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Construction of Solitary Wave Solutions and Rational Solutions for mKdV Equation with Initial Value Problem by Homotopy Perturbation Method

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Abstract

The mKdV equation with the initial value problem is studied numerically by means of the homotopy perturbation method. The analytical approximate solutions of the mKdV equation are obtained. Choosing the form of the initial value, the single solitary wave, two solitary waves and rational solutions are presented, some of which are shown by the plots.

Subject Areas

Partial Differential Equation

Keywords

mKdV Equation, Homotopy Perturbation Method, Soliton Solution

1. Introduction

Partial differential equations widely describe many phenomena in the world. Although many mathematicians and physicists presented various methods to find the explicit solutions of the partial differential equations, it is a difficult and important task to build the solutions of initial and boundary value problem. Recently, the homotopy perturbation method (HPM) have been applied into many problems [1]-[10] and tested to be an effective tool. Here, the initial value problem of the mKdV equation is studied by using HPM.

The initial value problem of mKdV equation is as following:

$$\begin{cases} u_t + 6u^2 u_x + u_{xxx} = 0, \\ u(x,0) = f(x). \end{cases} \tag{1}$$

The mKdV equation arises in many different fields, such as shallow water model, plasma science, biophysics and so on. A Darboux transformation was developed for generating dark multi-soliton solutions of the mKdV equation [11]. Based on the factorization of soliton equations into two commuting integrable x- and t-constrained flows, Ref. [12] derived N-soliton solutions for mKdV equation via its x- and t-constrained flows. Ref. [13] obtained numerical and exact compacton solutions of Equation (1) using by the variational iteration method. By using the extended tanh method, Wazwaz [14] got the abundant solitary wave solutions of the mKdV equation. Wazwaz [15] introduced new schemes to study the solitary wave solutions of the mKdV equation. Ref. [16] obtained the exact periodic solitary-wave solutions of the mKdV equation by the extended homoclinic test method. Applying the nonlocal conservation theorem method and the partial Lagrangian approach to the mKdV equation, the conservation laws were presented in Ref. [17]. From the observations on the tanh-coth expansion method, Parkes [18] found new solutions of the mKdV equation. By the bilinear approach, Ref. [19] obtained a symmetry constraint system and N-soliton solutions as group invariant solutions for the mKdV equation. In Ref. [20], ehe authors obtained an efficient numerical method to study the asymptotic solution of Equation (1). The authors studied compact solitary waves of the mKdV equation by using the phase portrait theory [21]. From the known Lax pair, Ref. [22] studied the nonlocal symmetry, optimal systems, and explicit solutions of the mKdV equation.

This paper is arranged as follows: In Section 2, by using HPM, we obtain the analytical approximate solution of Equation (1). In Section 3, by taking the form of the initial value, some exact solutions of mKdV equation are obtained. And some pictures are given to show the structure of the obtained solutions. Finally, some conclusions and discussions are given in Section 4.

2. The Homotopy Perturbation Method to mKdV Equation

In order to obtain the analytical approximate solution of Equation (1), we consider the one-parameter family of Equation (1) as follows

$$\left(u-u_0\right)_t+p\left(6u^2u_x+u_{xxx}\right)=0, \tag{2}$$

where the parameter $p\in\left[0,1\right]$ and $u_0=f\left(x\right)$.

If $p=0$, we meet $u=u_0$.

If $p=1$, we come back to the original problem (1). Let the solution $u\left(x,t\right)$ of the system (2) be written in the form of an infinite series,

$$u\left(x,t\right)=\sum_{i=0}^{\infty}u_i\left(x,t\right)p^i. \tag{3}$$

Then $u\left(x,t\right)=\sum_{i=0}^{\infty}u_i\left(x,t\right)$ is a series solution of Equation (1).

