Properties of Fuzzy Length on Fuzzy Set

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Abstract

The definition of fuzzy length space on fuzzy set in this research was introduced after the studies and discussion of many properties of this space were proved, and then an example to illustrate this notion was given. Also the definition of fuzzy convergence, fuzzy bounded fuzzy set, and fuzzy dense fuzzy set space was introduced, and then the definition of fuzzy continuous operator was introduced.

Subject Areas

Fuzzy Mathematics

Keywords

Fuzzy Length Space on Fuzzy Set, Fuzzy Convergence, Fuzzy Cauchy Sequence of Fuzzy Point, Fuzzy Bounded Fuzzy Set and Fuzzy Continuous Operator

1. Introduction

Zadeh in 1965 [1] introduced the theory of fuzzy sets. Many authors have introduced the notion of fuzzy norm in different ways [2]-[9]. Cheng and Mordeson in 1994 [10] defined fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Mickalek type [11] as follows:

The order pair (X, N) is said to be a fuzzy normed space if X is a linear space and N is a fuzzy set on $X \times [0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t \in [0, \infty)$.

(i) $N(x, 0) = 0$, for all $x \in X$.

(ii) For all $t > 0$, $N(x, t) = 1$ if and only if $x = 0$.

(iii) $N(\alpha x, t) = N\left(x, \dfrac{t}{|\alpha|}\right)$, for all $\alpha \neq 0$ and for all $t > 0$.

(iv) For all $s, t > 0$, $N(x + y, t + s) \geq N(x, t) \wedge N(y, s)$ where $a \wedge b = \min\{a, b\}$.

(v) $\lim_{t \to \infty} N(x, t) = 1$.

The definition of continuous *t*-norm was introduced by George and Veeramani in [12]. Bag and Samanta in [2] modified the definition of Cheng and Mordeson of fuzzy norm as follows:

The triple $(X, N, *)$ is said to be a fuzzy normed space if X is a linear space, $*$ is a continuous *t*-norm and N is a fuzzy set on $X \times [0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t \in [0, \infty)$.

(i) $N(x, 0) = 0$, for all $x \in X$.

(ii) For all $t > 0$, $N(x, t) = 1$ if and only if $x = 0$.

(iii) $N(\alpha x, t) = N\left(x, \dfrac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$.

(iv) For all $s, t > 0$, $N(x, t) * N(y, s) \leq N(x + y, t + s)$.

(v) For $x \neq 0$, $N(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

(vi) $\lim_{t \to \infty} N(x, t) = 1$.

The definition of fuzzy length space is introduced in this research as a modification of the notion of fuzzy normed space due to Bag and Samanta. In Section 1, we recall basic concepts of fuzzy set and the definition of continuous *t*-norm. Then in Section 2 we define the fuzzy length space on fuzzy set after we give an example; then we prove that every ordinary norm induces a fuzzy length, and also the definition of fuzzy open fuzzy ball, fuzzy convergent sequence, fuzzy open fuzzy set, fuzzy Cauchy sequence, and fuzzy bounded fuzzy set is introduced. In Section 3, we prove other properties of fuzzy length space. Finally in Section 4, we define a fuzzy continuous operator between two fuzzy length spaces. Also we prove several properties for fuzzy continuous operator.

2. Basic Concept about Fuzzy Set

Definition 2.1:

Let X be a classical set of object, called the universal set, whose generic elements are denoted by x. The membership in a classical subject A of X is often viewed as a characteritic function μ_A from X onto {0, 1} such that $\mu_A = 1$ if $x \in A$ and $\mu_A = 0$ if $x \notin A$, {0, 1} is called a valuation set. If a valuation set is allowed to be real interval [0, 1] then A is called a fuzzy set which is denoted in this case by $\tilde{A}$ and $\mu_{\tilde{A}}$ is the grade of membership of x in $\tilde{A}$. Also, it is remarkable that the closer the value of $\mu_{\tilde{A}}(x)$ to 1, the more belong to $\tilde{A}$. Clearly, $\tilde{A}$ is a subset of X that has no sharp boundary. The fuzzy set $\tilde{A}$ is completely characterized by the set of pairs:

$$\tilde{A} = \left\{\left(x, \mu_{\tilde{A}}(x)\right) : x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\right\} \ [1].$$

Definition 2.2:

Suppose that $\tilde{D}$ and $\tilde{B}$ are two fuzzy sets in Y. Then

(i) $\tilde{D} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{D}}(y) \leq \mu_{\tilde{B}}(y) \ \forall y \in Y$.

(ii) $\tilde{D} = \tilde{B} \Leftrightarrow \mu_{\tilde{D}}(y) = \mu_{\tilde{B}}(y) \ \forall y \in Y$.

(iii) $\tilde{C} = \tilde{D} \cup \tilde{B} \Leftrightarrow \mu_{\tilde{C}}(y) = \mu_{\tilde{D}}(y) \vee \mu_{\tilde{B}}(y) \ \forall y \in Y$.

(iv) $\tilde{E} = \tilde{A} \cap \tilde{B} \Leftrightarrow \mu_{\tilde{E}}(y) = \mu_{\tilde{D}}(y) \wedge \mu_{\tilde{B}}(y) \ \forall y \in Y$.

(v) $\mu_{\tilde{D}^c}(x) = 1 - \mu_{\tilde{D}}(y) \; \forall y \in Y$ [4].

Definition 2.3:

Suppose that $\tilde{D}$ and $\tilde{B}$ be two fuzzy sets in $V \neq \varnothing$ and $W \neq \varnothing$ respectively then $\tilde{D} \times \tilde{B}$ is a fuzzy set whose membership is defined by:

$$\mu_{\tilde{D} \times \tilde{B}}(a,b) = \mu_{\tilde{D}}(a) \wedge \mu_{\tilde{B}}(b) \quad \forall (a,b) \in V \times W \quad [8].$$

Definition 2.4:

A fuzzy point p in Y is a fuzzy set with single element and is denoted by x_α or (x, α). Two fuzzy points x_α and y_β are said to be different if and only if $x \neq y$ [11].

Definition 2.5:

Suppose that d_β is a fuzzy point and $\tilde{D}$ is a fuzzy set in Y. then d_β is said to belongs to $\tilde{D}$ which is written by $d_\beta \in \tilde{D} \Leftrightarrow$ if $\mu_{\tilde{D}}(x) > \beta$ [2].

Definition 2.6:

Suppose that h is a function from the set $V \neq \varnothing$ into the set $W \neq \varnothing$. Let $\tilde{D}$ be a fuzzy set in W then $h^{-1}(\tilde{D})$ is a fuzzy set in V its membership is:

$\mu_{h^{-1}(\tilde{D})}(d) = (\mu_{\tilde{D}} \circ h)(d)$ for all d in V. Also when $\tilde{E}$ is a fuzzy set in V then $h(\tilde{E})$ is a fuzzy set in W its membership is given by: $\mu_{h(\tilde{E})}(y) = \vee \{\mu_{\tilde{E}}(d) : d \in f^{-1}(y)\}$ when $f^{-1}(y) \neq \varnothing$ and $\mu_{h(\tilde{E})}(y) = 0$ otherwise [5].

Proposition 2.7:

Suppose that $h : V \to W$ is a function. Then the image of the fuzzy point d_β in V, is the fuzzy point $h(d_\beta)$ in W with $h(d_\beta) = (h(d), \beta)$ [11].

