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Fixed Point Results for Weakly *C*-Contraction Mapping in Modular Metric Spaces

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Abstract

In this paper, we introduce the concept of weakly *C*-contraction mapping in modular metric spaces. And we established some fixed point results in *w*-complete spaces. Our results encompass various generalizations of Banach contraction.

Subject Areas

Mathematical Analysis

Keywords

Modular Metric Spaces Weakly *C*-Contraction Fixed Point Theory

1. Introduction

Fixed point theory has absorbed many mathematicians since 1922 with the celebrated Banach contraction principle (see [1]). It is one of the most useful results in nonlinear analysis, functional analysis and topology. Due to its application in mathematics, the Banach contraction principle has been generalized in many directions (see [2] [3] [4]).

Chatteriea in [5] introduced the notion of *C*-contraction which is a generalization of the Banach contraction.

Definition 1.1. [5] A mapping $T: X \to X$ where (X,d) is a metric space is said to be a *C*-contraction if there exists $\alpha \in \left[0,\frac{1}{2}\right)$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx,Ty) \le \alpha\left(d(x,Ty),d(y,Tx)\right) \tag{1}$$

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Chatteriea in [5] proved that if X is complete, then every C-contraction mapping have a unique fixed point.

The notion of C-contraction was generalized to a weak C-contraction by Choudhury in [6].

Definition 1.2. [6] Let (X,d) be a metric space and $T:X\to X$ be a map. Then T is called a weakly C-contraction (or a weak C-contraction) if there exists $\varphi:[0\to\infty)^2\to[0\to\infty)$ which is continuous, and $\varphi(x,y)=0$ if and only if $x=y=0$ such that

$$d(Tx,Ty)\le\frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]-\varphi\big(d(x,Ty),d(y,Tx)\big), \tag{2}$$

for all $x,y\in X$.

In [6] the author proved that if X is a complete metric space, then every weakly C-contraction has a unique fixed point. This fixed point theory was generalized to a complete, partially ordered metric space in [7] and a ordered 2-metric space in [8].

In 2006, Chistyakov introduced the notion of modular metric space in [9]. Recently, there have been many interesting results in the field of existence and uniqueness of fixed point in complete modular metric (see [10] [11]). In this paper, we will establish fixed point theorems for weakly C-contraction in modular metric space. The presented results extend some recent results in the literature.

2. Preliminaries

Throughout this paper $\mathbb{N}$ will denote the set of natural numbers.

The notion of modular metric space was introduced by Chistyakov in [9] [12] [13], who proved some fixed point results in such kind of spaces.

Let X be a nonempty set. Throughout this paper, for a function $w:(0,\infty)\times X\times X\to[0,\infty)$, we write

$$w_\lambda(x,y)=w(\lambda,x,y), \tag{3}$$

for all $\lambda>0$ and $x,y\in X$.

Definition 2.1. [9] Let X be a nonempty set. A function $w:(0,\infty)\times X\times X\to[0,\infty)$ is said to be a metric modular on X if it satisfies, for all $x,y,z\in X$, the following condition:

1) $w_\lambda(x,y)=0$ for all $\lambda>0$ if and only if $x=y$;
2) $w_\lambda(x,y)=w_\lambda(y,x)$ for all $\lambda>0$;
3) $w_{\lambda+\mu}(x,y)\le w_\lambda(x,z)+w_\mu(z,y)$ for all $\lambda,\mu>0$.

If instead of (i) we have only the condition (i')

$$w_\lambda(x,x)=0 \text{ for all } \lambda>0, x\in X,$$

then w is said to be a pseudomodular (metric) on X.

An important property of the (metric) pseudomodular on set X is that the mapping $\lambda\mapsto w_\lambda(x,y)$ is non increasing for all $x,y\in X$.

Definition 2.2. [9] Let w is a pseudomodular on X. Fixed $x_0\in X$. The set

$$X_w=X_w(x_0)=\left\{x\in X: w_\lambda(x,x_0)\to 0 \text{ as } \lambda\to\infty\right\}$$

is said to be a modular metric space (around x_0).

Definition 2.3. [14] Let X_w be a modular metric space.

