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Solvability of a Class of Operator-Differential Equations of Third Order with Complicated Characteristic on the Whole Real Axis

Abdel Baset I. Ahmed, Mohamed A. Labeeb

Egyptian Russian University, Badr City, Egypt
Email: abdel2007@yandex.ru, m.labeeb85@yahoo.com

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Abstract

On the whole real axis, we demonstrate sufficient conditions of regular solvability of third order operator-differential equations with complicated characteristics. These conditions were formulated only by the operator coefficients of the equation. In addition, by the principal part of the equation, the norms of the operators of intermediate derivative were estimated.

Subject Areas

Ordinary Differential Equation

Keywords

Operator-Differential Equation, Hilbert Space, Self-Adjoint Operator, Intermediate Derivative Operator

1. Introduction

In a separable Hilbert space H, we have the following equation:

$$Pu(x) \equiv p_0 u(x) + p_1 u(x) = f(x), x \in R, \quad (1)$$

where

$$p_0 u(x) = \left(\frac{\mathrm{d}}{\mathrm{d}t} - A\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + A\right)^2 u(x).$$

$$p_1 u(x) = \sum_{s=1}^{2} A_s \frac{\mathrm{d}^{3-s} u(x)}{\mathrm{d}x^{3-s}},$$

A is a self-adjoint positive-definite operator, and $A_s, s = 1,2$ are generally linear unbounded operators. All derivatives are understood in the sense of distributions theory.

We consider $f(x) \in L_2(R;H)$, where

$$L_2(R;H) = \left\{ f(x) : \|f(x)\|^2_{L_2(R;H)} \left(\int_{-\infty}^{+\infty} \|f(x)\|^2_H \, \mathrm{d}t \right)^{\frac{1}{2}} < +\infty \right\}$$

(see [1] [2]), and $u(x) \in W_2^3(R;H)$, which are determined as follows:

$$W_2^3(R;H) = \left\{ u(x) : \frac{\mathrm{d}^3 u(x)}{\mathrm{d}x^3} \in L_2(R_+;H), A^3 u(x) \in L_2(R;H) \right\}.$$

With the norm

$$\|u\|_{W_2^3(R_+;H)} = \left(\left\| \frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \right\|^2_{L_2(R;H)} + \|A^3 u\|^2_{L_2(R;H)} \right)^{\frac{1}{2}},$$

See [2].

Notice that the principal part of the investigated equation possesses complicated characteristic, not multiple characteristics as in [3].

Definition 1. If for any $f(x) \in L_2(R;H)$ there exists a vector function $u(x) \in W_2^2(R;H)$ that satisfies Equation (1) almost everywhere in *R*, then it is called a regular solution of Equation (1)

Definition 2. If for any $f(x) \in L_2(R;H)$ there exists a regular solution of Equation (1), and satisfies the inequality

$$\|u\|_{W_2^3(R;H)} \le const \|f\|_{L_2(R;H)}, \tag{2}$$

then Equation (1) is called regularly solvable.

It is known that if $p_1 : W_2^3(R;H) \to L_2(R;H)$, then $A^{3-s} \frac{\mathrm{d}^s u(x)}{\mathrm{d}x^s} \in L_2(R;H)$, $s = 1, 2$.

And the following inequalities are valid (see [2]).

$$\left\| A^{3-s} \frac{\mathrm{d}^s u(x)}{\mathrm{d}x^s} \right\|_{L_2(R;H)} \le c_s \|u\|_{W_2^2(R;H)}, \quad s = 1, 2. \tag{3}$$

Definition 3. Parseval's equality

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \tilde{f}(\zeta) \right|^2 \mathrm{d}\zeta$$

where,

$$\tilde{f}(\zeta) = \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{-ix\zeta} \, \mathrm{d}x.$$

2. Main Results

Theorem 1. The operator P_0 is an isomorphism from the space $W_2^3(R;H)$ to the space $L_2(R;H)$.