Definition 2.8:

A binary operation $* : [0,1]^2 \to [0,1]$ is said to be t-norm (or continuous triangular norm) if $\forall p, q, t, r \in [0,1]$ the conditions are satisfied:

(i) $p * q = p * q$.

(ii) $p * 1 = p$.

(iii) $(p * q) * t = p * (q * t)$.

(iv) If $p \leq q$ and $t \leq r$ then $p * t \leq q * r$ [12].

Examples 2.9:

When $p * q = p \cdot q$ and $p * q = p \wedge q$ $\forall p, q \in [0,1]$ then $*$ is a continuous t-norm [10].

Remark 2.10:

$\forall p > q$, there is t such that $p * t \geq q$ and for every r, there is e such that $r * r \geq e$, where p, q, t, r and e belongs to [0, 1] [12].

Definition 2.11:

Let $\tilde{D}$ be a fuzzy set in Z and let τ be a collection of all subset fuzzy set in $\tilde{D}$. then $(\tilde{D}, \tau)$ is called a fuzzy topological space on the fuzzy set $\tilde{D}$ if (i) $\tilde{D}, \phi \in \tau$.

(ii) $\bigcap_{i=1}^n \tilde{E}_i \in \tau$ for any $\tilde{E}_i \in \tau$, $i = 1, 2, \cdots, n$.

(ii) $\bigcup_{i=1}^\infty \tilde{E}_i \in \tau$, for $\{\tilde{E}_i : \tilde{E}_i \in \tau, i = 1, 2, \cdots, \infty\}$ [10].

Proposition 2.12:

Let $T : \tilde{A} \to \tilde{D}$ be an arbitrary operator and $\tilde{E} \subseteq \tilde{A}$ and $\tilde{C} \subseteq \tilde{D}$. Then $T(\tilde{E}) \subseteq \tilde{C}$ if and only if $\tilde{E} \subseteq T^{-1}(\tilde{C})$ [9].

3. On Fuzzy Length Space

First we introduce the main definition in this paper.

Definition 3.1:

Let X be a linear space over field $\mathbb{K}$ and let $\tilde{A}$ be a fuzzy set in X. let $*$ be a t-norm and $\tilde{F}$ be a fuzzy set from $\tilde{A}$ to [0,1] such that:

(FL$_1$) $\tilde{F}(x_\alpha) > 0$ for all $x_\alpha \in \tilde{A}$.

(FL$_2$) $\tilde{F}(x_\alpha) = 1$ if and only if $x_\alpha = 0$.

(FL$_3$) $\tilde{F}(cx, \alpha) = \tilde{F}\left(x, \frac{\alpha}{|c|}\right)$ where $0 \neq c \in \mathbb{K}$.

(FL$_4$) $\tilde{F}(x_\alpha + y_\beta) \geq \tilde{F}(x_\alpha) * \tilde{F}(y_\beta)$.

(FL$_5$) $\tilde{F}$ is a continuous fuzzy set for all $x_\alpha, y_\beta \in \tilde{A}$ and $\alpha, \beta \in [0,1]$.

Then the triple $(\tilde{A}, \tilde{F}, *)$ is called a fuzzy length space on the fuzzy $set \tilde{A}$.

Definition 3.2:

Suppose that $(\tilde{D}, \tilde{F}, *)$ is a fuzzy length space on the fuzzy set $\tilde{D}$ then $\tilde{F}$ is continuous fuzzy set if whenever $\{(x_n, \propto_n)\} \to x_\alpha$ in $\tilde{D}$ then $\tilde{F}\{(x_n, \propto_n)\} \to \tilde{F}(x_\alpha)$ that is $\lim_{n \to \infty} \tilde{F}[(x_n, \propto_n)] = \tilde{F}(x_\alpha)$.

Proposition 3.3:

Let $(Y, .)$ be a normed space, suppose that $\tilde{D}$ be a fuzzy set in Y. Put $\|x_\alpha\| = \|x\|$. Then $(\tilde{D}, .)$, is a fuzzy normed space.

Proof:

Let $x_\alpha, y_\beta \in \tilde{D}$ and $0 \neq c \in \mathbb{K}$ then

1) $\|x_\alpha\| > 0$ for all $x_\alpha \in \tilde{D}$.

2) $\|x_\alpha\| = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0 \Leftrightarrow x_\alpha = 0$.

3) $\|(cx, \alpha)\| = \|cx\| = \|c\| \|x\| = \|c\| \|x_\alpha\|$.

4) $\|x_\alpha + y_\beta\| = \|x + y\| \leq \|x\| + \|y\| = \|x_\alpha\| + \|y_\beta\|$.

Hence $(\tilde{D}, \|.\|)$, is a fuzzy normed space.

Example 3.4:

Suppose that $(Y, \|.\|)$ is a normed space and assume that $\tilde{D}$ is a fuzzy set in Y. Put $p * q = p \cdot q$ for all $p, q \in [0,1]$. Define $\tilde{F}_{\|\|}(x_\alpha) = \frac{\alpha}{\alpha + \|x\|}$.

Then $(\tilde{D}, \tilde{F}_{\|\|}, *)$ is a fuzzy length space on the fuzzy set $\tilde{D}$, is called the fuzzy length induced by $\|.\|$.

Proof:

To prove $(\tilde{D}, \tilde{F}_{\|\|}, *)$ is a fuzzy length space on the fuzzy set $\tilde{D}$ we must prove the five conditions of Definition 3.1:

(FL$_1$) Since $\|x\| > 0$ for all $x \in X$ so $\tilde{F}_{\|\|}(x_\alpha) > 0$ for all $x \in \tilde{D}$, where $\alpha > 0$.

(FL$_2$) It is clear that $\tilde{F}_{\|\|}(x_\alpha) = 1$ for each $\alpha > 0$ if and only if $x = 0$.

(FL$_3$) If $0 \neq c \in \mathbb{K}$ then for each $x \in X$,

$$\tilde{F}_{\|\|}(cx,\alpha) = \frac{\alpha}{\alpha + \|cx\|} = \frac{\alpha}{\alpha + \|c\|\|x\|} = \tilde{F}_{\|\|}\left(x, \frac{\alpha}{|c|}\right) \text{ for each } \alpha > 0.$$

(FL$_4$)
$$\tilde{F}_{\|\|}(x_\alpha, y_\beta) = \tilde{F}_{\|\|}(x+y, \lambda) = \frac{\lambda}{\lambda + \|x+y\|} \geq \frac{\lambda}{\lambda + \|x\| + \|y\|} \geq \frac{\lambda}{\lambda + \|x\|} \cdot \frac{\lambda}{\lambda + \|y\|}$$
for
$$> \frac{\alpha}{\alpha + \|x\|} \cdot \frac{\beta}{\beta + \|y\|} = \tilde{F}_{\|\|}(x_\alpha) * \tilde{F}_{\|\|}(y_\beta)$$

each $x_\alpha, y_\beta \in \tilde{A}$, where $\lambda = \min\{\alpha, \beta\}$.

(FL$_5$) $\tilde{F}_{\|\|}(x_\alpha)$ is continuous since $\|x\|$ is a continuous function.

Hence $\left(\tilde{D}, \tilde{F}_{\|\|}, *\right)$ is a fuzzy length space on the fuzzy $set\tilde{D}$.

