Open Access Library Journal
2018, Volume 5, e4096
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Extensions of the System *N* of Natural Numbers Assigned to Primary Teachers

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How to cite this paper: Marjanović, M.M. (2018) Extensions of the System *N* of Natural Numbers Assigned to Primary Teachers. *Open Access Library Journal*, **5**: e4096.
https://doi.org/10.4236/oalib.1104096

Received: October 30, 2017
Accepted: January 2, 2018
Published: January 5, 2018

Abstract

This paper has resulted from a discussion conducted about which content of mathematics course should be the most suitable for the students of the institutions where primary teachers are educated and when such a course exists on the curriculum. It is normal to require that a primary teacher knows, in a deeper way, mathematics at proximate levels (preschool and upper classes of elementary school). In particular, this supposes a good acquaintance of the teacher with extensions of number systems. We propose that algebra is used as a tool of expressing properties of operations and the order relation in the system N of natural numbers. The jottings $m:n,(n \neq 0)$ and $m-n$ have their meaning in N only when m is divisible by n and when $m \geq n$, respectively. But these jottings are potentially numbers in an extended sense and we use former of them to extend the system N to the system Q_+ of non-negative rational numbers and the latter of them to extend N to the system Z of integers. This natural way of extension is based on the permanence of properties already established in N and they are used to define operations and the order relation in the extended systems (Peacock's principle). In this paper, we present the extension of N to Q_+ in detail and we just sketch the extension of N to Z. Let us note that the fractions are interpreted as the result of equal sharing of quantities and they are represented on the number axis as the sequences of adjacent intervals of equal length whose union is the interval representing the corresponding rational number. We leave open the question up to which degree these extensions should be supplied with all technical details and when some of these details should be just suggested. The answers to such a question certainly depend on many specific factors.

Subject Areas

Education

Keywords

Properties of Operations and the Order Relation in the System N of Natural Numbers, Non-Negative Rational Numbers, Integers, Permanence of Properties

1. Introduction

This paper partly represents the standpoint of this author held in a discussion about the content of mathematics course at the institutions where primary teachers are educated and where such a course exists on the curriculum. According to our opinion, formed upon our own experience, the content of a math course at these institutions has to be subordinate to the goals of the course of didactics of mathematics and we consider repetition and deepening of school mathematics to be an important subject of learning mathematics for this kind of students. In particular, a systematic treatment of number systems should be in the front of that content (without any intension to develop technical skills). Yes, of course, everyone would agree that such a treatment has to be systematic and a view from above, but when the professional needs of these students are concerned, the right question is the degree up to which intuition is abandoned in favor of logical rigor. (If the reader of this paper finds that some of our arguments are polemic in tone, then it is certainly due to the mentioned discussion).

Axiomatically, the system of integers Z can be introduced as the smallest order ring, the system of rational numbers Q as the smallest order field and the system of real numbers R as the unique continuous order field. In the case of each of these systems a construction (not the axioms) provides their existence. An important aspect of the corresponding axioms of these systems is the fact that they express the minimum of the properties of the operations and the order relation from which all other their properties can be deduced. Also, by means of generalization, axiomatising of number systems has produced the axioms of such mathematical structures as are groups, rings, fields etc., each of which has a variety of important non-isomorphic examples. Well, we can easily agree with H. Freudenthal when he says: "…as popular as axioms and axiomatising might be in modern mathematics, they are only the highlights and the finishing touches in the course of an activity where the stress is on the form rather than on the content" (See [1]). Thus, a recommendation for a math student could be to read the excellent Landau's book [2], but not to a student of primary education. In this paper which is a continuation of our paper [3], we describe how the system N of natural numbers is extended to the system Q_+ of non-negative rational numbers and we just give a sketch how such an extension goes when the system Z of integers is constructed.