Proof. From (2), it is easy to prove that the operator P_0 acts from $W_2^3(R;H)$ to $L_2(R;H)$ be bounded. Using Fourier transforms for the equation $P_0 u(x) = f(x)$, we obtain

$$\left(-i\xi E-A\right)\left(-i\xi E+A\right)^{2}\tilde{u}\left(\xi\right)=\tilde{f}\left(\xi\right). \tag{4}$$

(E is the unit operator), where $\tilde{u}\left(\xi\right),\tilde{f}\left(\xi\right)$ are Fourier transform for the functions $u\left(x\right),f\left(x\right)$, respectively. The operator pencil $\left(-i\xi E-A\right)\left(-i\xi E+A\right)^{2}$ is invertible and moreover

$$\tilde{u}\left(\xi\right)=\left(-i\xi E-A\right)^{-1}\left(-i\xi E+A\right)^{-2}\tilde{f}\left(\xi\right), \tag{5}$$

Hence,

$$u\left(x\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(-i\xi E-A\right)^{-1}\left(-i\xi E+A\right)^{-2}\tilde{f}\left(\xi\right)\mathrm{e}^{i\zeta x}\mathrm{d}\zeta. \tag{6}$$

We show that $u\left(x\right)\in W_{2}^{3}\left(R;H\right)$. By using the Parseval equality and (3), we obtain:

$$\begin{aligned}\left\|u\right\|_{W_2^3(R;H)}^{2}&=\left\|\frac{\mathrm{d}^3u}{\mathrm{d}t^3}\right\|_{L_2(R;H)}^{2}+\left\|A^3u\right\|_{L_2(R;H)}^{2}=\left\|-i\zeta^{3}\tilde{u}\left(\xi\right)\right\|_{L_2(R;H)}^{2}+\left\|A^{3}\tilde{u}\left(\xi\right)\right\|_{L_2(R;H)}^{2}\\&=\left\|-i\zeta^{3}\left(-i\xi E-A\right)^{-1}\left(-i\xi E+A\right)^{-2}\tilde{f}\left(\xi\right)\right\|_{L_2(R;H)}^{2}\\&\quad+\left\|A^{3}\left(-i\xi E-A\right)^{-1}\left(-i\xi E+A\right)^{-2}\tilde{f}\left(\xi\right)\right\|_{L_2(R;H)}^{2}\\&\le\sup_{\zeta\in R}\left\|-i\zeta^{3}\left(-i\zeta E-A\right)^{-1}\left(-i\zeta E+A\right)^{-2}\right\|_{H\to H}^{2}\left\|\tilde{f}\left(\zeta\right)\right\|_{L_2(R;H)}^{2}\\&\quad+\sup_{\zeta\in R}\left\|A^{3}\left(-i\zeta E-A\right)^{-1}\left(-i\zeta E+A\right)^{-2}\right\|_{H\to H}^{2}\left\|\tilde{f}\left(\zeta\right)\right\|_{L_2(R;H)}^{2}\end{aligned} \tag{7}$$

If $\sigma\left(A\right)$ is a spectrum of the operator A, then we consider

$$\begin{aligned}&\sup_{\xi\in R}\left\|-i\zeta^{3}\left(-i\zeta E-A\right)^{-1}\left(-i\zeta E+A\right)^{-2}\right\|_{H\to H}\\&\le\sup_{\xi\in R}\sup_{\sigma\in\sigma(A)}\left|-i\zeta^{3}\left(-i\zeta-\sigma\right)^{-1}\left(-i\zeta+\sigma\right)^{-2}\right|\\&=\sup_{\xi\in R}\frac{\left|\zeta\right|^{3}}{\left(\zeta^2+\sigma^2\right)^{\frac{3}{2}}}\le1,\end{aligned} \tag{8}$$

$$\begin{aligned}&\sup_{\xi\in R}\left\|A^{3}\left(-i\zeta E-A\right)^{-1}\left(i\zeta E+A\right)^{-2}\right\|_{H\to H}\\&\le\sup_{\xi\in R}\sup_{\sigma\in\sigma(A)}\left|\sigma^{3}\left(-i\zeta-\sigma\right)^{-1}\left(-i\zeta+\sigma\right)^{-2}\right|\\&=\sup_{\sigma\in\sigma(A)}\frac{\sigma^{3}}{\left(\zeta^2+\sigma^2\right)^{\frac{3}{2}}}\le1\end{aligned} \tag{9}$$