Definition 3.5:

Let $\tilde{A}$ be a fuzzy set in X, and assume that $\left(\tilde{A}, \tilde{F}, *\right)$ is a fuzzy length space on the fuzzy $set\tilde{A}$. Let $\tilde{B}(x_\alpha, r) = \left\{y_\beta \in \tilde{A} : \tilde{F}(y_\beta - x_\alpha) > (1-p)\right\}$. So $\tilde{B}(x_\alpha, p)$ is said to be a fuzzy open fuzzy ball of center $x_\alpha \in \tilde{A}$ and radius r.

We omitted the proof of the next result since it is clear.

Proposition 3.6:

In the fuzzy length space $\left(\tilde{A}, \tilde{F}, *\right)$ on the fuzzy $set\tilde{A}$, Let $\tilde{B}(x_\alpha, r_1)$ and $\tilde{B}(x_\alpha, r_2)$ with $x_\alpha \in \tilde{A}$ and $r_1, r_2 \in (0,1)$. Then either $\tilde{B}(x_\alpha, r_1) \subseteq \tilde{B}(x_\alpha, r_2)$ or $\tilde{B}(x_\alpha, r_2) \subseteq \tilde{B}(x_\alpha, r_1)$.

Definition 3.7:

The sequence $\{(x_n, \propto_n)\}$ in a fuzzy length space $\left(\tilde{A}, \tilde{F}, *\right)$ on the fuzzy $set\tilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if for a given ε, $0 < \varepsilon < 1$, then there exists a positive number K such that $\tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] > (1-\varepsilon) \; \forall n \geq K$.

Definition 3.8:

The sequence $\{(x_n, \propto_n)\}$ in a fuzzy length space $\left(\tilde{A}, \tilde{F}, *\right)$ on the fuzzy $set\tilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if $\lim_{n \to \infty} \tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] = 1$.

Theorem 3.9:

The two Definitions 3.8 and 3.7 are equivalent.

Proof:

Let the sequence $\{(x_n, \propto_n)\}$ is fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$, then for a given $0 < \varepsilon < 1$ then there is a number K with $\tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] > (1-r)$ for all $n \geq K$, and hence $1 - \tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] < r$. Therefore $\tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] \to 1$ when $n \to \infty$.

To prove the converse, let $\tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] \to 1$ when $n \to \infty$.

Hence when $0 < \varepsilon < 1$, there is K such that $1 - \tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] < r \; \forall n \geq K$.

So $\tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] > (1-r) \; \forall n \geq K$. Therefore $\{(x_n, \propto_n)\} \to x_\alpha$ by Definition 3.7.

∎

Lemma 3.10:

Suppose that $\left(\tilde{A}, \tilde{F}, *\right)$ is a fuzzy length space on the fuzzy $set\tilde{A}$. Then $\tilde{F}(x_\alpha - y_\beta) = \tilde{F}(y_\beta - x_\alpha)$ for any $x_\alpha, y_\beta \in \tilde{A}$.

Proof:

Let $x_\alpha, y_\beta \in \tilde{A}$, then $\tilde{F}(x_\alpha - y_\beta) = \tilde{F}((x-y), t)$ where $t = \alpha \wedge \beta$

$$= \tilde{F}\left(-1(y-x),t\right) = \tilde{F}\left((y-x),\frac{t}{|-1|}\right) = \tilde{F}\left((y-x),t\right) = \tilde{F}\left(y_\beta - x_\alpha\right).$$

Lemma 3.11:

If $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy length space. Then,

(a) The operator $\left(x_\alpha, y_\beta\right) \to x_\alpha + y_\beta$ is continuous.

(b) The operator $\left(r, x_\alpha\right) \to r x_\alpha$ is continuous.

Proof:

If $\left\{(x_n, \propto_n)\right\} \to x_\alpha$ and $\left\{(y_n, \beta_n)\right\} \to y_\alpha$ as $n \to \infty$ then

$$\tilde{F}\left(\left[(x_n, \propto_n) + y_n, \beta_n\right] - \left(x_\alpha + y_\beta\right)\right)$$
$$\geq \tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] * \tilde{F}\left[(y_n, \beta_n) - y_\beta\right] = 1*1 = 1$$

Hence the operator addition is continuous function.

Now if $\left\{(x_n, \propto_n)\right\} \to x_\alpha$, and $r_n \to r$ and $r_n \neq 0$ then

$$\lim_{n\to\infty} \tilde{F}\left[r_n\left(x_n, \propto_n\right) - r x_\alpha\right]$$
$$= \lim_{n\to\infty} \tilde{F}\left[\left(r_n x_n, \propto_n\right) - r x_\alpha\right]$$
$$= \lim_{n\to\infty} \tilde{F}\left[\left(r_n x_n, \propto_n\right) - r_n x_\alpha + r_n x_\alpha - r x_\alpha\right]$$
$$= \lim_{n\to\infty} \tilde{F}\left[r_n\left(x_n - x_\alpha, \propto_n\right) + x_\alpha\left(r_n - r\right)\right]$$
$$\geq \lim_{n\to\infty} \tilde{F}\left[x_n - x_\alpha, \frac{\propto_n}{|r_n|}\right] * \lim_{n\to\infty} \tilde{F}\left[x, \frac{\alpha}{|r_n - r|}\right] \geq 1*1 = 1$$

And this proves (b).

Definition 3.12:

Suppose that $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy length space and $\tilde{D} \subseteq \tilde{A}$ then $\tilde{D}$ is called fuzzy open if for every $y_\beta \in \tilde{D}$ there is $\tilde{B}\left(y_\beta, q\right) \subseteq \tilde{D}$. A subset $\tilde{E} \subseteq \tilde{A}$ is called fuzzy closed if $\tilde{E}^c = \tilde{A} - \tilde{E}$ is fuzzy open.

The proof of the following theorem is easy and so is omitted.

Theorem 3.13:

Any $\tilde{B}\left(y_\beta, q\right)$ in a fuzzy length space $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy open.

Definition 3.14:

Suppose that $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy length space, and assume that $\tilde{D} \subseteqq \tilde{A}$. Then the fuzzy closure of $\tilde{D}$ is denoted by $\overline{\tilde{D}}$ or $FC\left(\tilde{D}\right)$ and is defined by $\overline{\tilde{D}}$ is the smallest fuzzy closed fuzzy set that contains $\tilde{D}$.

Definition 3.15:

Suppose that $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy length space, and assume that $\tilde{D} \subseteq \tilde{A}$. Then $\tilde{D}$ is said to be fuzzy dense in $\tilde{A}$ if $\overline{\tilde{D}} = \tilde{A}$ or $FC\left(\tilde{D}\right) = \tilde{A}$.

Lemma 3.16:

Suppose that $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy length space, and assume that $\tilde{D} \subseteq \tilde{A}$, Then $d_\alpha \in \overline{\tilde{D}}$ if and only if we can find $\left\{(d_n, \propto_n)\right\}$ in $\tilde{D}$ such that $(d_n, \propto_n) \to d_\alpha$.

Proof:

Let $d_\alpha \in \overline{\tilde{D}}$, if $d_\alpha \in \tilde{D}$ then we take the sequence of fuzzy points of the type $\left(d_\alpha, d_\alpha, \cdots, d_\alpha\right)$. If $d_\alpha \notin \tilde{D}$, we construct the sequence of fuzzy points $(d_n, \propto_n) \in \tilde{D}$ as

follows: $\tilde{F}\left[\left(d_n,\propto_n\right)-d_\alpha\right]>\left(1-\dfrac{1}{n}\right)$ for each $n\in\mathbb{N}$.