When the system N is extended to the system Q_+ and when "the stress is on the form", then two ordered pairs of natural numbers (k,l) and

$(m,n),(l \neq 0, n \neq 0)$ are taken to be equivalent if $kn = lm$. The sum and the product of these pairs are taken to be $(kn + lm, ln)$ and (km, ln), respectively. Then the set of all such ordered pairs splits into equivalent classes which are taken to be numbers in Q_+ and the sum and the product of pairs induce these operations on the set Q_+. These definitions seem to be artificial and arbitrary chosen if it is not said that they are motivated by the properties of fractions (seeing (m,n) to be m/n). Instead of such a "finishing touch", we propose here an interpretation of the quotient $m:n,(n \neq 0)$ in the case when m is not divisible by n, understanding $m:n$ as a number in a larger sense and establishing the properties of those numbers being guided by the Peacock's principle of permanence of equivalent forms.

Writing the relations as, for example, $8+5=5+8$, $3\times(5+4)=3\times5+3\times4$, etc., the rule of interchange of summands and the rule of multiplication of the sum are expressed procedurally. This means that these relations continue to hold true when, in the former case, the numbers 8 and 5 are replaced by any two other natural numbers and, in the latter case, the numbers 3, 5 and 4 are replaced by any three other natural numbers. For such relations we say that they have invariant form, while, for example, the true relations $8+5=13$ and $3\times(5+4)=15+12$ do not have such a form. These rules can also be expressed rhetorically (in words) and symbolically as $m+n=n+m$, $k(m+n)=km+kn$. When the rules are established in the system N of natural numbers and expressed symbolically, than they continue to hold true when the letters representing variables are taken to represent elements of extended systems: Q_+ of non-negative rational numbers, Q of rational numbers, Z of integers and R of real numbers. This fact expresses the Peacock's principle of permanence of equivalent forms. Guided by this principle, the operations and the order relation are defined in the extended systems and their properties are established. In this way, the differences and the quotients, being in N the jottings without meaning, become numbers in a more general sense.

Let m and n be two natural numbers, then the jotting $m:n$ has the meaning in N, only when m is divisible by n. Algorithm of division with remainders produces two unique natural numbers q and r, ($0 \leq r < n$) such that $m = qn + r$, (q is called the quotient and r the remainder on dividing m by n). Thus, when $m:n$ is not a natural number, this jotting gets the meaning of a relationship between m and n (being a blurred idea how many times m is greater than n or, better to say, being the comparison of the values of these numbers). Without any restriction, a jotting of the form $m:n$ is called the ratio of numbers m and n and potentially, the ratios are conveyors of a more general meaning of the number than the natural numbers are.

Interpreting division as partition of a set having m elements into n subsets of equal cardinality, when the set to be partitioned consists of real world objects, then such partition is also called *equal sharing*. Let us consider the case of equal sharing, when m is not divisible by n and when r objects remain unshared. If these objects are homogeneous continuous quantities, equal to each other in shape and size, such a process of equal sharing continues by dividing the remaining objects into n equal parts. Then, each share consists of q whole object and r parts. Such a part is denoted by $1/n$ and r such parts by r/n. Then, a share has $qn+r$ such parts, what is denoted writing $(qn+r)/n$ or m/n. Thus, the notation m/n suggests a relationship between the number m of all parts and the number n of those parts which make a whole. The notations as $1/n, r/n, m/n$ etc. which denote the relationship of the number of selected parts and the number of equal parts of a whole are called fractions. (Due to the typographic convenience, when denoting fractions we use the oblique fraction bar instead of the horizontal one). In the context of a sharing, a fraction m/n denotes a quantity, what makes it easier acceptable to be a number, than when it indicates a ratio $m:n$. Nevertheless, the fractions are just a specific interpretation of the ratios and in that particular case, m/n is just a specific notation for $m:n$. (The paper [4] is a thorough analysis of the difficulties that a school student encounters learning rational numbers).