1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_w is said to be *w*-convergent to $x \in X_w$ if and only if $w_\lambda(x_n, x) \to 0$, as $n \to \infty$ for some $\lambda > 0$;

2) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_w is said to be *w*-Cauchy if $w_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$ for some $\lambda > 0$;

3) A subset *C* of X_w is said to be *w*-complete if any *w*-Cauchy sequence in *C* is a convergent sequence and its limit is in *C*.

Definition 2.4. [15] Let w be a metric modular on *X* and X_w be a modular metric space induced by *w*. If X_w is a w-complete modular metric space and $T: X_w \to X_w$ be an arbitrary mapping T is called a contraction if for each $x, y \in X_w$ and for all $\lambda > 0$ there exists $0 \le k < 1$ such that

$$w_\lambda(Tx, Ty) \le k w_\lambda(x, y). \tag{4}$$

In [15] Chirasak proved that if X_w is a *w*-complete modular metric space, then contraction mapping *T* has a unique fixed point. At the same time, the author proved the following theorem.

Theorem 2.5. [15] *Let w be a metric modular on X,* X_w *be a w-complete modular metric space induced by w and* $T: X_w \to X_w$. *If*

$$w_\lambda(Tx, Ty) \le k\left(w_{2\lambda}(Tx, x) + w_{2\lambda}(Ty, y)\right), \tag{5}$$

for all $x, y \in X_w$ *and for all* $\lambda > 0$, *where* $k \in \left[0, \frac{1}{2}\right)$, *then T has a unique fixed point in* X_w. *Moreover, for any* $x \in X_w$, *iterative sequence* $\{T^n x\}$ *converges to the fixed point.*

3. Main Results

Theorem 3.1. *Let w be a metric modular on X,* X_w *be a w-complete modular metric space induced by w and* $T: X_w \to X_w$. *If*

$$w_\lambda(Tx, Ty) \le k\left(w_{2\lambda}(x, Ty) + w_{2\lambda}(y, Tx)\right), \tag{6}$$

for all $x, y \in X_w$ *and for all* $\lambda > 0$, *where* $k \in \left[0, \frac{1}{2}\right)$, *then T has a unique fixed point in* X_w.

Proof. Let x_0 be an arbitrary point in X_w and we write $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$, and in general, $x_n = Tx_{n-1} = T^2 x_0$ for all $n \in \mathbb{N}$. If $Tx_{n_0 - 1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$, then $Tx_{n_0} = x_{n_0}$. Thus x_{n_0} is a fixed point of *T*. Suppose that $Tx_{n-1} \ne Tx_n$ for all $n \in \mathbb{N}$. For $k \in \left[0, \frac{1}{2}\right)$, we have

$$\begin{aligned} w_\lambda(x_{n+1}, x_n) &= w_\lambda(Tx_n, Tx_{n-1}) \\ &\le k\left(w_{2\lambda}(x_n, Tx_{n-1}) + w_{2\lambda}(x_{n-1}, Tx_n)\right) \\ &= k w_{2\lambda}(x_{n-1}, x_{n+1}) \\ &\le k\left(w_\lambda(x_{n-1}, x_n) + w_\lambda(x_n, x_{n+1})\right), \end{aligned} \tag{7}$$

for all $\lambda > 0$ and all $n \in \mathbb{N}$. Hence,

$$w_\lambda(x_{n+1}, x_n) \le \frac{k}{1-k} w_\lambda(x_n, x_{n-1}), \tag{8}$$

for all $\lambda > 0$ and all $n \in \mathbb{N}$. Put $\beta := \frac{k}{1-k}$, since $k \in \left[0, \frac{1}{2}\right)$, we get $\beta \in [0,1)$ and hence

$$w_\lambda(x_{n+1}, x_n) \le \beta w_\lambda(x_n, x_{n-1}) \le \beta^2 w_\lambda(x_{n-1}, x_{n-2}) \le \cdots \le \beta^n w_\lambda(x_1, x_0), \tag{9}$$