Substituting Equation (3) into Equation (2), and equating the coefficients of $p, p^2,\cdots$, we have

$$u_{1,t}+6u_0^2u_{0,x}+u_{0,xxx}=0, \tag{4}$$

$$u_{2,t} + 6u_0^2 u_{1,x} + u_{1,xxx} + 12u_0 u_1 u_{0,x} = 0, \tag{5}$$

$$u_{3,t} + 6u_0^2 u_{2,x} + 6u_1^2 u_{0,x} + u_{2,xxx} + 12u_0 u_1 u_{1,x} + 12u_0 u_2 u_{0,x} = 0, \tag{6}$$

and so on. Solving Equations (4), (5) and (6), one can obtain

$$u_1(x,t) = -\left(6u_0^2 u_{0,x} + u_{0,xxx}\right)t, \tag{7}$$

$$\begin{aligned} u_2(x,t) = \frac{1}{2}\Big(&144u_0^3 u_{0,x}^2 + 36u_0^4 u_{0,xx} + 12u_0^2 u_{0,xxxx} + 72u_{0,x}^2 u_{0,xx} \\ &+ 36u_0 u_{0,xx}^2 + 60u_0 u_{0,x} u_{0,xxx} + u_{0,xxxxxx}\Big)t^2, \end{aligned} \tag{8}$$

$$\begin{aligned} u_3(x,t) = -\frac{1}{6}\Big(&900u_{0,x}u_{0,xxx}^2 + 6480u_0^4 u_{0,x}^3 + 864u_{0,x}^5 + 504u_0 u_{0,xxx} u_{0,xxxx} \\ &+ 2376u_0^3 u_{0,xx} u_{0,xxx} + 6264u_0^2 u_{0,x}^2 u_{0,xxx} + 1512u_0^3 u_{0,x} u_{0,xxxx} \\ &+ 3888u_0^5 u_{0,x} u_{0,xx} + 7992u_0^2 u_{0,x} u_{0,xx}^2 + 144u_0 u_{0,x} u_{0,xxxxxx} + 324u_0 u_{0,xx} u_{0,xxxxx} \\ &+ 1404u_{0,x} u_{0,xx} u_{0,xxxx} + 324u_{0,x}^2 u_{0,xxxxx} + 108u_0^4 u_{0,xxxxx} + 216u_0^6 u_{0,xxx} \\ &+ 1296u_{0,xx}^2 u_{0,xxx} + 18u_0^2 u_{0,xxxxxxx} + 9504u_0 u_{0,x}^3 u_{0,xx} + u_{0,xxxxxxxxx}\Big)t^3. \end{aligned} \tag{9}$$

Hence, we obtain the solution of Equation (1)

$$u(x,t) = f(x) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots,$$

where $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ are given by Equations (7), (8) and (9) respectively.

3. Application

In this section, we will study the single soliton, two-soliton and rational solutions of mKdV equation.

3.1. Single Solitary Wave Solution

Consider the following case:

$$\begin{cases} u_t + 6u^2 u_x + u_{xxx} = 0, \\ u(x,0) = -\dfrac{2k\exp(kx)}{\exp(2kx)+1}. \end{cases}$$

From the above section, we can have

$$u_0(x,t) = -\frac{2k\exp(kx)}{\exp(2kx)+1},$$

$$u_1(x,t) = -\frac{2k^4\exp(kx)\left(\exp(2kx)-1\right)t}{\left(\exp(2kx)+1\right)^2},$$

$$u_2(x,t) = -\frac{k^7\exp(kx)\left(\exp(4kx)-6\exp(2kx)+1\right)t^2}{\left(\exp(2kx)+1\right)^3},$$

$$u_3(x,t) = -\frac{k^{10}\exp(kx)\left(\exp(6kx)-23\exp(4kx)+23\exp(2kx)-1\right)t^3}{3\left(\exp(2kx)+1\right)^4},$$

$$u(x,t) = -\frac{2k\exp(kx)}{\exp(2kx)+1} - \frac{2k^4\exp(kx)(\exp(2kx)-1)}{(\exp(2kx)+1)^2}t$$
$$-\frac{k^7\exp(kx)(\exp(4kx)-6\exp(2kx)+1)}{(\exp(2kx)+1)^3}t^2$$
$$-\frac{k^{10}\exp(kx)(\exp(6kx)-23\exp(4kx)+23\exp(2kx)-1)}{3(\exp(2kx)+1)^4}t^3+\cdots.$$

Using Taylor series, one can obtain the exact solution

$$u(x,t) = -\frac{2k\exp\left(k\left(x-k^2t\right)\right)}{\exp\left(2k\left(x-k^2t\right)\right)+1}. \tag{10}$$

Figure 1 shows the single soliton (10) for $k=-1$, $-4\le x\le 4$ and $-4\le t\le 4$. **Figure 2** shows the single soliton (10) for $k=-1$, $-4\le x\le 4$ and $t=0$.