Equal sharing in the form of various practical activities of the regulation of the trade and the market led to the use of fractions in the ancient civilizations of Egypt, Babylon, China and India. In the period of Roman Empire no symbols were used to denote fractions. Instead, several Latin words were used to denote some fractions (*uncia* for 1/12, *semis for* 6/12, *semuncia for* 1/24, etc.). Fractions did not exist in European mathematics until late Renaissance (Simon Stevin, John Napier). An explanation for such state of affairs might be a strict following of the ideas of ancient Greek mathematics. Books V and VII-IX of the Euclid's Elements [5] present theory of natural numbers, ratios and proportions and the theory of (positive) real numbers with rigor and logical purity. Ancient Greeks conceived numbers as ratios of quantities of the same kind. Geometric objects were considered to be such quantities and particularly, the line segments were considered to be the purest representatives of that kind. Operations were not performed with numbers but instead, with the quantities. For example the product of two line segments was defined to be the rectangle with its sides being these segments.

Being rhetorical (expressed in words), with its operations performed as constructions on geometric objects, Greek arithmetic was rather complicated and tedious. Lacking ease and elegance, this arithmetic lasted until creation of Viète's symbolic algebra (François Viète, (1540-1603)) and the appearance of Descartes' La Géométrie, (René Descartes, (1596-1650)). Namely, in 1637, Descartes published his famous "Discours de la method", with one of its appendices being "La Géomètrie". (See, for example, its translation from the French and Latin by D. E. Smith and M. L. Latham, [6]). A Descartes' contribution of fundamental importance was the creation of number axis as a model for the system of real numbers—a half-line with the point O as its origin and a segment fixed to be the unit of length. Then, each segment with one end fixed at O is the conveyor of the meaning of a real number and, what is particularly important, sums, differences, products and quotients of two segments are a segment again. (Having the unite segment fixed, for two given segments their product and quotient is obtained constructing the fourth proportional. In particular, those segments commensurable with the unit segment represent (positive) rational numbers).

Intuition provided by this Descartes' geometric model has given impetus to all further developments of the theory of number systems and this model certainly deserves to find its place in school books. Thus, the number line is the main model for description of number systems, starting with natural numbers as intervals which are union of adjacent intervals of length 1.

Summarizing, let us say briefly that the main goal of this paper is the use of algebra as a tool to establish relations which express properties of operations and the order relation in N and to take these properties to have a permanent meaning when the extended systems Q_+ of non-negative rational numbers and Z of integers are constructed. The distinction between a fraction as a collection of equal parts and the rational number which that fraction determines as a quantity which is the amount that these parts make when taken together, play a key role in a good understanding of these concepts. Representation of fractions and rational numbers that they determine on the number line is a standard procedure since the time of Descartes on and we inevitably suggest it in this paper. We also think that the variety of other existing interpretations contribute further to the meaning of these concepts. In particularly, interpretation of integers on the number line as being positively and negatively oriented intervals is deeply rooted in the human space intuition.

2. Further Properties of Operations in *N*

For the purposes of extending the system N to the system Q_+, we will establish here some further properties of operations and the order relation in N, not being already established in [3].

In [3], 3.13, the rule of preservation of the quotient (the rule p. q.) has been derived: Let $m:n$ be a quotient defined in N, then for each natural number k, $(k>0)$, $m:n=(km):(kn)$. This rule states that the value of a quotient stays unchanged, when it is expended, multiplying its components by a non-zero natural number or, reading in the reversed direction, when it is reduced, cancelling a common factor of its components.

Now we use the rule of preservation of the quotient to prove the following statement:

2.1. *Let* $k:l$ *and* $m:n$ *be two quotients defined in N. Then*,

(i) $k:l>m:n$ *if and only if* $kn>lm$,

(ii) $k:l=m:n$ *if and only if* $kn=lm$,

(iii) $k:l<m:n$ *if and only if* $kn<lm$.

Being $k:l=(kn):(ln)$ and $m:n=(lm):(ln)$, comparing $(kn):(ln)$ and $(lm):(ln)$, (i), (ii) and (iii) immediately follow.

In [3], 3.12 the rule of interdependence of multiplication and division (the rule i. m. d.) has been stated: $mn = p$ is true whenever one of the relations $p : m = n$, $p : n = m$ is true and vice versa. We use this rule to prove a number of properties of quotients in N (i.e. we suppose that all these quotients have the meaning in N).