for all $\lambda > 0$ and each $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} w_\lambda(x_{n+1}, x_n) = 0$ for all $\lambda > 0$. So for each $\lambda > 0$, we have for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $w_\lambda(x_{n+1}, x_n) < \varepsilon$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Without loss of generality, suppose $m, n \in \mathbb{N}$ and $m > n$. Observe that, for $\frac{\lambda}{m-n} > 0$ and for above-mentioned ε, there exists $n_{\lambda/(m-n)} \in \mathbb{N}$ such that

$$w_{\frac{\lambda}{m-n}}(x_{n+1}, x_n) < \frac{\varepsilon}{m-n}, \tag{10}$$

for all $n \ge n_{\lambda/(m-n)}$. Now we have

$$\begin{aligned} w_\lambda(x_n, x_m) &\le w_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + w_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + w_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n} = \varepsilon, \end{aligned} \tag{11}$$

for all $m, n \ge n_{\lambda/(m-n)} \in \mathbb{N}$. This implies $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of X_w, there exists point $x \in X_w$, such that $x_n \to x$ as $n \to \infty$.

By the notion of metric modular w and the contraction of *T*, we get

$$\begin{aligned} w_\lambda(Tx, x) &\le w_{\frac{\lambda}{2}}(Tx, Tx_n) + w_{\frac{\lambda}{2}}(Tx_n, x) \\ &\le k\left(w_\lambda(x, Tx_n) + w_\lambda(x_n, Tx)\right) + w_{\frac{\lambda}{2}}(Tx_n, x) \\ &= k\left(w_\lambda(x, x_{n+1}) + w_\lambda(x_n, Tx)\right) + w_{\frac{\lambda}{2}}(x_{n+1}, x), \end{aligned} \tag{12}$$

for all $\lambda > 0$ and for all $n \in \mathbb{N}$. Taking $n \to \infty$ in inequality (12), we obtained that

$$w_\lambda(Tx, x) \le k w_\lambda(Tx, x). \tag{13}$$

Since $k \in \left[0, \frac{1}{2}\right)$, we have $Tx = x$. Thus, x is a fixed point of *T*. Next, we prove that x is a unique fixed point. Suppose that z be another fixed point of *T*. We note that

$$\begin{aligned} w_\lambda(x, z) &= w_\lambda(Tx, Tz) \\ &\le k\left(w_{2\lambda}(x, Tz) + w_{2\lambda}(z, Tx)\right) \\ &\le k\left(w_\lambda(x, z) + w_\lambda(z, Tz) + w_\lambda(z, x) + w_\lambda(x, Tx)\right) \\ &= 2k w_\lambda(x, z), \end{aligned} \tag{14}$$

for all $\lambda > 0$. Therefore we have

$$(1-2k)w_\lambda(x,z)\le 0.$$

Since $1-2k>0$, we can imply that $x=z$. Therefore, x is a unique fixed point of *T*. □

Next, we will introduce the notion of weakly *C*-contraction in modular metric space.

Definition 3.2. Let w be a metric modular on *X*, X_w be a modular metric space induced by *w*. A mapping $T:X_w\to X_w$ is said to be a weak C-contraction in X_w if for all $x,y\in X_w$ and for all $\lambda>0$, the following inequality holds:

$$w_\lambda(Tx,Ty)\le\frac{1}{2}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right)-\varphi\left(w_\lambda(x,Ty),w_\lambda(y,Tx)\right), \tag{15}$$

where $\varphi[0,\infty)^2\to[0,\infty)$ is a continuous mapping such that $\varphi(x,y)=0$ if and only if $x=y$.

Theorem 3.3. *Let w be a metric modular on X,* X_w *be a w-complete modular metric space induced by w. Let* $T:X_w\to X_w$ *be a weak C-contraction in* X_w *such that T is continuous and non-decreasing. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X_w and we write $x_1=Tx_0$, $x_2=Tx_1=T^2x_0$, and in general, $x_n=Tx_{n-1}=T^2x_0$ for all $n\in\mathbb{N}$. If $Tx_{n_0-1}=Tx_{n_0}$ for some $n_0\in\mathbb{N}$, then $Tx_{n_0}=x_{n_0}$. Thus x_{n_0} is a fixed point of *T*. Suppose that $Tx_{n-1}\ne Tx_n$ for all $n\in\mathbb{N}$, we have