The fuzzy ball $\tilde{B}\left(d_\alpha,\dfrac{1}{n}\right)$ contains $\left(d_n,\propto_n\right)\in\tilde{D}$ and $\left(d_n,\propto_n\right)\to d_\alpha$ since $\lim_{n\to\infty}\tilde{F}\left[\left(d_n,\propto_n\right)-d_\alpha\right]=1$.

Conversely assume that $\left(d_n,\propto_n\right)\in\tilde{D}$ and $\left(d_n,\propto_n\right)\to d_\alpha$ then $d_\alpha\in\tilde{D}$ or the fuzzy open fuzzy ball of d_α contains $\left(d_n,\propto_n\right)$ with $\left(d_n,\propto_n\right)\neq d_\alpha$, so d_α is a fuzzy limit fuzzy point of $\tilde{D}$. Hence $d_\alpha\in\overline{\tilde{D}}$.

4. Other Properties of Fuzzy Length on Fuzzy Set

Theorem 4.1:

Suppose that $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy length space and let that $\tilde{D}\subseteq\tilde{A}$, then $\tilde{D}$ is fuzzy dense in $\tilde{A}$ if and only if for any $a_\alpha\in\tilde{A}$ we can find $d_\beta\in\tilde{D}$ with $\tilde{F}\left[a_\alpha-d_\beta\right]\geq\left(1-\varepsilon\right)$ for some $0<\varepsilon<1$.

Proof:

Let $\tilde{D}$ be a fuzzy dense in $\tilde{A}$, and $a_\alpha\in\tilde{A}$ so $a_\alpha\in\overline{\tilde{D}}$ then using 3.16 we can find $\left\{\left(d_n,\propto_n\right)\right\}\in\tilde{D}$ with $\tilde{F}\left[\left(d_n,\propto_n\right)-a_\alpha\right]>\left(1-\varepsilon\right)$ for all $n\geq K$. Take $d_\beta=\left(d_k,\propto_k\right)$, $\tilde{F}\left[d_\beta-a_\alpha\right]>\left(1-\varepsilon\right)$.

To prove the converse, we must prove that $\tilde{A}\subseteq\overline{\tilde{D}}$. Let $a_\alpha\in\tilde{A}$ then there is $\left(d_j,\propto_j\right)\in\tilde{D}$ such that $\tilde{F}\left[\left(d_j,\propto_j\right)-a_\alpha\right]>\left(1-\dfrac{1}{j}\right)$, where $j\in\mathbb{N}$. Now take $0<\varepsilon<1$ such that $\dfrac{1}{j}<\varepsilon$ for each $j\geq K$. Hence we have a sequence of fuzzy points $\left\{\left(d_j,\propto_j\right)\right\}\in\tilde{D}$ such that $\tilde{F}\left[\left(d_j,\propto_j\right)-a_\alpha\right]>\left(1-\dfrac{1}{j}\right)>\left(1-\varepsilon\right)$ for all $j\geq K$ that is $\left(d_j,\propto_j\right)\to a_\alpha$ so $a_\alpha\in\overline{\tilde{D}}.$ ∎

Definition 4.2:

Suppose that $\left(\tilde{A},\tilde{F},*\right)$ is a fuzzy length space. A sequence of fuzzy points $\left\{\left(x_n,\propto_n\right)\right\}$ is said to be a fuzzy Cauchy if for any given $\varepsilon,0<\varepsilon<1$, there is a positive number K such that $\tilde{F}\left[\left(x_n,\propto_n\right)-\left(x_m,\propto_m\right)\right]>\left(1-\varepsilon\right)$ for all $n,m\geq K$.

Theorem 4.3:

If $\left\{\left(x_n,\propto_n\right)\right\}$ is a sequence in a fuzzy length space $\left(\tilde{A},\tilde{F},*\right)$ with $\left(x_n,\propto_n\right)\to x_\alpha$ then $\left\{\left(x_n,\propto_n\right)\right\}$ is fuzzy Cauchy.

Proof:

If $\left\{\left(x_n,\propto_n\right)\right\}$ is a sequence in a fuzzy length space $\left(\tilde{A},\tilde{F},*\right)$ with $\left(x_n,\propto_n\right)\to x_\alpha$. So for any $\varepsilon,0<\varepsilon<1$ there is an integer K such that $\tilde{F}\left[\left(x_n,\propto_n\right)-x_\alpha\right]>\left(1-\varepsilon\right)$ for all $n\geq K$. Now by Remark 2.10 there is $\left(1-r\right)\in\left(0,1\right)$ such that $\left(1-\varepsilon\right)*\left(1-\varepsilon\right)>\left(1-r\right)$. Now for each $m,j\geq K$, we have

$$\tilde{F}\left[\left(x_m,\propto_m\right)-\left(x_j,\propto_j\right)\right]\geq\tilde{F}\left[\left(x_m,\propto_m\right)-x_\alpha\right]*\tilde{F}\left[x_\alpha-\left(x_j,\propto_j\right)\right]$$
$$\geq\left(1-\varepsilon\right)*\left(1-\varepsilon\right)>\left(1-r\right)$$

Hence $\{(x_n, \propto_n)\}$ is fuzzy Cauchy. ∎

Definition 4.4:

Suppose that $\{(x_n, \propto_n)\}$ is a sequence in a fuzzy length space $(\tilde{A}, \tilde{F}, *)$ let (n_k) be a sequence of positive numbers such that $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$, then the sequence $\{(x_{n_k}, \propto_{n_k})\}$ of fuzzy points is said to be a subsequence of $\{(x_n, \propto_n)\}$.

Theorem 4.5:

Suppose that $\{(x_n, \propto_n)\}$ is a sequence in a fuzzy length space $(\tilde{A}, \tilde{F}, *)$, if it is fuzzy converges to $x_\alpha \in \tilde{A}$ then every subsequence $\{(x_{nk}, \propto_{nk})\}$ of $\{(x_n, \propto_n)\}$ is fuzzy converges to x_α.

Proof:

Let $\{(x_n, \propto_n)\}$ be a sequence of fuzzy points in $(\tilde{A}, \tilde{F}, *)$ fuzzy converges to x_α then $\lim_{n \to \infty} \tilde{F}\left[(x_n, \propto_n) - x_\alpha\right] = 1$. But $\{(x_n, \propto_n)\}$ is fuzzy Cauchy by Theorem (4.3) hence, $\lim \tilde{F}\left[(x_n, \propto_n) - (x_m, \propto_m)\right] = 1$ as $n \to \infty$ and $m \to \infty$. Now

$$\tilde{F}\left[(x_{nk}, \propto_{nk}) - x_\alpha\right] \geq \tilde{F}\left[(x_{nk}, \propto_{nk}) - (x_n, \propto_n)\right] * \tilde{F}\left[(x_n, \propto_n) - x_\alpha\right].$$

Taking the limit to both sides as $n \to \infty$ we get

$$\lim_{n \to \infty} \tilde{F}\left[(x_{nk}, \propto_{nk}) - x_\alpha\right]$$
$$\geq \lim_{n \to \infty} \tilde{F}\left[(x_{nk}, \propto_{nk}) - (x_n, \propto_n)\right] * \lim_{n \to \infty} \tilde{F}\left[(x_n, \propto_n) - x_\alpha\right]$$
$$\geq 1 * 1 = 1$$

Hence, $\{(x_{nk}, \propto_{nk})\}$ fuzzy converges to x_α, since $\{(x_{nk}, \propto_{nk})\}$ was an arbitrary subsequence of $\{(x_n, \propto_n)\}$. Therefore all subsequence $\{(x_{nk}, \propto_{nk})\}$ of $\{(x_n, \propto_n)\}$ is fuzzy converges to x_α.