2.2. $(k : l)m = (km) : l$.

To prove statement 2.2, let us put $(k : l)m = x$. Applying the rule i. m. d., we have $k : l = x : m$ and applying 2.1 (ii), we obtain $km = lx$ what can be written as $(km) : l = x$.

2.3. $(k : l) : m = k : (lm)$.

Indeed, from $(k : l) : m = x$ applying the rule i. m. d., it follows that $k : l = mx$ and then, $k = (mx)l = (lm)x$ and $k : (lm) = x$.

2.4. $k : (l : m) = (km) : l$.

Applying the rule i.m.d. several times, starting with $k : (l : m) = x$ we get $k = (l : m)x$ and by 2.2, $k = (lx) : m$ from which it follows that $km = lx$ and finally $(km) : l = x$.

2.5. $(k : l)(m : n) = (km) : (ln)$.

Multiplying $m : n$ by the number $k : l$ and applying 2.2 and 2.3, we get $(k : l)(m : n) = ((k : l)m) : n = ((km) : l) : n = (km) : (ln)$.

2.6. $(k : l) : (m : n) = (kn) : (lm)$.

Indeed, applying 2.3 then 2.2 and then 2.4, we have $(k : l) : (m : n) = k : (l(m : n)) = k : ((lm) : n) = (kn) : (lm)$.

2. 7.(i) $(l + k) : m = l : m + k : m$.

(ii) $(l - k) : m = l : m - k : m$.

Let us put $l : m = x, k : m = y$ then $x + y = l : m + k : m$. From $l = mx$ and $k = my$, applying the rule of multiplication of a sum, it follows that $l + k = mx + my = m(x + y)$. Hence, $x + y = (l + k) : m$.

Similarly (ii) is proved, supposing that $l \geq k$.

2.8. (i) $k : l + m : n = (kn + lm) : (ln)$,

(ii) $k : l - m : n = (kn - lm) : (\ln)$.

Indeed, applying the rule i. q. and 2. 7, we have $k : l + m : n = (kn) : (ln) + (lm) : (ln) = (kn + lm) : (ln)$.

Similarly, (ii) is proved, when $kn \geq lm$ or equivalently $k : l \geq m : n$ is supposed to hold.

3. System of Non-Negative Rational Numbers Q_+

Extending the system N to the system Q_+, we will be guided by the Peacock's principle, particularly when operations and order relation are defined in Q_+. As this paper is mainly aimed to project a math course for students of primary education our presentation will be somewhat less demanding than it would be in the case of math students, who could be supposed to be acquainted with elementary theory of numbers (Euclid's lemma, the unique factorization theorem, etc.).

Let m and n, $(n \neq 0)$ be two natural numbers, then the jotting m/n is called a *fraction*. From the view point of phenomenology, a fraction m/n represents m parts $1/n$, when a whole is partitioned into n equal parts. In the case of number line, m/n represents a sequence of m adjacent segments of the length $1/n$ into which the unite segment is divided. For a variation of possible interpretations of fractions, see H. Freudenthal, [7], Guided by 2.1(II) we take two fractions k/l and m/n to be *equal* if and only if $kn = lm$.

Now we prove that the equality of fractions is an equivalence relation on the set of all fractions.

It is trivial to prove that this relation is reflexive and symmetric. Let us assume that $k/l = m/n$ and $m/n = r/s$. Then by the definition of equality of fractions, it follows that $kn = lm$ and $ms = rn$. Multiplying the first of these equalities by s and the second by l, we obtain: $kns = lms$ and $lms = lrn$, what implies $kns = lrn$ or else $ks = lr$ what means that $k/l = r/s$.

Thus, the equality of fractions is a relation which provides a partition of the set of all fractions into (disjoint) equivalence classes. We take as a definition that all mutually equivalent fractions represent one and the same *rational number*.

(More formally, we could say that a rational number is an equivalence class of mutually equivalent fractions).