3.2. Two Solitary Waves Solution

In this case, we take

$$f(x) = \frac{4\exp(x)}{\exp(2x)+1}.$$

Then from the above section, one can have

$$u_0(x,t) = \frac{4\exp(x)}{\exp(2x)+1},$$

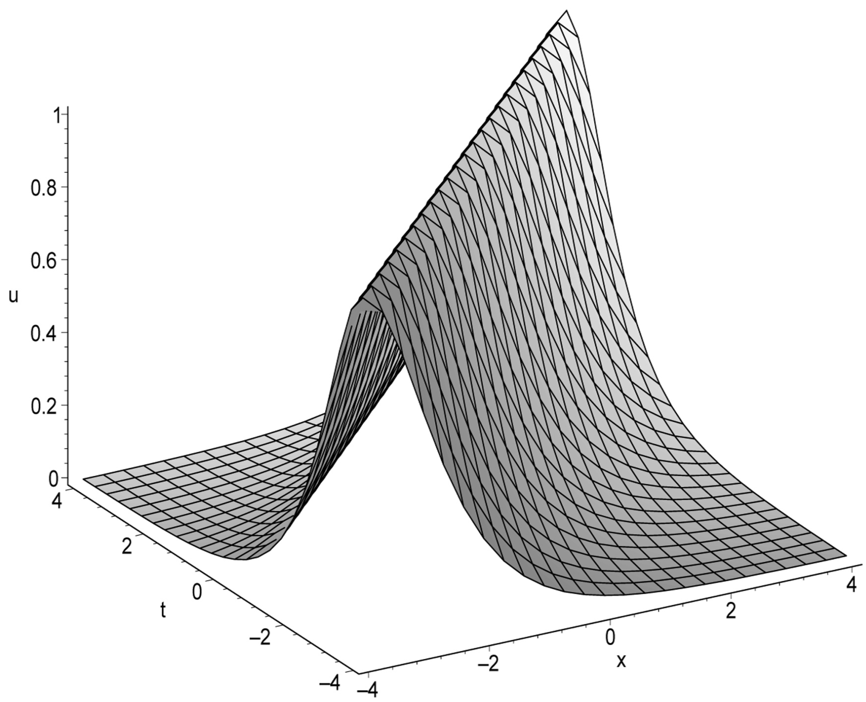

Figure 1. 3D plot of solution (10) for $k=-1$.