Proposition 4.6:

Let $\{(x_n, \propto_n)\}$ be a fuzzy Cauchy sequence in a fuzzy length space $(\tilde{A}, \tilde{F}, *)$ contains a $\{(x_{n_k}, \propto_{n_k})\}$ such that $\{(x_{n_k}, \propto_{n_k})\} \to x_\alpha$, then $\{(x_n, \propto_n)\}$ fuzzy converges to x_α.

Proof:

Assume that $\{(x_n, \propto_n)\}$ is a fuzzy Cauchy sequence in $(\tilde{A}, \tilde{F}, *)$. So for all ε, $0 < \varepsilon < 1$ there is an integer K such that $\tilde{F}\left[(x_m, \propto_m) - (x_j, \propto_j)\right] > (1 - \varepsilon)$ whenever $m, j \geq K$. Let $\{(x_{n_j}, \propto_{n_j})\}$ be a subsequence of $\{(x_n, \propto_n)\}$ and $(x_{n_j}, \propto_{n_j}) \to x_\alpha$. It follows that $\tilde{F}\left[(x_{n_j}, \propto_{n_j}) - x_\alpha\right] \to 1$ whenever $n_j \to \infty$. Since (n_j) is increasing sequence of positive integers. Now

$$\tilde{F}\left[x_\alpha - (x_n, \propto_n)\right] \geq \tilde{F}\left[x_\alpha - (x_{n_m}, \propto_{n_m})\right] * \tilde{F}\left[(x_{n_m}, \propto_{n_m}) - (x_n, \propto_n)\right]$$
$$> \tilde{F}\left[x_\alpha - (x_{n_m}, \propto_{n_m})\right] * (1 - \varepsilon)$$

Letting $m \to \infty$, we have $\tilde{F}\left[x_\alpha - (x_n, \propto_n)\right] \geq 1 * (1 - \varepsilon) = (1 - \varepsilon)$.

Hence, the sequence $\{(x_n, \propto_n)\}$ it fuzzy converges to x_α.

Definition 4.7:

Suppose that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space and $\tilde{D} \subseteq \tilde{A}$. Then $\tilde{D}$ is said to be fuzzy bounded if we can find q, $0 < q < 1$ 1 such that, $\tilde{F}(x_\alpha) > (1 - q) \forall x_\alpha \in \tilde{A}$.

Lemma 4.8:

Assume that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space. If a sequence $\{(x_n, \infty_n)\}$ with $(x_n, \infty_n) \to x_\alpha$ then it is fuzzy bounded and its fuzzy limit is unique.

Proof:

Let $(x_n, \infty_n) \to x_\alpha$, that is for a given $r > 0$ then we can find K with $\tilde{F}[(x_n, \infty_n) - x_\alpha] > (1 - r)$ for all $n \geq K$.

Put $t = \min\{\tilde{F}[(x_1, \infty_1) - x_\alpha], \tilde{F}[(x_2, \infty_2) - x_\alpha], \cdots, \tilde{F}[(x_K, \infty_K) - x_\alpha]\}$.

Using Remark 2.10 we can find $0 < \varepsilon < 1$ with $t * (1 - r) > (1 - \varepsilon)$. Now for $n \geq K$

$$\tilde{F}[(x_n, \infty_n) - x_\alpha] \geq \tilde{F}[(x_n, \infty_n) - (x_K, \infty_K)] * \tilde{F}[(x_K, \infty_K) - x_\alpha]$$
$$\geq t * (1 - r) > (1 - \varepsilon).$$

Hence (x_n, ∞_n) is fuzzy bounded in $(\tilde{A}, \tilde{F}, *)$.

Suppose that $(x_n, \infty_n) \to x_\alpha$ and $(x_n, \infty_n) \to y_\beta$. Therefore $\lim_{n \to \infty} \tilde{F}[(x_n, \infty_n) - x_\alpha] = 1$ and $\lim_{n \to \infty} \tilde{F}[(x_n, \infty_n) - y_\beta] = 1$. Now $\tilde{F}[x_\alpha - y_\beta] \geq \tilde{F}[x_\alpha - (x_n, \infty_n)] * \tilde{F}[(x_n, \infty_n) - y_\beta]$. By taking the limit to both sides, as $n \to \infty$, $\tilde{F}[x_\alpha - y_\beta] \geq 1 * 1 = 1$. So $\tilde{F}[x_\alpha - y_\beta] = 1$, hence $x_\alpha = y_\beta$.

Definition 4.9:

Suppose that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space on the fuzzy set $\tilde{A}$. Then $\tilde{B}[x_\alpha, q] = \{y_\beta \in \tilde{A} : \tilde{F}[x_\alpha - y_\beta] \geq (1 - q)\}$ is said to be a fuzzy closed fuzzy ball with center $x_\alpha \in \tilde{A}$ and radius q, $0 < q < 1$.

The proof of the following lemma is clear and hence is omitted.

Lemma 4.10:

Any $\tilde{B}[x_\alpha, q]$ in a fuzzy length space $(\tilde{A}, \tilde{F}, *)$ is a fuzzy closed fuzzy set.

Theorem 4.11:

A fuzzy length space $(\tilde{A}, \tilde{F}, *)$ is a fuzzy topological space.

Proof:

Suppose that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space. Put $\tau_{\tilde{F}} = \{\tilde{D} \subseteq \tilde{A} : x_\alpha \in \tilde{D}\}$ if and only if there is $0 < r < 1$ such that $\tilde{B}(x_\alpha, r) \in \tilde{D}$. We prove that $\tau_{\tilde{F}}$ is a fuzzy topology on $\tilde{A}$.

(i) Clear that ϕ and $\tilde{A} \in \tau_{\tilde{F}}$.

(ii) let $\tilde{D}_1, \tilde{D}_2, \cdots, \tilde{D}_n \in \tau_{\tilde{F}}$ and put $\tilde{V} = \bigcap_{j=1}^n \tilde{D}_j$. We will show that $\tilde{V} \in \tau_{\tilde{F}}$, let $a_\alpha \in \tilde{V}$ then $a_\alpha \in \tilde{D}_j$ for each $1 \leq j \leq n$. Hence there is $0 \leq q_j \leq 1$ such that $\tilde{B}(a_\alpha, q_j) \subseteq \tilde{D}_i$. Put $q = \min\{q_j : 1 \leq j \leq n\}$ so $q \leq q_j$ for all $1 \leq j \leq n$, this implies that $(1 - q) \geq (1 - q_j)$ for all $1 \leq j \leq n$. Therefore, $\tilde{B}(a_\alpha, q_j) \subseteq \bigcap_{j=1}^n \tilde{D}_j = \tilde{V}$, thus $\tilde{V} \in \tau_{\tilde{F}}$.