When a number is represented by an interval on the number axis, then the extension in the space of this interval, that is its length is the quantity which that interval represents. When two equal fractions k/l and m/n are represented on number axis, the former consists of k subintervals of length $1/l$ and the latter of m subintervals of the length $1/n$. When each subinterval of the lengthy $1/l$ is divided into n subintervals of equal length and each subinterval of the length $1/n$ into l subintervals of equal length, then this new fraction represents the same rational number, being a sequence of kn subintervals of the length $1/(\ln)$. Summarizing, we can say that geometrically, the fraction m/n is a sequence of m adjacent intervals of length $1/n$ while the rational number m/n (the value of the fraction m/n) is the interval which is the union of this sequence of m subintervals.

Guided by 2.1 (iii), for two fractions k/l and m/n, we define that $k/l < m/n$ if and only if $kn < lm$. Assume that $k/l = k'/l'$ and $m/n = m'/n'$, that is $kl' = k'l$ and $mn' = m'n$. Then, $(kl')(m'n) = (k'l)(mn')$ or else $(kn)(l'm') = (lm)(k'n')$. Being $kn < lm$ it follows that $k'n' < l'm'$. What implies that $k'/l' < m'/n'$. Hence, each fraction representing the rational number k/l is less than each fraction representing the rational number m/n. Geometrically, this means that the interval on the number axis which represents the rational number k/l is contained in the interval which represents the rational number m/n.

Now we verify that the relation "<", (less than) is an order relation on the set Q_+. ("→" denotes the implication and "↔" the equivalence). Indeed, the relation

"<" is:

1) ir-reflexive: $\neg(k/l < k/l) \leftrightarrow \neg(kl < kl)$.

2) anti-symmetric: $(k/l < m/n) \leftrightarrow \neg(kn \geq lm) \leftrightarrow \neg(m/n \leq k/l)$.

3) transitive: $(k/l < m/n$ and $m/n < r/s) \leftrightarrow (kn < lm$ and $ms < nr) \leftrightarrow$ $(kns < lms$ and $lms < lnr) \rightarrow (kns < lnr) \leftrightarrow (ks < lr) \leftrightarrow (k/l < r/s)$.

Given k/l and m/n, then one and only one of the relations $kn < lm$, $kn = lm$ or $kn > lm$ holds true, what implies that the *relation* "<" *is a linear order*.

Guided by 2.8. (i), we define the *sum of fractions* k/l and m/n to be $k/l + m/n = (kn + lm)/ln$. Now we prove that when $k/l = k'/l'$ and $m/n = m'/n'$, then $k/l + m/n = k'/l' + m'/n'$ what makes the meaning of $k/l + m/n = (kl + mn)/ln$ to be a definition of the *sum of two rational numbers* represented by k/l and m/n respectively. Let $kl' = k'l$ and $mn' = m'n$. Then,

$$(l'n')(kn + lm) = (kl')(nn') + (ll')(mn') = (k'l)(nn') + (ll')(m'n)$$
$$= (ln)(k'n') + (ln)(l'm') = (ln)(k'n' + l'm')$$
$$(kn + lm)/(ln) = (k'n' + l'm')/(l'n').$$

In a similar way, following 2.8 (ii) the *difference of the rational numbers* $k/l - m/n$, $(k/l \geq m/n)$ is defined by $k/l - m/n = (kn - lm)/ln$ and similarly, it is proved that this difference does not depend on the choice of the representatives l/k and m/n.

When the fractions k/l and m/n are expended: kn/ln and lm/ln, their sum consists of $kn + lm$ intervals of the length $1/(\ln)$. This shows clearly that sum of rational numbers represented by the corresponding intervals is the sum of those intervals (as it is constructed by means of a pair of compasses). A similar remark concerns the difference of two rational numbers.

Guided by 2.5, we define the product of two fractions k/l and m/n to be $(k/l) \times (m/n) = (km)/(ln)$. When $k/l = k'/l'$ and $m/n = m'/n'$, *i.e.* when $kl' = k'l$ and $mn' = m'n$, then $(km)(l'n') = (kl')(mn') = (k'l)(m'n) = (ln)(k'm')$. Hence, $km/ln = (k'm')/(l'n')$, and we have proved that the *product of two rational numbers* k/l and m/n does not depend on the chosen representatives.