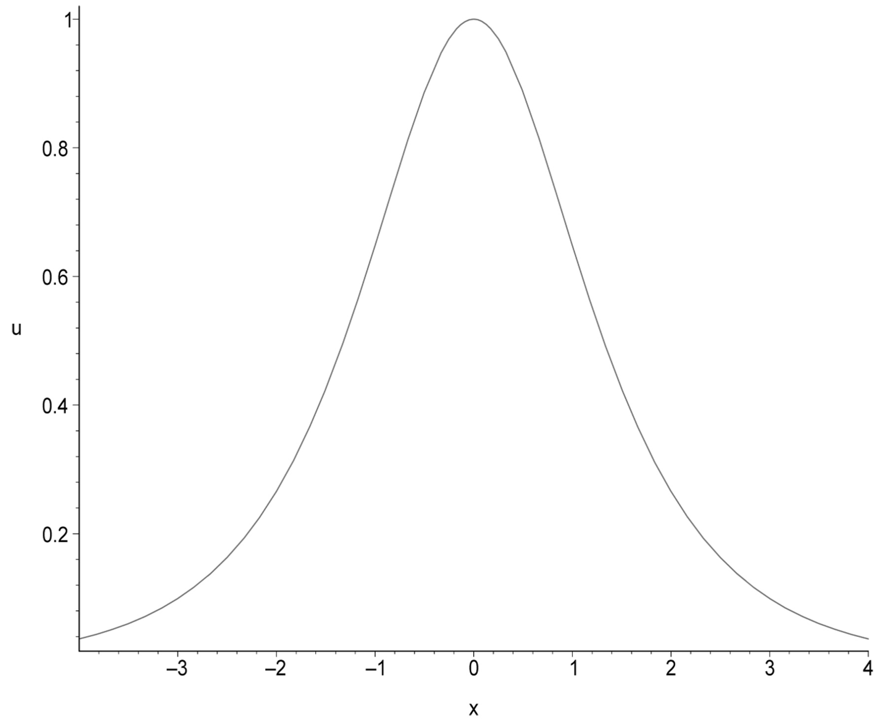

Figure 2. Plot of solution (10) for $k=-1$ and $t=0$.

$$u_1(x,t)=\frac{4\exp(x)}{(\exp(2x)+1)^4}(\exp(6x)+73\exp(4x)-73\exp(2x)-1)t,$$

$$u_2(x,t)=\frac{2\exp(x)}{(\exp(2x)+1)^7}(\exp(12x)+2158\exp(10x)+2863\exp(8x)$$
$$-26236\exp(6x)+2863\exp(4x)+2158\exp(2x)+1)t^2,$$

$$u_3(x,t)=\frac{2\exp(kx)}{3(\exp(2x)+1)^{10}}(\exp(18x)+58951\exp(16x)+225620\exp(14x)$$
$$-1999268\exp(12x)-6147250\exp(10x)+6147250\exp(8x)$$
$$+1999268\exp(6x)-225620\exp(4x)-58951\exp(2x)-1)t^3,$$

$$u(x,t)=\frac{4\exp(x)}{\exp(2x)+1}+\frac{4\exp(x)}{(\exp(2x)+1)^4}(\exp(6x)+73\exp(4x)-73\exp(2x)-1)t$$
$$+\frac{2\exp(x)}{(\exp(2x)+1)^7}(\exp(12x)+2158\exp(10x)+2863\exp(8x)$$
$$-26236\exp(6x)+2863\exp(4x)+2158\exp(2x)+1)t^2$$
$$+\frac{2\exp(kx)}{3(\exp(2x)+1)^{10}}(\exp(18x)+58951\exp(16x)+225620\exp(14x)$$
$$-1999268\exp(12x)-6147250\exp(10x)+6147250\exp(8x)$$
$$+1999268\exp(6x)-225620\exp(4x)-58951\exp(2x)-1)t^3+\cdots.$$

Using Taylor series, one can obtain the exact solution

$$u(x,t)=\frac{4\left(\exp(t-x)+3\exp(27t-3x)+3\exp(29t-5x)+\exp(55t-7x)\right)}{1+4\exp(2t-2x)+6\exp(28t-4x)+4\exp(54t-6x)+\exp(56t-8)}. \tag{11}$$

Figure 3 shows the two-soliton solution (11) for $-5\le x\le 5$ and $-0.5\le t\le 0.5$. Figure 4 shows the two-soliton solution (11) for $-6\le x\le 6$ and $t=-0.2$. Figure 5 shows the two-soliton solution (11) for $-4\le x\le 4$ and $t=0$. Figure 6 shows the two-soliton solution (11) for $-6\le x\le 6$ and $t=0.2$.

