(x,y,z) \leq s\left[d(x,y,a)+d(y,z,a)+d(z,x,a)\right]$ for all $x,y,z,a \in X$.

Then d is called a b_2-metric on X and (X,d) is called a b_2-metric space with parameter s. Obviously, for $s = 1$, b_2-metric reduces to 2-metric.

Definition 2.5 [8] *Let* $\{x_n\}$ *be a sequence in a* b_2*-metric space* (X,d).

1) A sequence $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.

2) $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$, when $n, m \to \infty$.

3) (X,d) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

3. Main Results

These are the main results of the paper.

Lemma 3.1 [6] *For each nonnegative sequence* $\{t_n\}$ *satisfying the condition*: $t_{n+1} \le \varphi(t_n)$, $n = 1, 2, \cdots$, *then* $\lim_{n\to\infty} \varphi^n(t) = 0$

Proof. For each $t > 0$, $\varphi^n(t) < \varphi^{n-1}(t) < \cdots < \varphi(t) < t$

Since φ is sequentially continuous from the right, we can get

$$\lim_{n\to\infty} \varphi^n(t) = \lim_{n\to\infty} \varphi(\varphi^{n-1}(t)) = \varphi\left(\lim_{n\to\infty} \varphi^{n-1}(t)\right)$$

$$\text{let } \tilde{t} = \lim_{n\to\infty} \varphi^n(t),\ \therefore \tilde{t} = \varphi(\tilde{t})\ \therefore \lim_{n\to\infty} \varphi^n(t) = 0.$$

$$\because t_{n+1} \le \varphi(t_n) \le \varphi^2(t_{n-1}) \le \cdots \le \varphi^n(t_1)\ \therefore \lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} \varphi^n(t_1) = 0.$$

which shows that $\lim_{n\to\infty} t_n = 0$.

Theorem 3.2 *Let* (X,d) *be a* b_2 *metric spaces,* $f, g : X \to X$ *are two self mappings on X, and satisfy the condition*:

$$s \cdot d(fx, gy, a) \le \varphi\left(\max\left\{s \cdot d(x, y, a), s \cdot d(x, fx, a), s \cdot d(y, gy, a), \frac{1}{2} d(x, gy, a) + \frac{1}{2} d(y, fx, a)\right\}\right), \tag{1}$$

with $s > 1$, if fX or gX is complete, then f and g have an unique common fixed point.

Proof. $\forall x_0 \in X$, from the condition (1), we can construct a sequence as follow:

$$x_{2n+1} = f(x_{2n}), x_{2n+2} = g(x_{2n+1}) \tag{2}$$

we can easily get:

$$d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0 \tag{3}$$

In fact, from the condition (1,2), we know that:

$$s \cdot d(x_{2n}, x_{2n+1}, x_{2n+2}) = s \cdot d(fx_{2n}, gx_{2n+1}, x_{2n}) \le \varphi(s \cdot d(x_{2n}, x_{2n+1}, x_{2n+2}))$$

$\because \varphi(t) < t$ holds for each $t > 0$

Suppose that $s \cdot d(x_{2n}, x_{2n+1}, x_{2n+2}) > 0$

Then, we can get:

$$s \cdot d(x_{2n}, x_{2n+1}, x_{2n+2}) \le \varphi(s \cdot d(x_{2n}, x_{2n+1}, x_{2n+2})) < s \cdot d(x_{2n}, x_{2n+1}, x_{2n+2})$$

which is a contradiction.

Therefore,we can get $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$

Hence, we can get:

$$\begin{aligned} & s \cdot d(x_{2n+1}, x_{2n+2}, a) = d(fx_{2n}, gx_{2n+1}, a) \\ & \leq \varphi\left(\max\left(s \cdot d(x_{2n}, x_{2n+1}, a), s \cdot d(x_{2n+1}, x_{2n+2}, a), \frac{1}{2} d(x_{2n}, x_{2n+2}, a)\right)\right) \end{aligned} \tag{4}$$

From now on, we will proof the sequence we construct is a cauchy sequence through two cases as follows.