Similarly, guided by 2. 6, the *quotient of two fractions* k/l and m/n is defined to be $(k/l):(m/n) = (kn)/(lm)$. Similarly the quotient of rational numbers k/l and m/n is defined to be the rational number kn/lm and it is also proved, in the similar way that this product is correctly defined.

Mapping n onto $n/1$, *i.e.* identifying the natural number n with the fraction $n/1$, the set N of natural numbers is seen as a subset of the set Q_+ of non-negative rational numbers. (Due to this fact we speak of Q_+ as an extension of N). This mapping preserves the order relation "<" and all four operations:

(i) $m < n$ *implies* $m/1 < n/1$,

(ii) $m + n$ *maps onto* $(m + n)/1 (= m/1 + n/1)$,

(iii) $m - n$ *maps onto* $(m - n)/1 (= m/1 - n/1)$,

(iv) mn *maps onto* $mn/1 (= (m/1) \times (n/1))$,

(v) $m:n$ *maps onto* $(m:n)/1\left(=\left((m:n)n\right)/n=m/n=(m/1):(n/1)\right)$.

The following properties of operations and the order relation in Q_+ are based on the corresponding properties which hold in *N*. Their verification is just a matter of easy exercises.

(i) $(\forall k/l)(\forall m/n)(k/l+m/n=m/n+k/l)$,

(ii) $(\forall k/l)(\forall m/n)(\forall r/s)\left((k/l+m/n)+r/s=k/l+(m/n+r/s)\right)$,

(iii) $(\forall k/l)(k/l+0/1=k/l)$,

(iv) $(\forall k/l)(\forall m/n)\left(k/l<m/n\to k/l+(m/n-k/l)=m/n\right)$,

(v) $(\forall k/l)(\forall m/n)\left((k/l)\times(m/n)=(m/n)\times(k/l)\right)$,

(vi) $(\forall k/l)(\forall m/n)(\forall r/s)\left((k/l\times m/n)\times r/s=k/l\times(m/n\times r/s)\right)$,

(vii) $(\forall k/l)\left((k/l)\times(1/1)=k/l\right)$,

(viii) $(\forall k/l\neq 0)\left((k/l)\times(l/k)=1/1\right)$,

(ix) $(\forall k/l)(\forall m/n)(\forall r/s)\left((k/l)\times(m/n+r/s)=(k/l)\times(m/n)+(k/l)\times(r/s)\right)$,

(x) $$(\forall k/l)(\forall m/n)(\forall r/s)\big(ms\leq nr\to(k/l)\times(m/n-r/s)$$
$$=(k/l)\times(m/n)-(k/l)\times(r/s)\big),$$

(xi) $(\forall k/l)(\forall m/n)(\forall r/s)(k/l<m/n\leftrightarrow k/l+r/s<m/n+r/s)$,

(xii) $(\forall k/l)(\forall m/n)(\forall r/s\neq 0)\left(k/l<m/n\leftrightarrow(k/n)\times(r/s)<(m/n)\times(r/s)\right)$.

((i): Using the definitions of the sum and the difference of rational numbers and the properties of operations in *N*, we have

$$k/l+m/n=(kn+ln)/(ln),\ m/n+k/l=(ml+nk)/(nl)$$

and being $kn+lm=ml+nk$ and $ln=nl$, the equality under (i) has been proved.

...

(iv): $k/l<m/n$ implies $kn<lm$, what ensures that $lm-kn\in N$. Then

$$k/l+(m/n-k/l)=k/l+(ml-kn)/nl=\left(k(nl)+l(ml-nk)\right)/l(nl)$$
$$=(llm)/(lln)=m/n.$$

(v): $(k/l)\times(m/n)=(km)/(ln)=(mk)/(nl)=(m/n)\times(k/l)$.

...