Taking into account (5) and (6) into (4) we obtain:

$$\left\|u\right\|_{W_2^3(R;H)}^{2}\le2\left\|\overline{f}\left(\zeta\right)\right\|_{L_2(R;H)}^{2}=2\left\|f\left(x\right)\right\|_{L_2(R;H)}^{2}. \tag{10}$$

Consequently, $u\left(x\right)\in W_{2}^{3}\left(R;H\right)$.

Applying Banach theorem on the inverse operator, we get that the operator P_0 is an isomorphism from $W_{2}^{3}\left(R;H\right)$ to $L_{2}\left(R;H\right)$.

Now, we estimate the norms of intermediate derivative operators participating in the main part of the Equation (1) for finding exact conditions on regular solvability of the given equation, expressed only by its operator coefficients.

From theorem 1, we have that the norms $\|p_0 u\|_{L_2(R;H)}$ and $\|u\|_{W_2^3(R;H)}$ are equivalent in the space $W_2^3(R;H)$. Therefore by the norm $\|p_0 u\|_{L_2(R;H)}$ the theorem on intermediate derivatives is valid as well.

Theorem 2. Let $u(x) \in W_2^3(R;H)$. Then there hold the following inequalities:

$$\left\| A^{3-s} \frac{\mathrm{d}^s u(x)}{\mathrm{d}x^s} \right\|_{L_2(R;H)} \le a_s \|p_0 u\|_{L_2(R;H)}, \quad s = 1,2. \tag{11}$$

where $a_1 = a_2 = \dfrac{2}{3\sqrt{3}}$.

Proof. To establish the validity of inequality (11) we make change $p_0 u(x) = f(x)$ and apply the Fourier transformation. We get

$$\begin{aligned} &\left\| A^{3-s} (-i\zeta)^s (-i\zeta E - A)^{-1} (-i\zeta E + A)^{-2} \bar{f}(\zeta) \right\|_{L_2(R;H)} \\ &\le \sup_{\xi \in R} \left\| A^{3-s} (-i\zeta)^s (-i\zeta E - A)^{-1} (-i\zeta E + A)^{-2} \right\|_{H \to H} \left\| \bar{f}(\zeta) \right\|_{L_2(R;H)}, \quad s = 1,2. \end{aligned} \tag{12}$$

For $\zeta \in R$ we estimate the following norms:

$$\begin{aligned} &\left\| A^{3-s} (-i\zeta)^s (-i\zeta E - A)^{-1} (-i\zeta E + A)^{-2} \right\|_{H \to H} \\ &\le \sup_{\sigma \in \sigma(A)} \left| \sigma^{3-s} (-i\zeta)^s (-i\zeta - \sigma)^{-1} (i\zeta + \sigma)^{-2} \right| \\ &= \sup_{\sigma \in \sigma(A)} \left| \sigma^{-s} (-i\zeta)^s \left(-i\frac{\zeta}{\sigma} - 1 \right)^{-1} \left(-i\frac{\zeta}{\sigma} + 1 \right)^{-2} \right| \\ &\le \sup_{\mu = \frac{\xi^2}{\sigma^2} \ge 0} \frac{\mu^{s/2}}{(\mu+1)^{3/2}} = \frac{1}{3\sqrt{3}} s^{s/2} (3-s)^{(3-s)/2}, \quad s = 1,2. \end{aligned} \tag{13}$$

Finally, from (12), we have

$$\left\| A^{3-s} (-i\zeta)^s (-i\zeta E - A)^{-1} (-i\zeta E + A)^{-2} \tilde{f}(\zeta) \right\|_{L_2(R;H)} \le a_s \left\| \tilde{f}(\xi) \right\|_{L_2(R;H)}, \quad s = 1,2. \tag{14}$$

Lemma. The operator P_1 continuously acts from $W_2^3(R;H)$ to $L_2(R;H)$ provided that the operators $A_s A^{-s}, s = 1,2$ are bounded in *H*.