Case 1: if $d(x_{2n+1}, x_{2n+2}, a) \geq d(x_{2n}, x_{2n+1}, a)$

Meanwhile, we notice that:

$$\frac{1}{2} d(x_{2n}, x_{2n+2}, a) \leq \frac{1}{2} s \cdot d(x_{2n+1}, x_{2n+2}, a) + \frac{1}{2} s \cdot d(x_{2n}, x_{2n+1}, a) \tag{5}$$

In this case, with condition (5), we can get:

$$\frac{1}{2} d(x_{2n}, x_{2n+2}, a) \leq s \cdot d(x_{2n+1}, x_{2n+2}, a) \tag{6}$$

with condition (4, 6), we can easily get the following holds:

$$\begin{aligned} & \max\left\{s \cdot d(x_{2n}, x_{2n+1}, a), s \cdot d(x_{2n+1}, x_{2n+2}, a), \frac{1}{2} \cdot d(x_{2n}, x_{2n+2}, a)\right\} \\ & = s \cdot d(x_{2n+1}, x_{2n+2}, a) \end{aligned} \tag{7}$$

From condition (4, 7), suppose that $s \cdot d(x_{2n+1}, x_{2n+2}, a) > 0$ we can get:

$$s \cdot d(x_{2n+1}, x_{2n+2}, a) \leq \varphi\left(s \cdot d(x_{2n+1}, x_{2n+2}, a)\right) < s \cdot d(x_{2n+1}, x_{2n+2}, a)$$

which shows $d(x_{2n+1}, x_{2n+2}, a) = 0$,
since $d(x_{2n+1}, x_{2n+2}, a) \geq d(x_{2n}, x_{2n+1}, a)$
we can get $d(x_{2n}, x_{2n+1}, a) = 0$

Hence, we can get $d(x_n, x_{n+1}, a) = 0$ holds for each $a \in X$ and $n = 1, 2, \cdots$

$\therefore \{x_n\}$ is a constant sequence, obviously, it is a cauchy sequence.

Case 2: if $d(x_{2n+1}, x_{2n+2}, a) < d(x_{2n}, x_{2n+1}, a)$

Meanwhile, we noticed that:

$$\frac{1}{2} d(x_{2n}, x_{2n+2}, a) \leq \frac{1}{2} s \cdot d(x_{2n+1}, x_{2n+2}, a) + \frac{1}{2} s \cdot d(x_{2n}, x_{2n+1}, a)$$

In this case, we can get:

$$\frac{1}{2} d(x_{2n}, x_{2n+2}, a) \leq s \cdot d(x_{2n}, x_{2n+1}, a) \tag{8}$$

From the condition (4) and (8), we can get:

$$\begin{aligned} & \max\left\{s \cdot d(x_{2n}, x_{2n+1}, a), s \cdot d(x_{2n+1}, x_{2n+2}, a), \frac{1}{2} \cdot d(x_{2n}, x_{2n+2}, a)\right\} \\ & = s \cdot d(x_{2n}, x_{2n+1}, a) \end{aligned} \tag{9}$$

From condition (4, 9), we can get:

$$s \cdot d(x_{2n+1}, x_{2n+2}, a) \leq \varphi\left(s \cdot d(x_{2n}, x_{2n+1}, a)\right) \tag{10}$$

Let $s \cdot d(x_{2n}, x_{2n+1}, a) = t_{2n}$, we can mark the condition (10) as:
$t_{2n} \le \varphi(t_{2n+1})$, and from Lemma 3.1, we can get:
$\lim_{n\to\infty} t_{2n} = 0$, which shows that $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}, a) = 0$
In this case, as $d(x_{2n}, x_{2n+1}, a) < d(x_{2n+1}, x_{2n+2}, a)$
We can also get: $\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}, a) = 0$
Therefore, for each $a \in X$ and $n = 1, 2, \cdots$,

$$\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0 \text{ holds.} \tag{11}$$

From now on, we will prove the sequence in case 2 is a cauchy sequence through mathematical induction.