(ix):

$$(k/l)\times(m/n+r/s)=(k/l)\times\left((ms+nr)/ns\right)$$
$$=\left(k(ms+nr)\right)/lns=(kms+knr)/(lns),$$

$$(k/l)\times(m/n)+(k/l)\times(r/s)$$
$$=(km)/(ln)+(kr)/(ls)=\left((km)(ls)+(ln)(kr)\right)/(ln)(ls)$$
$$=(klms+klnr)/(llns)=(kms+knr)/(lns).$$

...

(xii): $k/l<m/n\leftrightarrow kn<lm$, $(k/l)\times(r/s)=(kr)/(ls)$, $(m/n)\times(r/s)=mr/ns$,

$$(k/l)\times(r/s)<(m/n)\times(r/s)\leftrightarrow kr/ls<mr/ns\leftrightarrow(kr)(ns)<(ls)(mr)$$
$$\leftrightarrow((kn)(rs)<(lm)(rs))\leftrightarrow(kn<lm).$$

4. Sketching a Construction of the System *Z* of Integers

Historically, in mathematics which followed the spirit of the Euclid's Elements negative numbers appeared as solutions of some algebraic equations, but they were rejected and considered to be just obscuring the doctrine of equation solving. Their contents had not been logically organized until 19th century (work of G. Peacock, A. De Morgan, et al.).

In the second century BC, used for commercial and tax calculations, positive numbers were represented in Chinese rod system in red and negative ones in black. In India, in the work of Brahmagupta (593-670) the idea of "fortunes" was used for positive and the idea of "debts" for negative numbers. He established the rule of dealing with these numbers, stating, for example, that the product or the quotient of a debt (fortune) and a fortune (debt) is a debt and that the product or quotient of two debts is one fortune! (The paper [8] gives a short and neatly account of the history of negative numbers).

Potentially, the integers exist as jottings $k-l$ in *N*, when the restriction $k\ge l$ is neglected. The rules of operating with such jottings are taken to be the same rules established in *N* when $k\ge l$ is assumed.

It is easy to establish the following statements for differences having their meaning in *N*.

4.1 (i) $k-l<m-n\leftrightarrow k+n<l+m,$

(ii) $k-l=m-n\leftrightarrow k+n=l+m,$

(iii) $k-l>m-n\leftrightarrow k+n>l+m.$

Let *D* be the set of all jottings $k-i$, where *k* and *l* are arbitrary natural numbers (also including 0). Guided by 4.1 (ii), we take that two jottings $k-l$ and $m-n$ are *equal* if and only if $k+n=l+m$. It is easy to verify that this equality is an equivalence relation on the set *D* and the set of equivalence classes with respect to this relation is denoted by *Z* and its elements are called the *integers*. When we say the integer $k-l$, we mean the equivalence class determined by $k-l$.

When $k\ge l, k-l=(k-l)-0$ and when $k<l, k-l=0-(l-k)$. Thus, each equivalence class has its standard representative of the form $n-0,(n\ge 0)$ or $0-n,(n>0)$. The equivalence classes determined by $n-0,(n\ge 0)$ are called non-negative integers, they are identified with natural numbers (including 0) and simply denoted by *n* The equivalence classes $0-n,(n>0)$ are called *negative integers* and they are denoted writing simply $-n$ (instead of $0-n$). In both of these cases, the natural number *n* is called the *absolute value* of the integers $n-0$ or $0-n$.

On the number line, $n-0$ is represented by the interval *OM* of length *n*, which is positively oriented (*M* stands on the right side of *O*) and $0-n$ by the interval *MO* of length *n*, which is negatively oriented (*M* stands on the left side of *O*).

Guided by 4. 1 (i), the order relation in *Z* is defined taking that $k-l<m-n$ if and only if $k+n<l+m$. This definition also becomes correct after proving that this relation does not depend on the choice of the representatives $k-l$ and $m-n$.