Taking into account the results found up [4] to now we get possibility to establish regular solvability conditions of Equation (1).

Theorem 3. Let the operators $A_s A^{-s}, s = 1,2$ be bounded in H and it holds the inequality $\sum_{s=1}^{2} a_s \left\| A_{3-s} A^{-(3-s)} \right\|_{H \to H} \prec 1$, where the numbers $a_s, s = 1,2$ are determined in theorem 2. Then the Equation (1) is regularly solvable.

Proof. By theorem 1, provided that the operator P_0 has a bounded inverse operator P_0^{-1} acting from $L_2(R;H)$ to $W_2^3(R;H)$, then after replacing $p_0 u(x) = v(x)$ in Equation (1) can be written as $\left(E + p_1 p_0^{-1}\right) v(x) = f(x)$.

Now we prove under the theorem conditions (see [5]), that the norm

$$\left\| p_1 p_0^{-1} \right\|_{L_2(R;H) \to L_2(R;H)} < 1.$$

By theorem (2), we have:

$$\begin{aligned}\left\| p_1 p_0^{-1} v \right\|_{L_2(R;H)} = \left\| p_1 u \right\|_{L_2(R;H)} &\le \sum_{s=1}^{2} \left\| A^s \frac{\mathrm{d}^{3-s} u}{\mathrm{d}x^{3-s}} \right\|_{L_2(R;H)} \\ &\le \sum_{s=1}^{2} \left\| A_s A^{-s} \right\|_{H \to H} \left\| A^s \frac{\mathrm{d}^{3-s} u}{\mathrm{d}x^{3-s}} \right\|_{L_2(R;H)} \\ &\le \sum_{s=1}^{2} a_{3-s} \left\| A_s A^{-s} \right\|_{H \to H} \left\| p_0 u \right\|_{L_2(R;H)} \\ &= \sum_{s=1}^{2} a_{3-s} \left\| A_s A^{-s} \right\|_{H \to H} \left\| v \right\|_{L_2(R;H)}\end{aligned}$$

Consequently,

$$\left\| p_1 p_0^{-1} \right\|_{L_2(R;H) \to L_2(R;H)} \le \sum_{s=1}^{2} a_{3-s} \left\| A_s A^{-s} \right\|_{H \to H} \prec 1.$$

Thus, the operator $E + p_1 p_0^{-1}$ is invertible in $L_2(R;H)$ and hence $u(x)$ can be determined by $u(x) = p_0^{-1} \left(E + p_1 p_0^{-1} \right)^{-1} f(x)$, moreover

$$\begin{aligned}\left\| u \right\|_{W_2^3(R;H)} &\le \left\| p_0^{-1} \right\|_{L_2(R;H) \to W_2^3(R;H)} \left\| \left(\left(E + p_1 p_0^{-1} \right) \right)^{-1} \right\|_{L_2(R;H) \to L_2(R;H)} \left\| f \right\|_{L_2(R;H)} \\ &\le const \left\| f \right\|_{L_2(R;H)}.\end{aligned} \tag{15}$$

The theorem is proved.

3. Conclusion

We formulated exact conditions on regular solvability of Equation (1), expressed only by its operator coefficients. We estimated the norms of intermediate derivative operators participating in the principle part of the given equation. In the case when in the perturbed part of Equation (1), the participant variable operator coefficients, *i.e.* $A_s(x), s = 1,2$ are linear operators, which determined for all $x \in R$, are investigated in a similar way.

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