From the condition(11), we can get: For arbitrary $\varepsilon > 0, \exists n_0 \in N_+$, when $n \ge n_0$, the following holds:

$$d(x_n, x_{n+1}, a) < \frac{1}{3s}\varepsilon < \varepsilon, \forall a \in X \tag{12}$$

Then, we use the mathematical induction for m to prove $d(x_m, x_n, a) < \varepsilon$, $(\forall m > n > n_0)$ $\forall a \in X$ holds.

1) When $m = n+1$, $d(x_{n+1}, x_n, a) < \varepsilon$ holds

2) Suppose that when $m = n + k(k \ge 1)$, $d(x_m, x_n, a) < \varepsilon$ holds, from this, we will prove the condition holds for $m+1$.

From the condition (12) and the inductive hypothesis, we can easily get:
$d(x_m, x_{m+1}, a) < \frac{1}{3s}\varepsilon, d(x_n, x_m, a) < \frac{1}{3s}\varepsilon$ holds for all $a \in X$

From the rectangle inequality, we can get:

$$\begin{aligned} d(x_n, x_{m+1}, a) &\le s \cdot \left[d(x_m, x_{m+1}, a) + d(x_n, x_m, a) + d(x_m, x_{m+1}, x_n) \right] \\ &< s \cdot \left(\frac{1}{3s}\varepsilon \times 3 \right) = \varepsilon \end{aligned} \tag{13}$$

From the condition (13) and inductive principle, we claim that:
$d(x_n, x_m, a) < \varepsilon$ holds for all $m > n > n_0$ and $a \in X$

Therefore, from the definition 2.5, we can get the conclusion that the sequence $\{x_n\}$ we construct in case 2 is a cauchy sequence.

Therefore, from the proof above, we can get the conclusion that the sequence we construct in condition (2) is a cauchy sequence.

If fX is complete, then from the condition (2), we can get:

$$\lim_{n\to\infty} x_{2n+1} = u = fz \in fX$$

From the rectangle inequality, we can get:

$$d(x_{2n+2}, u, a) \le s \cdot \left[d(x_{2n+2}, x_{2n+1}, u) + d(x_{2n+2}, x_{2n+1}, a) + d(x_{2n+1}, u, a) \right]$$

Since the $\{x_n\}$ is a cauchy sequence, let $n \to \infty$, we can get:

$$\lim_{n\to\infty} d(x_{2n+2}, u, a) \le \lim_{n\to\infty} s \cdot d(x_{2n+1}, x_{2n+2}, a) < s \times \frac{1}{3s}\varepsilon < \varepsilon$$

Therefore, we can get $x_{2n+2} \to u$

Then, we will prove the point u is the unique common fixed point for the mappings f and g.

$$\because s\cdot d\left(fu,x_{2n+2},a\right)$$
$$\le \varphi\left(\max\left\{s\cdot d\left(u,x_{2n+2},a\right),s\cdot d\left(fu,u,a\right),s\cdot d\left(x_{2n+1},x_{2n+2},a\right),\frac{1}{2}d\left(u,x_{2n+2},a\right)+\frac{1}{2}d\left(x_{2n+1},fu,a\right)\right\}\right)$$

Let $n\to\infty$, suppose that $s\cdot d\left(fu,u,a\right)>0$, we can get:

$$s\cdot d\left(fu,u,a\right)\le\varphi\left(s\cdot d\left(fu,u,a\right)\right)<s\cdot d\left(fu,u,a\right)$$

which is a contradiction.

Therefore, we claim that $d\left(fu,u,a\right)=0$ holds for $\forall a\in X$, which means $fu=u$.

$$\because s\cdot d\left(u,gu,a\right)=s\cdot d\left(fu,gu,a\right)\le\varphi\left(\max\left(s\cdot d\left(u,gu,a\right),\frac{1}{2}d\left(u,gu,a\right)\right)\right)$$

Suppose that $s\cdot d\left(u,gu,a\right)>0$, we can get:

$$s\cdot d\left(gu,u,a\right)\le\varphi\left(s\cdot d\left(gu,u,a\right)\right)<s\cdot d\left(gu,u,a\right)$$

which is a contradiction. Therefore, we claim that $d\left(gu,u,a\right)=0$ holds for $\forall a\in X$,which means $gu=u$.