Guided by the rule of adding differences in *N*, ([3], 3.16), the sum of integers $k-l$ and $m-n$ is defined to be the integer $(k+m)-(l+n)$. To make this definition correct, one has to prove that $(k+m)-(l+n)$ determines the same equivalence class independently of the choice of the representatives.

Establishing $(k-l)(m-n)=(km+ln)-(kn+lm)$ in *N* and guided by this relation , the product of the integers $k-l$ and $m-n$ is defined to be $(k-l)\times(m-n)=(km+ln)-(kn+lm)$ and it is also easy to prove that this product does not depend on the chosen representatives.

As always somewhat mysterious to students, the rule of "multiplication of signs" is now seen to be a consequence of the definition of multiplication in *Z*. Indeed, we have

1) the product of two positive integers is a positive integer:

$$m\times n=(m-0)\times(n-0)=mn-0=mn.$$

2) the product of a negative (positive) and a positive (negative) integer is a negative integer: $(-m)\times n=(0\neg m)\times(n-0)=0-mn=-mn$ and $n\times(-m)=(n-0)\times(0-m)=0-mn=-mn.$

3) the product of two negative integers is a positive integer:

$$(-m)\times(-n)=(0-m)\times(0-n)=mn-0=mn.$$

As for the properties of operations and the order relation in *Z*, it is reasonable to verify only those which are axioms of the ordered ring.

(For example, to prove the distributive law in *Z*, we have to verify that

$$(k-l)\times((m-n)+(r-s))=(k-l)\times(m-n)+(k-l)\times(r-s).$$

Indeed,

$$(k-l)\times((m-n)+(r-s))=(k-l)\times((m+r)-(n+s))$$
$$=(k(m+r)+l(n+s))-(k(n+s)+l(m+r)),$$

$$(k-l)\times(m-n)+(k-l)\times(r-s)$$
$$=((km+ln)-(kn+lm))+((kr+ls)-(ks+lr))$$
$$=(km+ln+kr+ls)\tau-(kn+lm+ks+lr).$$

etc.).

5. Concluding Remarks

The system Q of rational numbers is contracted after the system Q_+ has been constructed, using the differences $a-b,(a\in Q_+,b\in Q_+)$ or extending the system *Z*, using the quotients $a:n,(a\in Z,n\in N\setminus\{0\})$. These two extensions are carried out analogously with the extensions of *N* to *Z* and Q_+. When all those extensions are done completely, with great attention to detail, then such an elaboration of this teaching theme could easily make the attending students to be lulled to sleep. In a text-book, the proofs of some statements can be assigned to students as exercises supplied with hints. In the case of all these extensions the same strategy is evidently present. After a pair of these extensions has been exemplified (as it has been done in this paper), only then the remark that a further extension can be carried out analogously will sound more convincing. Finally, teachers of a math course, who would elaborate the extensions of number systems, as they are suggested in this paper, could gather a useful experience and upon their reports a *modus optimum* could be found and this purely pedagogical question could be settled.

These theoretical touches of the number systems contribute certainly to their deeper understanding, but the exposition of purely theoretical facts should be combined with doing simple concrete exercises as they are found in the school text-books and with various practical uses of numbers belonging to these extended systems. It is particularly important to point out that in Q_+, the divisions $a:b,(b \neq 0)$ and in Z, the subtractions $a-b$ are feasible without any limitation. In Q, both of these operations are feasible without any limitation (excluding, of course, division with 0).

In the course of this reconsideration of number systems, some and at the end, all axioms of the order field should be listed and referred to as the fundamental properties of operations and the order relation. The exercises where these properties are taken as the basis upon which all main calculation rules are established will be particularly useful for deepening the knowledge of this content of arithmetic.

Geometric constructions of sums and differences of the intervals representing the numbers a and b, interpret sums and differences of these numbers, themselves. Carrying over to the sides of an angle, the unit interval and the intervals representing numbers a and b and then, by constructing the fourth proportional, the intervals representing the product ab and the quotient $a:b$ are constructed. This iconic representation of numbers and the operations with them makes the understanding better and easier (and as Aristotle says it: "The soul never thinks without an image").

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