$$\begin{aligned}
&w_\lambda(x_{n+1},x_n)=w_\lambda(Tx_n,Tx_{n-1})\\
&\le\frac{1}{2}\left(w_{2\lambda}(x_n,Tx_{n-1})+w_{2\lambda}(x_{n-1},Tx_n)\right)-\varphi\left(w_\lambda(x_n,Tx_{n-1}),w_\lambda(x_{n-1},Tx_n)\right)\\
&=\frac{1}{2}\left(w_{2\lambda}(x_n,x_n)+w_{2\lambda}(x_{n-1},x_{n+1})\right)-\varphi\left(w_\lambda(x_n,x_n),w_\lambda(x_{n-1},x_{n+1})\right)\\
&=\frac{1}{2}w_{2\lambda}(x_{n-1},x_{n+1})-\varphi\left(0,w_\lambda(x_{n-1},x_{n+1})\right)\\
&\le\frac{1}{2}w_{2\lambda}(x_{n-1},x_{n+1})\le\frac{1}{2}\left(w_\lambda(x_{n-1},x_n)+w_\lambda(x_n,x_{n+1})\right),
\end{aligned} \tag{16}$$

for all $\lambda>0$. The last inequality gives us

$$w_\lambda(x_n,x_{n+1})\le w_\lambda(x_{n-1},x_n),$$

for all $\lambda>0$ and for all $n\in\mathbb{N}$. Thus $\{w_\lambda(x_n,x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

For each $\lambda>0$, let

$$\lim_{n\to\infty}w_\lambda(x_n,x_{n+1})=r. \tag{17}$$

Letting $n\to\infty$ in (16) we have

$$r\le\lim_{n\to\infty}\frac{1}{2}w_\lambda(x_{n-1},x_{n+1})\le\frac{1}{2}(r+r)=r. \tag{18}$$

or, equivalently,

$$\lim_{n\to\infty}w_\lambda(x_{n-1},x_{n+1})=2r. \tag{19}$$

Again, making $n\to\infty$ in (17), (19) and the continuity of φ we have

$$r\le\frac{1}{2}2r-\varphi(0,2r)=r-\varphi(0,2r)\le r. \tag{20}$$

And, consequently, $\varphi(0,2r)=0$. This gives us that $r=0$ by our assumption about φ.

Thus, for all $\lambda>0$, we have

$$\lim_{n\to\infty} w_\lambda(x_n,x_{n+1})=0. \tag{21}$$

From the proof of theorem 3.1, we can prove that $\{x_n\}$ is a w-Cauchy sequence. By the completeness of X_w, there exists a point $x\in X_w$, such that $x_n\to x$ as $n\to\infty$.

By the notion of metric modular w and the contraction of T, we get

$$\begin{aligned} w_\lambda(Tx,x) &\le w_{\frac{\lambda}{2}}(Tx,Tx_n)+w_{\frac{\lambda}{2}}(Tx_n,x)\\ &\le \frac{1}{2}\left(w_\lambda(x,Tx_n)+w_\lambda(x_n,Tx)\right)\\ &\quad -\varphi\left(w_\lambda(x,Tx_n),w_\lambda(x_n,Tx)\right)+w_{\frac{\lambda}{2}}(Tx_n,x)\\ &= \frac{1}{2}\left(w_\lambda(x,x_{n+1})+w_\lambda(x_n,Tx)\right)\\ &\quad -\varphi\left(w_\lambda(x,x_{n+1}),w_\lambda(x_n,Tx)\right)+w_{\frac{\lambda}{2}}(x_{n+1},x), \end{aligned} \tag{22}$$

for all $\lambda>0$ and for all $n\in\mathbb{N}$. Taking $n\to\infty$ by (22), we obtained that

$$w_\lambda(Tx,x)\le\frac{1}{2}w_\lambda(Tx,x)-\varphi\left(0,w_\lambda(Tx,x)\right). \tag{23}$$