If there exist another $v\in X$, $s.t.\ v=gv=fv$, suppose that $d\left(u,v,a\right)>0$ we can get:

$$s\cdot d(u,v,a)=s\cdot d\left(fu,gv,a\right)\le\varphi\left(s\cdot d\left(u,v,a\right)\right)<s\cdot d\left(u,v,a\right)$$

which is a contradiction. Therefore, we can get: $d\left(u,v,a\right)=0$ holds for $\forall a\in X$, which means $u=v$.

Therefore, we claim that u is the unique common fixed point for the mappings f and g.

The proof is in the similar way for the case if gX is complete.

Now, we will generalize the theorem 3.2 into family of mappings.

Let $\{f_i\}_1^m,\{g_i\}_1^n:X\to X$, $m,n\in N_+$, Let $\delta=\left\{\{f_i\}_1^m,\{g_i\}_1^n\right\}$. If for each $A,B\in\delta, AB=BA$, then we call δ is pairwise commuting.

Theorem 3.3 *Let* (X,d) *be a* b_2 *metric space,* $\{f_i\}_1^m,\{g_i\}_1^n:X\to X$ *is the mapping family, let* $f=\prod_{i=1}^m f_i, g=\prod_{i=1}^n g_i$, *and satisfy the following condition*:

$$s\cdot d\left(fx,gy,a\right)\le\varphi\left(\max\left\{s\cdot d\left(x,y,a\right),s\cdot d\left(x,fx,a\right),s\cdot d\left(y,gy,a\right),\frac{1}{2}d\left(x,gy,a\right)+\frac{1}{2}d\left(y,fx,a\right)\right\}\right)$$

if fX or gX is complete, δ is pairwise commuting, then δ has an unique common fixed point.

Proof. From theorem 3.2, we know u is the unique common fixed point for $\{f,g\}$. Since δ is pairwise commuting, we can get: For each $1\le i\le m$

$$f_iu=f_ifu=ff_iu, f_iu=f_igu=gf_iu,$$

Therefore, f_iu is the common fixed point for $\{f,g\}$, since u is the unique common fixed point for $\{f,g\}$, we can obtain $f_i u = u\left(1 \le i \le m\right)$, we can prove $g_i u = u\left(1 \le i \le n\right)$ in the similar way. Therefore, u is the common fixed point for δ, if v is the common fixed point for δ, obviously, v is the common fixed point for $\{f,g\}$, therefore, $u = v$ is the unique common fixed point for δ.

Example 3.4 [8] *Let* $X = \{(\alpha,0): \alpha \in [0,1]\} \cup \{(0,2)\} \subset R^2$*, and let* $d(x,y,z)$ *denote the square of the area of triangle with vertices* $x,y,z \in X$*, e.g.*

$$d\left((\alpha,0),(\beta,0),(0,2)\right) = (\alpha-\beta)^2$$

It is easy to check that d is a b_2-metric with parameter $s = 2$.

Consider the mappings $f,g: X \to X$ given by:

$$\begin{cases} f(\alpha,0) = \left(\dfrac{\alpha}{3},0\right), \alpha \in [0,1], f(0,2) = (0,2). \\ g(\beta,0) = \left(\dfrac{\beta}{4},0\right), \beta \in [0,1], g(0,2) = (0,0). \end{cases}$$

and let the comparison function $\varphi(t) = \frac{3}{4}t, t \in [0,+\infty)$, in order to prove the mappings f and g satisfy the condition (1),we will divide it into 3 cases.