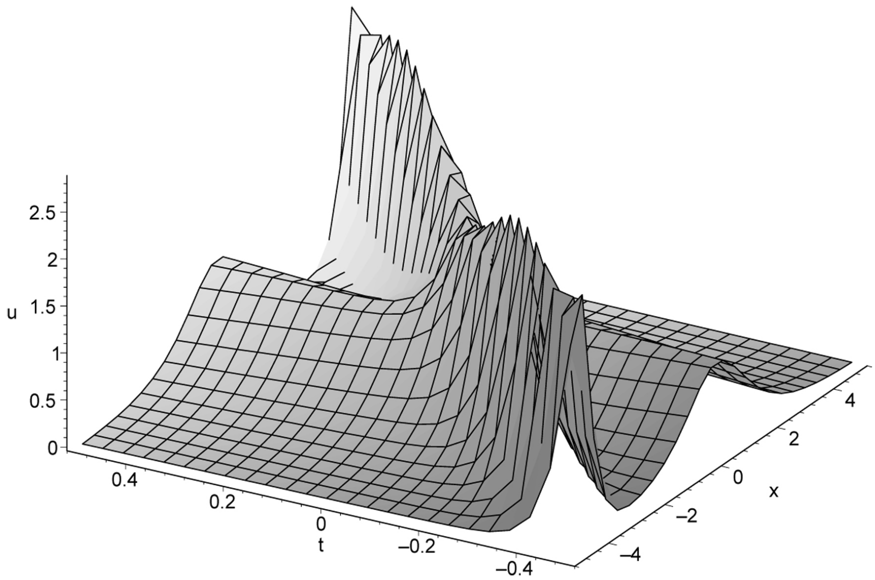

Figure 3. 3D plot of solution (11).

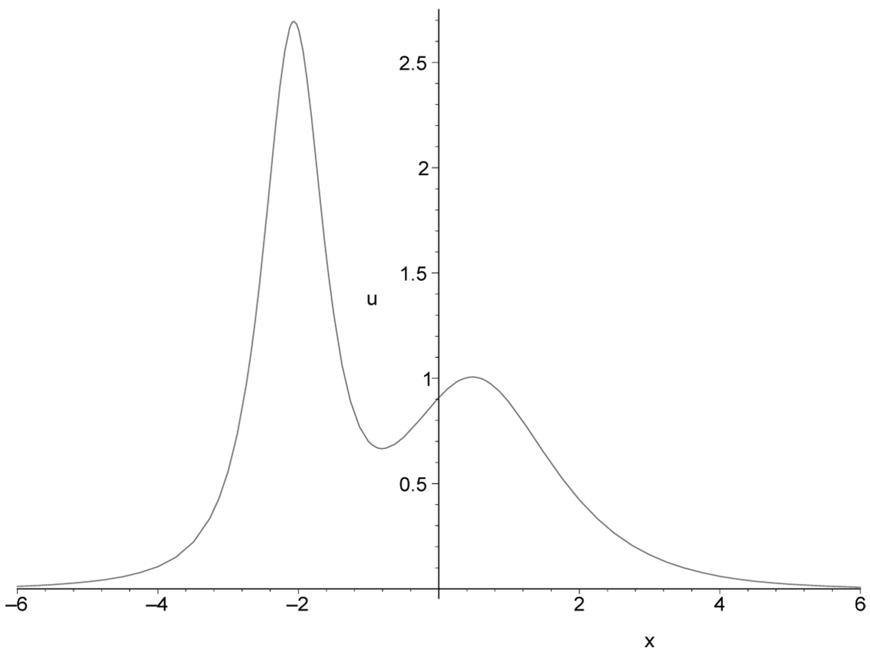

Figure 4. Plot of solution (11) for $t=-0.2$.