(iii) Let $\{\tilde{D}_j : j \in I\} \in \tau_{\tilde{F}}$. put $\tilde{U} = \bigcup_{j \in I} \tilde{D}_j$. We will show that $\tilde{U} \in \tau_{\tilde{F}}$. Let $b_\beta \in \tilde{U}$, then $b_\beta \in \tilde{D}_j$ for some $j \in I$, since $\tilde{D}_j \in \tau_{\tilde{F}}$ then there exists $0 < q < 1$ such that $\tilde{B}(b_\beta, p) \subseteq \tilde{D}_j$, Hence $\tilde{B}(b_\beta, p) \subseteq \tilde{D}_j \subset \bigcup_{j \in I} \tilde{D}_j = \tilde{U}$. This prove that $\tilde{U} \in \tau_{\tilde{F}}$. Hence, $(\tilde{A}, \tilde{F}, *)$ is a fuzzy topological space. $\tau_{\tilde{F}}$ is called the fuzzy topology induced by $\tilde{F}$.

Definition 4.12:

A fuzzy length space $(\tilde{A}, \tilde{F}, *)$ is said to be a fuzzy Hausdorff space if for any $x_\alpha, y_\beta \in \tilde{A}$ such that $x_\alpha \neq y_\beta$ we can find $\tilde{B}(x_\alpha, r)$ and $\tilde{B}(y_\beta, t)$ for some $0 < r < 1$

and $0 < t < 1$ such that $\tilde{B}(x_\alpha, r) \cap \tilde{B}(y_\beta, t) = \varnothing$.

Theorem 4.13:

Every fuzzy length space $(\tilde{A}, \tilde{F}, *)$ is a fuzzy Hausdorff space.

Proof:

Assume that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space and let $x_\alpha, y_\beta \in \tilde{A}$ with $x_\alpha \neq y_\beta$.

Put $\tilde{F}[x_\alpha - y_\beta] = p$, for some $0 < p < 1$. Then for each s, $0 < s < 1$, we can find q such that $q * q \geq s$ by Remark 2.10. Now consider the two fuzzy open fuzzy balls $\tilde{B}(x_\alpha, 1-q)$ and $\tilde{B}(y_\beta, 1-q)$. Then $\tilde{B}(x_\alpha, 1-q) \cap \tilde{B}(y_\beta, 1-q) = \varnothing$ Since if there exists $z_\delta \in \tilde{B}(x_\alpha, 1-q) \cap \tilde{B}(y_\beta, 1-q)$. Then

$$p = \tilde{F}[x_\alpha - y_\beta] \geq \tilde{F}[x_\alpha - z_\delta] * \tilde{F}[z_\delta - y_\beta] \geq q * q \geq s > p$$

is a contradiction, hence $(\tilde{A}, \tilde{F}, *)$ is a fuzzy Hausdorff space.

5. Fuzzy Continuous Operators on Fuzzy Length Spaces

In this section, we will suppose $\tilde{A}$ is fuzzy set of X and $\tilde{D}$ be fuzzy set of Y where X and Y are vector space.

Definition 5.1:

Let $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{D}, \tilde{F}_{\tilde{D}}, *)$ be two fuzzy length space on fuzzy set $\tilde{A}$ and $\tilde{D}$ respectively, let $\tilde{E} \subseteq \tilde{A}$ then The operator $T : \tilde{E} \to \tilde{D}$ is said to be fuzzy continuous at $a_\alpha \in \tilde{E}$, if for every $0 < \varepsilon < 1$ there exist $0 < \delta < 1$ such that $\tilde{F}_{\tilde{D}}[T(x_\beta) - T(a_\alpha)] > (1-\varepsilon)$ whenever $x_\beta \in \tilde{E}$ satisfying $\tilde{F}_{\tilde{A}}(x_\beta - a_\alpha) > (1-\delta)$. If T is fuzzy continuous at every fuzzy point of $\tilde{E}$, then T it is said to be fuzzy continuous on $\tilde{E}$.

Theorem 5.2:

Let $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{D}, \tilde{F}_{\tilde{D}}, *)$ be two fuzzy length space, let $\tilde{E} \subseteq \tilde{A}$. The operator $T : \tilde{E} \to \tilde{D}$ is fuzzy continuous at $a_\alpha \in \tilde{E}$ if and only if whenever a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{E}$ fuzzy converge to a_α, then the sequence of fuzzy points $\{(T(x_n), \alpha_n)\}$ fuzzy converges to $T(a_\alpha)$.

Proof:

Suppose that the operator $T : \tilde{E} \to \tilde{D}$ is fuzzy continuous at $a_\alpha \in \tilde{E}$ and let $\{(x_n, \alpha_n)\}$ be a sequence in $\tilde{E}$ fuzzy converge to a_α. Let $0 < \varepsilon < 1$ be given. By fuzzy continuity of T at a_α, then there exists $0 < \delta < 1$ such that whenever $x_\beta \in \tilde{E}$ and $\tilde{F}_{\tilde{A}}(x_\beta - a_\alpha) > (1-\delta)$, implies $\tilde{F}_{\tilde{D}}[T(x_\beta) - T(a_\alpha)] > (1-\varepsilon)$. Since $(x_n, \alpha_n) \to \alpha_n$ then we can find K with $n \geq K$ such that $\tilde{F}_{\tilde{A}}[(x_n, \alpha_n) - a_\alpha] > (1-\delta)$. Therefore when $n \geq K$ implies $\tilde{F}_{\tilde{D}}[T(x_n, \alpha_n) - T(a_\alpha)] > (1-\varepsilon)$. Thus $T(x_n, \alpha_n) \to T(a_\alpha)$.

Conversely, assume that every sequence $\{(x_n, \alpha_n)\}$ in $\tilde{E}$ fuzzy converging to a_α has the property that $T(x_n, \alpha_n) \to T(a_\alpha)$. Suppose that T is not fuzzy continuous at a_α. Then there is $0 < \varepsilon < 1$ and for which no δ, $0 < \delta < 1$ can satisfy the requirement that $x_\beta \in \tilde{E}$ and $\tilde{F}_{\tilde{A}}[x_\beta - a_\alpha] > (1-\delta)$ implies $\tilde{F}_{\tilde{D}}[T(x_\beta) - T(a_\alpha)] > (1-\varepsilon)$. This means that for every $\delta, 0 < \delta < 1$ there exists $x_\beta \in \tilde{E}$ such that $\tilde{F}_{\tilde{A}}[x_\beta - a_\alpha] > (1-\delta)$ but $\tilde{F}_{\tilde{D}}[T(x_\beta) - T(a_\alpha)] \leq (1-\varepsilon)$. For every $n \in K$, there exist

$(x_n, \alpha_n) \in \tilde{E}$ such that $\tilde{F}_{\tilde{A}}\left[(x_n, \alpha_n) - a_\alpha\right] \geq \left(1 - \dfrac{1}{n}\right)$, but

$\tilde{F}_{\tilde{D}}\left[T(x_n, \alpha_n) - T(a_\alpha)\right] \leq (1 - \varepsilon)$. The sequence $\{(x_n, \alpha_n)\}$ fuzzy converges to a_α but the sequence $\{T(x_n, \alpha_n)\}$ does not fuzzy converge to (a_α). This contradicts the assumption that every sequence $\{(x_n, \alpha_n)\}$ in $\tilde{E}$ fuzzy converging to a_α has the property that $T(x_n, \alpha_n) \to T(a_\alpha)$. Therefore the assumption that T is not fuzzy continuous at a_α must be false.