This prove that $x=Tx$. Thus x is a fixed point of T. Next, we prove that x is a unique fixed point. Suppose that z and x are different fixed points of T, then from (15), we have

$$\begin{aligned} w_\lambda(z,x)&=w_\lambda(Tz,Tx)\\ &\le\frac{1}{2}\left(w_{2\lambda}(z,Tx)+w_{2\lambda}(x,Tz)\right)-\varphi\left(w_\lambda(z,Tx),w_\lambda(x,Tz)\right)\\ &\le w_{2\lambda}(x,z)-\varphi\left(w_\lambda(z,x),w_\lambda(x,z)\right), \end{aligned} \tag{24}$$

for all $\lambda>0$ By the property of the φ, we have $x=z$. Hence x is a unique fixed point of T. □

Example 3.4 *Let* $X=\{(a,0)\in R^2\mid a\ge 0\}\cup\{(0,b)\in R^2\mid b\ge 0\}$. *Defined the mapping* $w:(0,\infty)\times X\times X\to[0,\infty)$ *by*

$$w_\lambda((a_1,0),(a_2,0))=\frac{3|a_1-a_2|}{\lambda},$$

$$w_\lambda((0,b_1),(0,b_2))=\frac{|b_1-b_2|}{\lambda},$$

and

$$w_\lambda((a,0),(0,b))=\frac{3a}{\lambda}+\frac{b}{\lambda}=w_\lambda((0,b),(a,0)).$$

We note that if we take $\lambda \to \infty$, then we see that $X = X_w$ and also T and φ is define by

$$T\left(\left(a,0\right)\right)=\left(0,\frac{a}{2}\right),$$

$$T\left(\left(0,b\right)\right)=\left(\frac{b}{24},0\right).$$

and

$$\varphi\left(x,y\right)=\frac{1}{20}\left(x+y\right).$$

We can imply that

$w_\lambda\left(Tx,Ty\right)\le\frac{1}{2}\left(w_{2\lambda}\left(x,Ty\right)+w_{2\lambda}\left(y,Tx\right)\right)-\varphi\left(w_\lambda\left(x,Ty\right),w_\lambda\left(y,Tx\right)\right)$ for all $x,y\in X$ and all $\lambda>0$.

Indeed, case1. let $x=\left(a_1,0\right), y=\left(a_2,0\right)$, then

$$w_\lambda\left(Tx,Ty\right)=w_\lambda\left(T\left(a_1,0\right),T\left(a_2,0\right)\right)=w_\lambda\left(\left(0,\frac{a_1}{2}\right),\left(0,\frac{a_2}{2}\right)\right)=\frac{\left|a_1-a_2\right|}{2\lambda}, \quad (25)$$

$$w_{2\lambda}\left(x,Ty\right)=w_{2\lambda}\left(\left(a_1,0\right),T\left(a_2,0\right)\right)=w_{2\lambda}\left(\left(a_1,0\right),\left(0,\frac{a_2}{2}\right)\right)=\frac{3a_1}{2\lambda}+\frac{a_2}{4\lambda}, \quad (26)$$

$$w_{2\lambda}\left(y,Tx\right)=w_{2\lambda}\left(\left(a_2,0\right),T\left(a_1,0\right)\right)=w_{2\lambda}\left(\left(a_2,0\right),\left(0,\frac{a_1}{2}\right)\right)=\frac{3a_2}{2\lambda}+\frac{a_1}{4\lambda}, \quad (27)$$

$$w_\lambda\left(Tx,Ty\right)\le\frac{2}{7}\left(w_{2\lambda}\left(x,Ty\right)+w_{2\lambda}\left(y,Tx\right)\right). \quad (28)$$

Case 2. let $x=\left(0,b_1\right), y=\left(0,b_2\right)$, we have

$$w_\lambda\left(Tx,Ty\right)=w_\lambda\left(T\left(0,b_1\right),T\left(0,b_2\right)\right)=w_\lambda\left(\left(\frac{b_1}{24},0\right),\left(\frac{b_2}{24},0\right)\right)=\frac{\left|b_1-b_2\right|}{8\lambda}, \quad (29)$$

$$w_{2\lambda}\left(x,Ty\right)=w_{2\lambda}\left(\left(0,b_1\right),T\left(0,b_2\right)\right)=w_{2\lambda}\left(\left(0,b_1\right),\left(\frac{b_2}{24},0\right)\right)=\frac{b_2}{16\lambda}+\frac{b_1}{2\lambda}, \quad (30)$$

$$w_\lambda\left(Tx,Ty\right)\le\frac{2}{9}\left(w_{2\lambda}\left(x,Ty\right)+w_{2\lambda}\left(y,Tx\right)\right). \quad (31)$$