case 1 $x = (\alpha,0), y = (\beta,0), a = (0,2)$, we can easily check

$$\begin{aligned} s\cdot d(fx,gy,a) &= 2\cdot\left(\frac{\beta}{4}-\frac{\alpha}{3}\right)^2 < 2\cdot\frac{3}{4}(\alpha-\beta)^2 = \varphi\left(s\cdot d(x,y,a)\right) \\ &\le \varphi\Bigg(\max\Big\{s\cdot d(x,y,a), s\cdot d(x,fx,a), s\cdot d(y,gy,a), \\ &\qquad \frac{1}{2}d(x,gy,a)+\frac{1}{2}d(y,fx,a)\Big\}\Bigg) \end{aligned}$$

case 2 $x = (\alpha,0), y = (0,2), a = (0,2)$, we can easily check

$$\begin{aligned} s\cdot d(fx,gy,a) &= 2\cdot\left(\frac{\alpha}{3}\right)^2 = 2\cdot\frac{3}{4}\left(\frac{2}{3}\alpha\right)^2 = \varphi\left(s\cdot d(x,fx,a)\right) \\ &\le \varphi\Bigg(\max\Big\{s\cdot d(x,y,a), s\cdot d(x,fx,a), s\cdot d(y,gy,a), \\ &\qquad \frac{1}{2}d(x,gy,a)+\frac{1}{2}d(y,fx,a)\Big\}\Bigg) \end{aligned}$$

case 3 $x = (0,2), y = (\beta,0), a = (\alpha,0)$, we can easily check

$$\begin{aligned} s\cdot d(fx,gy,a) &= 2\cdot\left(\frac{\beta}{4}-\alpha\right)^2 \le 2\cdot\left(\frac{\beta}{4}-\frac{\alpha}{4}\right)^2 \le 2\cdot\frac{3}{4}(\beta-\alpha)^2 = \varphi\left(s\cdot d(x,y,a)\right) \\ &\le \varphi\Bigg(\max\Big\{s\cdot d(x,y,a), s\cdot d(x,fx,a), s\cdot d(y,gy,a), \\ &\qquad \frac{1}{2}d(x,gy,a)+\frac{1}{2}d(y,fx,a)\Big\}\Bigg) \end{aligned}$$

Now, we have proved the mappings f and g satisfy condition (1) and we can obtain the point $(0,0)$ is the unique common fixed point for f and g by using theorem 3.2.

Since b_2-metric space is a generalization of 2-metric space, we can obtain the following theorem.

Theorem 3.5 *Let* (X,d) *be a 2 metric spaces,* $f,g:X\to X$ *are two self mappings on X, and satisfy the condition*:

$$d(fx,gy,a) \le \varphi\left(\max\left\{d(x,y,a),d(x,fx,a),d(y,gy,a),\frac{1}{2}d(x,gy,a)+\frac{1}{2}d(y,fx,a)\right\}\right)$$

if fX or gX is complete, then f and g have an unique common fixed point.

Proof. In the proof of theorem 3.2, let $s = 1$, then the b_2 metric space turn to 2 metric space. Meanwhile, we can easily check the proof for theorem 3.2 holds for 2-metric space, therefore, we get the conclusion.

Theorem 3.6 *Let* (X,d) *be a 2 metric spaces,* $f,g:X\to X$ *are two self mappings on X, and satisfy the condition*:

$$d(fx,gy,a) \le h\left(\max\left\{d(x,y,a),d(x,fx,a),d(y,gy,a),\frac{1}{2}d(x,gy,a)+\frac{1}{2}d(y,fx,a)\right\}\right)$$

$h\in[0,1)$, if fX or gX is complete, then f and g have an unique common fixed point.

Proof. Let $\varphi(t)=ht, h\in[0,1)$, we can easily check $\varphi(t)=ht$ satisfy the comparison function's condition, from the theorem 3.5, we can get the conclusion.

4. Conclusion

In this paper, we introduce the concept of φ-contraction, and proved that mappings or family of mappings satisfying the contraction have the unique common fixed point. Meanwhile, we give an example for the theorem we proved. The results we obtained generalized many results in the b_2 metric spaces.

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