Definition 5.3:

Let $\left(\tilde{A}, \tilde{F}_{\tilde{A}}, *\right)$ and $\left(\tilde{D}, \tilde{F}_{\tilde{D}}, *\right)$ be two fuzzy length spaces let $\tilde{E} \subseteq \tilde{A}$ and let $T : \tilde{E} \to \tilde{D}$ the fuzzy point a_α is a fuzzy limit of $\tilde{E}$ written by $\lim_{x_\beta \to a_\alpha} T(x_\beta) = b_\beta$ whenever $b_\beta \in \tilde{D}$ if for every $0 < \varepsilon < 1$ there is $0 < \delta < 1$ such that $\tilde{F}_{\tilde{D}}\left[T(x_\beta) - b_\beta\right] > (1 - \varepsilon)$ when $x_\beta \in \tilde{E}$ and $\tilde{F}_{\tilde{A}}\left[x_\beta - a_\alpha\right] > (1 - \delta)$.

Proposition 5.4:

Let $\left(\tilde{A}, \tilde{F}_{\tilde{A}}, *\right)$ and $\left(\tilde{D}, \tilde{F}_{\tilde{D}}, *\right)$ be two fuzzy length spaces, let $\tilde{E} \subseteq \tilde{A}$ and $T : \tilde{E} \to \tilde{D}$. Suppose that a_α is fuzzy limit fuzzy point of $\tilde{E}$. Then $\lim_{x_\beta \to a_\alpha} T(x_\beta) = b_\beta$ if and only if for every sequence $\{(x_n, \alpha_n)\}$ in $\tilde{E}$ such that $(x_n, \alpha_n) \neq a_\alpha$ and $(x_n, \alpha_n) \to a_\alpha$.

Proof: the argument is similar that of Theorem (5.3) and therefore is not included.

Proposition 5.5:

An operator T of the fuzzy length space $\left(\tilde{A}, \tilde{F}_{\tilde{A}}, *\right)$ into a fuzzy length space $\left(\tilde{D}, \tilde{F}_{\tilde{D}}, *\right)$ is fuzzy continuous at the fuzzy point $a_\alpha \in \tilde{A}$ if and only if for every $0 < \varepsilon < 1$ there exist $0 < \delta < 1$ such that $\tilde{B}(a_\alpha, \delta) \subseteq T^{-1}\left[\tilde{B}(a_\alpha, \varepsilon)\right]$.

Proof: the operator $T : \tilde{A} \to \tilde{D}$ is fuzzy continuous at $a_\alpha \in \tilde{A}$ if and only if for every $0 < \varepsilon < 1$ there exist $0 < \delta < 1$ such that $\tilde{F}_{\tilde{D}}\left[T(x_\beta) - T(a_\alpha)\right] > (1 - \varepsilon)$ for all x_β satisfying $\tilde{F}_{\tilde{A}}\left[x_\beta - a_\alpha\right] > (1 - \delta)$ *i.e.* $x_\beta \in \tilde{B}(a_\alpha, \delta)$ implies $T(x_\beta) \in \tilde{B}(T(a_\alpha), \varepsilon)$ or $T\left[\tilde{B}(a_\alpha, \delta)\right] \subseteq \tilde{B}(T(a_\alpha), \varepsilon)$ this is equivalent to the condition $\tilde{B}(a_\alpha, \delta) \subseteq T^{-1}\left[\tilde{B}(T(a_\alpha), \varepsilon)\right]$.

Theorem 5.6:

An operator $T : \tilde{A} \to \tilde{D}$ is fuzzy continuous if and only if $T^{-1}(\tilde{G})$ is fuzzy open in $\tilde{A}$ for all fuzzy open $\tilde{G}$ of $\tilde{D}$ where $\left(\tilde{A}, \tilde{F}_{\tilde{A}}, *\right)$ and $\left(\tilde{D}, \tilde{F}_{\tilde{D}}, *\right)$ are fuzzy length space.

Proof:

Suppose that T is fuzzy continuous operator and let $\tilde{G}$ be fuzzy open in $\tilde{D}$ we show that $T^{-1}(\tilde{G})$ is fuzzy open in $\tilde{A}$. Since $\varnothing$ and $\tilde{A}$ are fuzzy open, we may suppose that $T^{-1}(\tilde{G}) \neq \varnothing$ and $T^{-1}(\tilde{G}) \neq \tilde{A}$.

Let $x_\beta \in T^{-1}(\tilde{G})$ then $T(x_\beta) \in \tilde{G}$. Since $\tilde{G}$ is fuzzy open, then there is $0 < \varepsilon < 1$ such that $\tilde{B}(T(x_\beta), \varepsilon) \subseteq \tilde{G}$, since T is fuzzy continuous at x_β by proposition 4.5 for this ε there exist $0 < \delta < 1$ such that $\tilde{B}(x_\beta, \delta) T^{-1} \subseteq \left[\tilde{B}(T(x_\beta), \varepsilon)\right] \subseteq T^{-1}(\tilde{G})$.

Thus, every fuzzy point x_β of $T^{-1}(\tilde{G})$ is an interior fuzzy point and so $T^{-1}(\tilde{G})$ if a fuzzy open in $\tilde{A}$.

Conversely, suppose that $T^{-1}(\tilde{G})$ is fuzzy open in $\tilde{A}$ for any fuzzy open $\tilde{G}$ of $\tilde{D}$. Let $x_\alpha \in \tilde{A}$ for each $0 < \varepsilon < 1$, the fuzzy ball $\tilde{B}(T(x_\alpha), \varepsilon)$ is fuzzy open in $\tilde{A}$

since $x_\alpha \in T^{-1}\left[\tilde{B}\left(T\left(x_\alpha\right),\varepsilon\right)\right]$ it follows that there exist $0 < \delta < 1$ such that $\tilde{B}\left(x_\alpha,\delta\right) \subseteq T^{-1}\left[\tilde{B}\left(T\left(x_\alpha\right),\varepsilon\right)\right]$ by proposition 5.5, it follows that T is fuzzy continuous.

Theorem 5.7:

An operator $T : \tilde{A} \to \tilde{D}$ is fuzzy continuous on $\tilde{A}$ if and only if $T^{-1}\left(\tilde{E}\right)$ is fuzzy closed in $\tilde{A}$ for all fuzzy closed $\tilde{E}$ of $\tilde{D}$.

Proof:

Let $\tilde{E}$ be a fuzzy closed subset of $\tilde{D}$ then $\tilde{D} - \tilde{E}$ is fuzzy open in $\tilde{D}$ so that $T^{-1}\left[\tilde{D} - \tilde{E}\right]$ is fuzzy open in $\tilde{A}$ by theorem 4.6 but $T^{-1}\left[\tilde{D} - \tilde{E}\right] = \tilde{A} - T^{-1}\left(\tilde{E}\right)$ so $T^{-1}\left(\tilde{E}\right)$ is fuzzy closed in $\tilde{A}$.