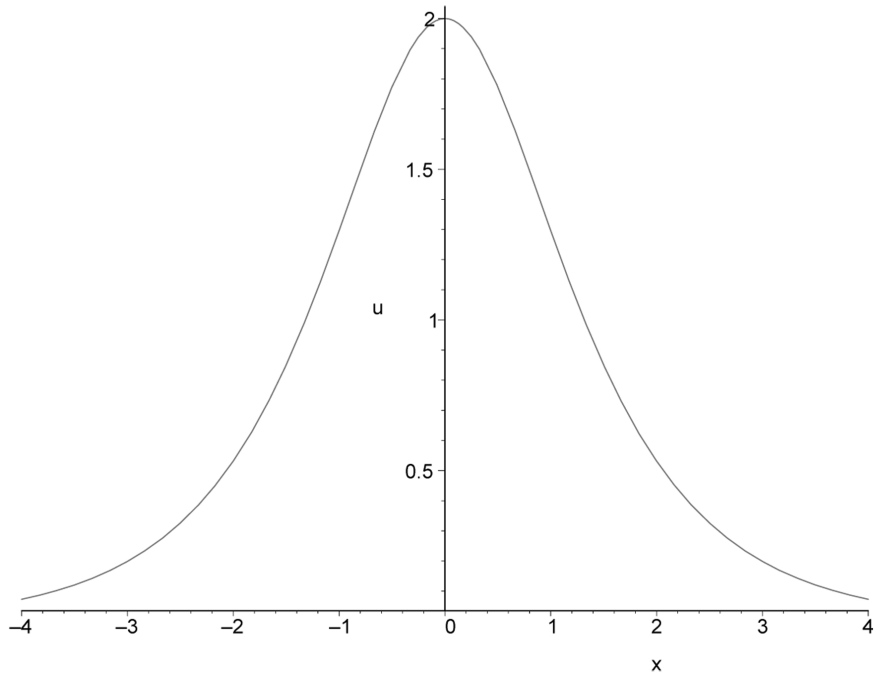

Figure 5. Plot of solution (11) for $t = 0$.

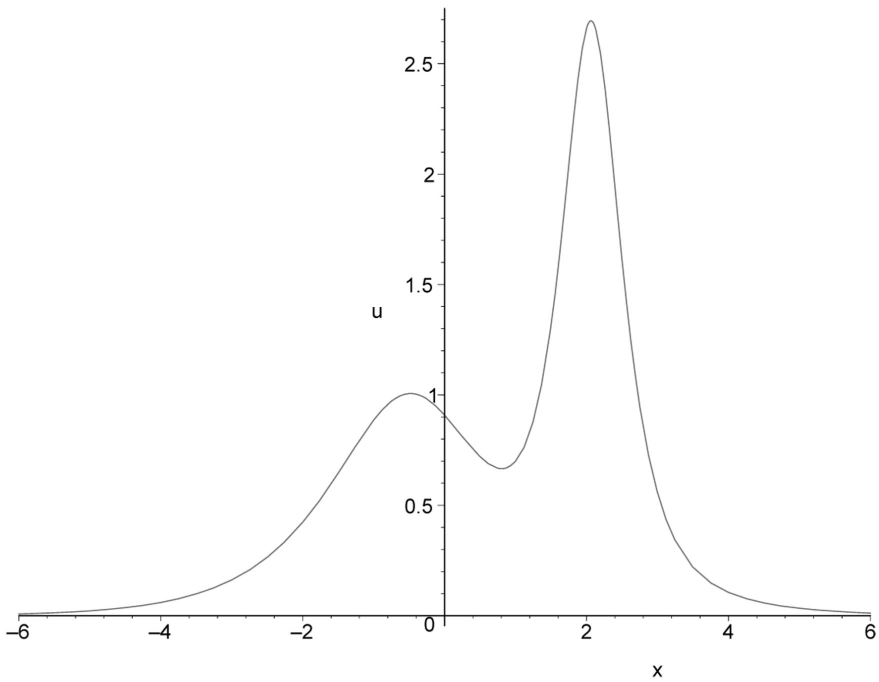

Figure 6. Plot of solution (11) for $t = 0.2$.

3.3. Rational Solution

Here, our goal is to find the rational solution of mKdV equation. To do this, we consider the form of the initial value as follows:

$$f(x) = \frac{2I}{x-a}$$

Due to the above section, it is obtained

$$u_0(x,t)=\frac{2I}{x-a},$$

$$u_1(x,t)=-\frac{36It}{(x-a)^4},$$

$$u_2(x,t)=\frac{432It^2}{(x-a)^7},$$

$$u_3(x,t)=-\frac{5184It^3}{(x-a)^{10}},$$

$$u(x,t)=\frac{2I}{x-a}-\frac{36I}{(x-a)^4}t+\frac{432I}{(x-a)^7}t^2-\frac{5184I}{(x-a)^{10}}t^3+\cdots.$$

From the knowledge of Taylor series, one can get the exact solution

$$u(x,t)=\frac{2I\left[(x-a)^3-6t\right]}{(x-a)\left[(x-a)^3+12t\right]},$$

which is singular at $x=a$ or $(x-a)^3+12t=0$.

4. Conclusion

In summary, we successfully apply homotopy perturbation method to the mKdV equation with the initial value problem and obtain the analytical approximate solution of the mKdV equation. Using the form of the initial value, the single solitary wave, two solitary waves and rational solutions of the mKdV are obtained. Here, we get the two solitary waves solution without using bilinear forms, Wronskian, etc. In our later works, we will focus on the form of the initial value that can create the two solitary waves solutions.

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