Case 3. Let $x=\left(a,0\right), y=\left(0,b\right)$, then

$$w_\lambda\left(Tx,Ty\right)=w_\lambda\left(T\left(a,0\right),T\left(0,b\right)\right)=w_\lambda\left(\left(0,\frac{a}{2}\right),\left(\frac{b}{24},0\right)\right)=\frac{b}{8\lambda}+\frac{a}{2\lambda}, \quad (32)$$

$$w_{2\lambda}\left(x,Ty\right)=w_{2\lambda}\left(\left(a,0\right),T\left(0,b\right)\right)=w_{2\lambda}\left(\left(a,0\right),\left(\frac{b}{24},0\right)\right)=\left|\frac{b}{16\lambda}-\frac{3a}{2\lambda}\right|, \quad (33)$$

$$w_{2\lambda}\left(y,Tx\right)=w_{2\lambda}\left(\left(0,b\right),T\left(a,0\right)\right)=w_{2\lambda}\left(\left(0,b\right),\left(0\frac{a}{2}\right)\right)=\left|\frac{b}{2\lambda}-\frac{a}{4\lambda}\right|, \quad (34)$$

$$w_\lambda\left(Tx,Ty\right)\le\frac{2}{5}\left(w_{2\lambda}\left(x,Ty\right)+w_{2\lambda}\left(y,Tx\right)\right). \quad (35)$$

$$\begin{aligned}\varphi\left(w_\lambda(x,Ty),w_\lambda(y,Tx)\right)&=\frac{1}{20}\left(w_\lambda((x,Ty)+w_\lambda(y,Tx)\right)\\&=\frac{1}{20}\left[2\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right)\right]\\&=\frac{1}{10}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right).\end{aligned}\tag{36}$$

Hence we have

$$w_\lambda(Tx,Ty)\le\frac{2}{5}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right),\tag{37}$$

for all $\lambda>0$ and $x,y\in X$. And

$$\begin{aligned}&\frac{1}{2}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right)-\varphi\left(w_\lambda(x,Ty),w_\lambda(y,Tx)\right)\\&=\frac{1}{2}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right)-\frac{1}{10}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right)\\&=\frac{2}{5}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right),\end{aligned}\tag{38}$$

for all $\lambda>0$ and $x,y\in X$. We can get

$$w_\lambda(Tx,Ty)\le\frac{1}{2}\left(w_{2\lambda}(x,Ty)+w_{2\lambda}(y,Tx)\right)-\varphi\left(w_\lambda(x,Ty),w_\lambda(y,Tx)\right),\tag{39}$$

for all $x,y\in X$ and all $\lambda>0$. Thus T is a weakly C-contractive mapping. Therefore, T has a unique fixed point that is $(0,0)\in X_w$.

On the Euclidean metric d on X_w, we see that

$$\begin{aligned}d\left(T(1,0),T\left(0,\frac{1}{2}\right)\right)>\frac{1}{2}&\left(d\left(T(1,0),T\left(0,\frac{1}{2}\right)\right)+d\left(\left(0,\frac{1}{2}\right),T(1,0)\right)\right)\\&-\varphi\left(d\left((1,0),T\left(0,\frac{1}{2}\right)\right),d\left(\left(0,\frac{1}{2}\right),T(1,0)\right)\right).\end{aligned}\tag{40}$$

Thus, T is not a weak C-contraction on standard metric space.

4. Conclusion

In this paper, we extend the fixed point results for the weakly C-contraction in modular metric space. Moreover, as example, we give a unique fixed point theorem for a mapping satisfying a weak C-contractive condition in modular metric space rather than in standard metric space. The main results of this article generalize and unify some recent results given by some authors.

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