Conversely, suppose that $T^{-1}\left(\tilde{E}\right)$ is fuzzy closed in $\tilde{A}$ for all fuzzy closed subset $\tilde{E}$ of $\tilde{D}$. But the empty fuzzy set and the whole space $\tilde{A}$ are fuzzy closed fuzzy set. Then $\tilde{A} - T^{-1}\left(\tilde{E}\right)$ is fuzzy open in $\tilde{A}$ and $T^{-1}\left[\tilde{D} - \tilde{E}\right] = \tilde{A} - T^{-1}\left(\tilde{E}\right)$ is fuzzy in $\tilde{A}$. Since every fuzzy open subset of $\tilde{D}$ is of the type $\tilde{D} - \tilde{E}$, where $\tilde{E}$ is suitable fuzzy closed fuzzy set, it follows by using theorem 5.6, that T is fuzzy continuous.

Theorem 5.8:

Let $\left(\tilde{A}, \tilde{F}_{\tilde{A}}, *\right)$ and $\left(\tilde{D}, \tilde{F}_{\tilde{D}}, *\right)$ and $\left(\tilde{Z}, \tilde{F}_{\tilde{Z}}, *\right)$ be three fuzzy length spaces let $T : \tilde{A} \to \tilde{D}$ and $S : \tilde{D} \to \tilde{Z}$ be fuzzy continuous operator. Then the composition SoT is a fuzzy continuous operator from $\tilde{A}$ into $\tilde{Z}$.

Proof:

Let $\tilde{G}$ be fuzzy open subset of $\tilde{Z}$. By theorem (5.6), $S^{-1}\left(\tilde{G}\right)$ is a fuzzy open subset of $\tilde{D}$ and another application of the same theorem shows that $T^{-1}\left(S^{-1}\left(\tilde{G}\right)\right)$ is a fuzzy open subset of $\tilde{A}$, since $\left(SoT\right)^{-1}\left(\tilde{G}\right) = T^{-1}\left(S^{-1}\left(\tilde{G}\right)\right)$ it follows from the same theorem again that (SoT) is a fuzzy continuous.

Theorem 5.9:

Let $\left(\tilde{A}, \tilde{F}_{\tilde{A}}, *\right)$ and $\left(\tilde{D}, \tilde{F}_{\tilde{D}}, *\right)$ be two fuzzy length spaces let $T : \tilde{A} \to \tilde{D}$, then the following statements are equivalent.

(i) T is fuzzy continuous on $\tilde{A}$.

(ii) $\overline{T^{-1}\left(\tilde{E}\right)} \subseteq T^{-1}\left(\overline{\tilde{E}}\right)$ For all subset $\tilde{E}$ of $\tilde{D}$.

(iii) $T^{-1}\left(\overline{\tilde{G}}\right) \subseteq \overline{T^{-1}\left(\tilde{G}\right)}$ For all subset $\tilde{G}$ of $\tilde{A}$.

Proof: (i) ⇒ (ii)

Let $\tilde{E}$ be a subset of $\tilde{D}$, since $\overline{\tilde{E}}$ is a fuzzy closed subset of $\tilde{D}$, $T^{-1}\left(\overline{\tilde{E}}\right)$ is fuzzy closed in $\tilde{A}$. Moreover $T^{-1}\left(\tilde{E}\right) \subseteq T^{-1}\left(\overline{\tilde{E}}\right)$ and so $\overline{T^{-1}\left(\tilde{E}\right)} \subseteq T^{-1}\left(\overline{\tilde{E}}\right)$.

[Recall that $\overline{T^{-1}\left(\tilde{E}\right)}$ is the smallest fuzzy closed fuzzy set containing $T^{-1}\left(\tilde{E}\right)$].

Proof: (ii) ⇒ (iii)

Let $\tilde{G}$ be a subset of $\tilde{A}$, then if $\tilde{E} = T\left(\tilde{G}\right)$, we have $\tilde{G} \subseteq T^{-1}\left(\tilde{E}\right)$ and $\overline{\tilde{G}} \subseteq \overline{T^{-1}\left(\tilde{E}\right)}$ thus $T\left(\overline{\tilde{G}}\right) \subseteq T\left(T^{-1}\left(\overline{\tilde{E}}\right)\right) = \overline{\tilde{E}} = \overline{T\left(\tilde{G}\right)}$.

Proof: (iii) ⇒ (i)

Let $\tilde{S}$ be a fuzzy closed fuzzy set in $\tilde{D}$ and fuzzy set $T^{-1}\left(\tilde{S}\right) = \tilde{S}_1$. By theorem

(4.7), it is sufficient to show that $\tilde{S}_1$ is a fuzzy closed in $\tilde{A}$, that is, $\tilde{S}_1 = \overline{\tilde{S}_1}$ Now $T\left(\overline{\tilde{S}_1}\right) \subseteq \overline{T\left(T^{-1}\left(\tilde{S}\right)\right)} \subseteq \overline{\tilde{S}} = \tilde{S}$ so that $\overline{\tilde{S}_1} \subseteq T^{-1}\left(T\left(\overline{\tilde{S}_1}\right)\right) \subseteq T^{-1}\left(\tilde{S}\right) = \tilde{S}_1$.

6. Conclusion

In this research the fuzzy length space of fuzzy points was defined as a generalization of the definition of fuzzy norm on ordinary points. Based on the defined fuzzy length, most of the properties of the ordinary norm were proved.

References

[1] Zadeh, L. (1965) Fuzzy Sets. *Information and Control*, **8**, 338-452. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Bag, T. and Samanta, S. (2003) Finite Dimensional Fuzzy Linear Spaces. *Fuzzy Mathematics*, **11**, 678-705.

[3] Bag, T. and Samanta, S. (2006) Fixed Point Theorems on Fuzzy Normed Spaces. *Information Sciences*, **176**, 2910-2931. https://doi.org/10.1016/j.ins.2005.07.013

[4] Wang, C.X. and Ming, M. (1993) Continuity and Boundness Mappings between Fuzzy Normed Spaces. *Fuzzy Mathematics*, **1**, 13-24.

[5] Felbin, C. (1992) Finite Dimensional Fuzzy Normed Linear Spaces. *Fuzzy Sets and Systems*, **48**, 239-248. https://doi.org/10.1016/0165-0114(92)90338-5

[6] Golet, I. (2010) On Generalized Fuzzy Normed Spaces and Coincidence Theorems. *Fuzzy Sets and Systems*, **161**, 1138-1144. https://doi.org/10.1016/j.fss.2009.10.004

[7] Kider, J. (2012) New Fuzzy Normed Spaces. *Baghdad Science Journal*, **9**, 559-564.

[8] Kider, J.R. (2013) Completeness of the Cartesian Product of Two Complete Fuzzy Normed Spaces. *Engineering and Technology Journal*, **31**, 310-315.

[9] Kider, J.K. and Hussain, Z.A. (2014) Continuous and Uniform Continuous Mapping on Standard Fuzzy Metric Spaces. *Engineering and Technology Journal*, **32**, 1111-1119.

[10] Cheng, S. and Mordeson, J. (1994) Fuzzy Linear Operators and Fuzzy Normed Linear Spaces. *Bulletin of Calcutta Mathematical Society*, **86**, 429-436.

[11] Kramosil, O. and Michalek, J. (1975) Fuzzy Metrics and Statistical Metric Spaces. *Kybernetika*, **11**, 326-334.

[12] George, A. and Veeramani, P. (1994) On Some Results in Fuzzy Metric Spaces. *Fuzzy Sets and Systems*, **64**, 395-399. https://doi.org/10.1016/0165-0114(94)90162-7